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$\mathcal{S}$-paracompactness modulo an ideal

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ABSTRACT

The notion of $\mathcal{S}$-paracompactness modulo an ideal was introduced and studied in [15]. In this paper, we introduce and investigate the notion of $\alpha\mathcal{S}$-paracompact subset modulo an ideal which is a generalization of the notions of $\alpha\mathcal{S}$-paracompact set [1] and α-paracompact set modulo an ideal [7].

RESUMEN

La noción de $\mathcal{S}$-paracompacidad módulo un ideal fue introducida y estudiada en [15]. En este artículo, introducimos e investigamos la noción de un subconjunto $\alpha\mathcal{S}$-paracompacto módulo un ideal, que es una generalización de las nociones de conjunto $\alpha\mathcal{S}$-paracompacto [1] y conjunto α-paracompacto módulo un ideal [7].

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1 Introduction

The concept of α-paracompact subset modulo an ideal was defined and investigated by Ergun and Noiri [7]. The notions of S-paracompact spaces and αS-paracompact subsets were introduced in 2006 by Al-Zoubi [1] and also have been studied by Li and Song [13]. Very recently, Sanabria, Rosas, Carpintero, Salas and García [15] have introduced and investigated the concept of S-paracompact space with respect to an ideal as a generalization of the S-paracompact spaces. In this paper, we introduce the notion of αS-paracompact subset modulo an ideal which is a generalization of both αS-paracompact subset [1] and α-paracompact subset modulo an ideal.

2 Preliminaries

Throughout this paper, (X, τ) always means a topological space on which no separation axioms are assumed unless explicitly stated. If A is a subset of (X, τ), we denote the closure of A and the interior of A by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. Also, we denote by $\wp(X)$ the class of all subset of X. A subset A of (X, τ) is said to be *semi-open* [11] (resp. *semi-preopen* [2]) if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$). The complement of a semi-open set is called a *semi-closed* set. The *semi-closure* of A, denoted by $\mathrm{sCl}(A)$, is defined by the intersection of all semi-closed sets containing A. The collection of all semi-open sets of a topological space (X, τ) is denoted by $\mathrm{SO}(X, \tau)$. A collection $\mathcal{V}$ of subsets of a space (X, τ) is said to be *locally finite*, if for each $x \in X$ there exists $U_x \in \tau$ containing x and U_x intersects at most finitely many members of $\mathcal{V}$. A space (X, τ) is said to be *paracompact* (resp. S-*paracompact* [1]), if every open cover of X has a locally finite open (resp. semi-open) refinement which covers to X (we do not require a refinement to be a cover).

Lemma 2.1. *Let (X, τ) be a space. Then, the following properties hold:*

(1) *If (A, τ_A) is a subspace of (X, τ), $B \subseteq A$ and $B \in \mathrm{SO}(X, \tau)$, then $B \in \mathrm{SO}(A, \tau_A)$ [11].*

(2) *If $A \in \tau$ and $B \in \mathrm{SO}(X, \tau)$, then $A \cap B \in \mathrm{SO}(X, \tau)$ [4].*

(3) *If (A, τ_A) is an open subspace of (X, τ), $B \subseteq A$ and $B \in \mathrm{SO}(A, \tau_A)$, then $B \in \mathrm{SO}(X, \tau)$ [5].*

An *ideal* $\mathcal{I}$ on a nonempty set X is a nonempty collection of subset of X which satisfies the following two properties:

(1) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$;

(2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.

In this paper, the triplet $(X, \tau, \mathcal{I})$ denote a topological space (X, τ) together with an ideal $\mathcal{I}$ on X and will simply called a space. Given a space $(X, \tau, \mathcal{I})$, a set operator $(.)^\star : \wp(X) \to \wp(X)$, called the *local function* [10] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subset X$,

$A^\star(\mathcal{I},\tau) = \{x \in X : U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$. When there is no chance for confusion, we will simply write $A^\star$ for $A^\star(\mathcal{I},\tau)$. In general, $X^\star$ is a proper subset of X. The hypothesis $X = X^\star$ is equivalent to the hypothesis $\tau \cap \mathcal{I} = \emptyset$. According to [14], we call the ideals which satisfy this hypothesis τ-*boundary* ideals. Note that $\mathrm{Cl}^\star(A) = A \cup A^\star$ defines a Kuratowski closure for a topology $\tau^\star(\mathcal{I})$, finer than τ. A basis $\beta(\mathcal{I},\tau)$ for $\tau^\star(\mathcal{I})$ can be described as follows: $\beta(\mathcal{I},\tau) = \{V \setminus J : V \in \tau \text{ and } J \in \mathcal{I}\}$. When there is no chance for confusion, we will simply write $\tau^\star$ for $\tau^\star(\mathcal{I})$ and β for $\beta(\mathcal{I},\tau)$. In the sequel, the ideal of nowhere dense (resp. meager) subsets of (X,τ) is denoted by $\mathcal{N}$ (resp. $\mathcal{M}$).

3 αS-paracompactness modulo an ideal

In this section, we shall introduce and study the αS-paracompact subsets modulo an ideal $\mathcal{I}$, which is a natural generalization of αS-paracompact subsets. First recall some notions of paracompactness.

Definition 3.1. A subset A of a space (X,τ) is said to be α-paracompact [3] (resp. α-almost paracompact [9]) if for any open cover $\mathcal{U}$ of A, there exists a locally finite collection $\mathcal{V}$ of open sets such that $\mathcal{V}$ refines $\mathcal{U}$ and $A \subset \bigcup\{V : V \in \mathcal{V}\}$ (resp. $A \subset \bigcup\{\mathrm{Cl}(V) : V \in \mathcal{V}\}$). A space (X,τ) is said to be paracompact (resp. almost-paracompact) if X is α-paracompact (resp. α-almost paracompact).

Definition 3.2. A subset A of a space $(X,\tau,\mathcal{I})$ is said to be α-paracompact modulo $\mathcal{I}$ [7] (briefly α-paracompact (mod $\mathcal{I}$)), if for any open cover $\mathcal{U}$ of A, there exist $I \in \mathcal{I}$ and a locally finite collection $\mathcal{V}$ of open sets such that $\mathcal{V}$ refines $\mathcal{U}$ and $A \subset \bigcup\{V : V \in \mathcal{V}\} \cup I$.

A space $(X,\tau,\mathcal{I})$ is said to be $\mathcal{I}$-paracompact or paracompact with respect to $\mathcal{I}$ [16], if X is α-paracompact modulo $\mathcal{I}$. In the present, it is called paracompact modulo $\mathcal{I}$ (or briefly paracompact (mod $\mathcal{I}$)).

Definition 3.3. A subset A of a space (X,τ) is said to be αS-paracompact [1] if for any open cover $\mathcal{U}$ of A, there exists a locally finite collection $\mathcal{V}$ of open sets such that $\mathcal{V}$ refines $\mathcal{U}$ and $A \subset \bigcup\{V : V \in \mathcal{V}\}$. A space (X,τ) is said to be S-paracompact if X is αS-paracompact.

Now, we give the definition of αS-paracompact subset modulo an ideal $\mathcal{I}$.

Definition 3.4. A subset A of a space $(X,\tau,\mathcal{I})$ is said to be αS-paracompact modulo $\mathcal{I}$ (briefly αS-paracompact (mod $\mathcal{I}$)), if for any open cover $\mathcal{U}$ of A, there exist $I \in \mathcal{I}$ and a locally finite collection $\mathcal{V}$ of semi-open sets such that $\mathcal{V}$ refines $\mathcal{U}$ and $A \subset \bigcup\{V : V \in \mathcal{V}\} \cup I$.

A space $(X,\tau,\mathcal{I})$ is said to be $\mathcal{I}$-S-paracompact or S-paracompact with respect to $\mathcal{I}$ [15], if X is αS-paracompact modulo $\mathcal{I}$. In the present, it is called S-paracompact modulo $\mathcal{I}$ (or briefly S-paracompact (mod $\mathcal{I}$)). We say that A is S-paracompact (mod $\mathcal{I}$) if $(A,\tau_A,\mathcal{I}_A)$ is S-paracompact (mod $\mathcal{I}_A$) as a subspace, where τ_A is the relative topology induced on A by τ and $\mathcal{I}_A = \{I \cap A : I \in \mathcal{I}\}$.

Proposition 3.1. Let A be a subset of a space (X, τ) and $\mathcal{I}$ an ideal on (X, τ). Then, the following properties hold:

(1) If A is α-paracompact (mod $\mathcal{I}$), then A is αS-paracompact (mod $\mathcal{I}$).

(2) Every $I \in \mathcal{I}$ is an αS-paracompact (mod $\mathcal{I}$).

(3) $(X, \tau, \mathcal{I})$ is S-paracompact (mod $\mathcal{I}$) if there exists $I \in \mathcal{I}$ such that $X - I$ is αS-paracompact (mod $\mathcal{I}$).

(4) A is αS-paracompact if and only if it is αS-paracompact (mod $\{\emptyset\}$).

Proof. (1) Follows from the fact that every open set is semi-open.
(2) Suppose that there exists $I \in \mathcal{I}$ such that I is not αS-paracompact (mod $\mathcal{I}$). Then, there exists an open cover $\mathcal{U}$ of I such that $I \not\subset \bigcup\{V : V \in \mathcal{V}\} \cup J$ for every $J \in \mathcal{I}$ and every locally finite collection $\mathcal{V}$ which refines $\mathcal{U}$. This is a contradiction, because $I \in \mathcal{I}$ and $I \subset \bigcup\{V : V \in \mathcal{V}\} \cup I$.
(3) Suppose that there exists $I \in \mathcal{I}$ such that $X - I$ is αS-paracompact (mod $\mathcal{I}$) and let $\mathcal{U}$ be an open cover of X. Then, $\mathcal{U}$ is an open cover of $X - I$ and hence there exist $J \in \mathcal{I}$ and a locally finite collection $\mathcal{V}$ of semi-open sets such that $\mathcal{V}$ refines $\mathcal{U}$ and $X - I \subset \bigcup\{V : V \in \mathcal{V}\} \cup J$. Thus, $X = (X - I) \cup I \subset \bigcup\{V : V \in \mathcal{V}\} \cup (J \cup I)$ and as $J \cup I \in \mathcal{I}$, we have $(X, \tau, \mathcal{I})$ is S-paracompact (mod $\mathcal{I}$).
(4) It is obvious. □

Now, we give some comments related with the Proposition 3.1.

Remark 3.1. According to Proposition 3.1(1), every α-paracompact (mod $\mathcal{I}$) (resp. αS-paracompact) subset is αS-paracompact (mod $\mathcal{I}$), and from this point of view, the notion of αS-paracompact (mod $\mathcal{I}$) subset is a natural generalization of the notion of α-paracompact (mod $\mathcal{I}$) (resp. αS-paracompact) subset. On the other hand, in Example 2.11 of [13], it is shows that there exists a semiregular Hausdorff space X and a regular closed subset M of X such that M is an αS-paracompact (mod $\{\emptyset\}$) subset of X, but M is not α-paracompact (mod $\{\emptyset\}$). Thus, the converse of Proposition 3.1(1) in general is not true.

Proposition 3.2. Let A be a subset of a space (X, τ) and $\mathcal{I}$ an ideal on (X, τ). Then, the following properties hold:

(1) If A is a semi-open and αS-paracompact (mod $\mathcal{I}$) set and $\mathcal{I}$ is τ-boundary, then A is α-almost paracompact.

(2) A semi-preopen set A is αS-paracompact (mod $\mathcal{N}$) if and only if it is α-almost paracompact.

Proof. (1) Let $\mathcal{U}$ be any open cover of A. Then there exist $I \in \mathcal{I}$ and a locally finite collection $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ of semi-open sets such that $\mathcal{V}$ refines $\mathcal{U}$ and $A \subset \bigcup\{V_\lambda : \lambda \in \Lambda\} \cup I$. Since A is semi-open, $A \subset \mathrm{Cl}(\mathrm{Int}(A))$ and as $\mathcal{I}$ is τ-boundary, $\mathrm{Int}(I) = \emptyset$. Now, by the locally finiteness of $\mathcal{V}$, the collection $\mathcal{V}' = \{\mathrm{Int}(V_\lambda) : \lambda \in \Lambda\}$ is also locally finite, it follows that

$$\begin{aligned}
A &\subset \mathrm{Cl}(\mathrm{Int}(A)) \subset \mathrm{Cl}\left(\mathrm{Int}\left(\bigcup_{\lambda\in\Lambda} V_\lambda \cup I\right)\right) \\
&\subset \mathrm{Cl}\left(\mathrm{Int}\left(\bigcup_{\lambda\in\Lambda} \mathrm{Cl}(\mathrm{Int}(V_\lambda)) \cup I\right)\right) \\
&= \mathrm{Cl}\left(\mathrm{Int}\left(\mathrm{Cl}\left(\bigcup_{\lambda\in\Lambda} \mathrm{Int}(V_\lambda)\right) \cup I\right)\right) \\
&= \mathrm{Cl}\left(\mathrm{Int}\left(\mathrm{Cl}\left(\bigcup_{\lambda\in\Lambda} \mathrm{Int}(V_\lambda)\right) \cup \mathrm{Int}(I)\right)\right) \\
&= \mathrm{Cl}\left(\mathrm{Int}\left(\mathrm{Cl}\left(\bigcup_{\lambda\in\Lambda} \mathrm{Int}(V_\lambda)\right)\right)\right) \\
&\subset \mathrm{Cl}\left(\bigcup_{\lambda\in\Lambda} \mathrm{Int}(V_\lambda)\right) = \bigcup_{\lambda\in\Lambda} \mathrm{Cl}(\mathrm{Int}(V_\lambda)).
\end{aligned}$$

If $W_\lambda = \mathrm{Int}(V_\lambda)$, then $A \subset \bigcup_{\lambda\in\Lambda} \mathrm{Cl}(W_\lambda)$. Observe that W_λ is open for each $\lambda \in \Lambda$ and $W_\lambda \subset V_\lambda \subset U$ for some $U \in \mathcal{U}$, hence $\mathcal{W} = \{W_\lambda : \lambda \in \Lambda\}$ is a locally finite open refinement of $\mathcal{U}$. Therefore, A is α-almost paracompact.

(2) Similar to the proof of (1), if A is semi-preopen, then

$$\begin{aligned}
A &\subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))) \subset \mathrm{Cl}\left(\mathrm{Int}\left(\mathrm{Cl}\left(\bigcup_{\lambda\in\Lambda} V_\lambda \cup I\right)\right)\right) \\
&= \mathrm{Cl}\left(\mathrm{Int}\left(\mathrm{Cl}\left(\bigcup_{\lambda\in\Lambda} V_\lambda\right) \cup \mathrm{Cl}(I)\right)\right) \\
&= \mathrm{Cl}\left(\mathrm{Int}\left(\mathrm{Cl}\left(\bigcup_{\lambda\in\Lambda} V_\lambda\right) \cup \mathrm{Int}(\mathrm{Cl}(I))\right)\right) \\
&= \mathrm{Cl}\left(\mathrm{Int}\left(\mathrm{Cl}\left(\bigcup_{\lambda\in\Lambda} V_\lambda\right)\right)\right) \\
&\subset \mathrm{Cl}\left(\mathrm{Int}\left(\mathrm{Cl}\left(\bigcup_{\lambda\in\Lambda} \mathrm{Cl}(\mathrm{Int}(V_\lambda))\right)\right)\right) \\
&= \mathrm{Cl}\left(\mathrm{Int}\left(\mathrm{Cl}\left(\bigcup_{\lambda\in\Lambda} \mathrm{Int}(V_\lambda)\right)\right)\right) \\
&\subset \mathrm{Cl}\left(\bigcup_{\lambda\in\Lambda} \mathrm{Int}(V_\lambda\right) = \bigcup_{\lambda\in\Lambda} \mathrm{Cl}(\mathrm{Int}(V_\lambda)).
\end{aligned}$$

Therefore, the proof follows. $\square$

As a consequence of Proposition 3.2, we obtain the following result.

Corollary 3.1. (Sanabria et al. [15]) Let $\mathcal{I}$ be an ideal on a space (X,τ). Then, the following properties hold:

(1) If $\mathcal{I}$ is τ-boundary and (X,τ) is S-paracompact (mod $\mathcal{I}$), then (X,τ) is almost-paracompact.

(2) (X,τ) is S-paracompact (mod $\mathcal{N}$) if and only if it is almost-paracompact.

Theorem 3.1. If every open subset of a space $(X,\tau,\mathcal{I})$ is αS-paracompact (mod $\mathcal{I}$), then every subspace of $(X,\tau,\mathcal{I})$ is S-paracompact (mod $\mathcal{I}$).

Proof. Suppose that A is any subspace of $(X,\tau,\mathcal{I})$ and let $\mathcal{U}=\{U_\mu:\mu\in\Delta\}$ be a τ_A-open cover of A. For every $\mu\in\Delta$ there exists $V_\mu\in\tau$ such that $U_\mu=V_\mu\cap A$. Put $V=\bigcup\{V_\mu:\mu\in\Delta\}$, then $V\in\tau$ and $\mathcal{V}=\{V_\mu:\mu\in\Delta\}$ is a τ-open cover of V. By hypothesis, there exist $I\in\mathcal{I}$ and a τ-locally finite collection $\mathcal{W}=\{W_\lambda:\lambda\in\Lambda\}$ of τ-semi-open sets such that $\mathcal{W}$ refines $\mathcal{V}$ and $V\subset\bigcup\{W_\lambda:\lambda\in\Lambda\}\cup I$. Then, we have

$$\begin{aligned} A &= \bigcup_{\mu\in\Delta} U_\mu = \bigcup_{\mu\in\Delta}(V_\mu\cap A) = \left(\bigcup_{\mu\in\Delta} V_\mu\right)\cap A \\ &= V\cap A \subset \left(\bigcup_{\lambda\in\Lambda} W_\lambda \cup I\right)\cap A = \bigcup_{\lambda\in\Lambda}(W_\lambda\cap A)\cup I_A, \end{aligned}$$

where $I_A=I\cap A\in\mathcal{I}_A$. If $x\in A$, then there exists $G_x\in\tau$ containing x such that $W_\lambda\cap G_x=\emptyset$ for all $\lambda\neq\lambda_1,\lambda_2,\dots,\lambda_n$ and so $(W_\lambda\cap G_x)\cap A=\emptyset$ for all $\lambda\neq\lambda_1,\lambda_2,\dots,\lambda_n$. It follows that $(W_\lambda\cap A)\cap(G_x\cap A)=\emptyset$ for all $\lambda\neq\lambda_1,\lambda_2,\dots,\lambda_n$ and hence, the collection $\mathcal{H}=\{W_\lambda\cap A:\lambda\in\Lambda\}$ is τ_A-locally finite. If $W_\lambda\cap A\in\mathcal{H}$, then $W_\lambda\in\mathcal{W}$ and since $\mathcal{W}$ refines $\mathcal{V}$, $W_\lambda\subseteq V_\mu$ for some $V_\mu\in\mathcal{V}$, which implies that $W_\lambda\cap A\subset V_\mu\cap A=U_\mu\in\mathcal{U}$. Therefore, $\mathcal{H}$ refines $\mathcal{U}$. This shows that $\mathcal{H}=\{W_\lambda\cap A:\lambda\in\Lambda\}$ is a τ_A-locally finite collection of τ_A-semi-open sets which refines $\mathcal{U}$ such that $A\subset\bigcup\{H:H\in\mathcal{H}\}\cup I_A$. Thus, every subspace of $(X,\tau,\mathcal{I})$ is S-paracompact (mod $\mathcal{I}$). $\square$

The following result is an immediate consequence of Theorem 3.2.

Corollary 3.2. If every open subset of a space $(X,\tau,\mathcal{I})$ is αS-paracompact (mod $\mathcal{I}$), then $(X,\tau,\mathcal{I})$ is S-paracompact (mod $\mathcal{I}$).

Recall that a subset A of a space (X,τ) is said to be g-closed [12] if $\mathrm{Cl}(A)\subset U$ whenever $A\subset U$ and $U\in\tau$.

Theorem 3.2. If $(X,\tau,\mathcal{I})$ is S-paracompact (mod $\mathcal{I}$) and A is a g-closed subset of X, then A is αS-paracompact (mod $\mathcal{I}$).

Proof. Suppose that A is a g-closed subset of an S-paracompact (mod $\mathcal{I}$) space $(X, \tau, \mathcal{I})$. Let $\mathcal{U} = \{U_\mu : \mu \in \Delta\}$ be an open cover of A. Since A is g-closed and $A \subset \bigcup\{U_\mu : \mu \in \Delta\}$, then $\mathrm{sCl}(A) \subset \bigcup\{U_\mu : \mu \in \Delta\}$. For each $x \notin \mathrm{Cl}(A)$ there exists a τ-open set G_x containing x such that $A \cap G_x = \emptyset$. Put $\mathcal{U}' = \{U_\mu : \mu \in \Delta\} \cup \{G_x : x \notin \mathrm{Cl}(A)\}$. Then $\mathcal{U}'$ is an open cover of the S-paracompact (mod $\mathcal{I}$) space X and so, there exist $I \in \mathcal{I}$ and a locally finite collection $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ of semi-open sets such that $\mathcal{V}$ refines $\mathcal{U}$ and $X = \bigcup\{V_\lambda : \lambda \in \Lambda\} \cup I$. For each $\lambda \in \Lambda$, either $V_\lambda \subset U_{\mu(\lambda)}$ for some $\mu(\lambda) \in \Delta$ or $V_\lambda \subset G_{x(\lambda)}$ for some $x(\lambda) \notin \mathrm{Cl}(A)$. Now, put $\Lambda_0 = \{\lambda \in \Lambda : V_\lambda \subset U_{\beta(\lambda)}\}$. Then $\mathcal{V}' = \{V_\lambda : \lambda \in \Lambda_0\}$ is a collection of semi-open sets which is locally finite and refines $\mathcal{U}$. Also,

$$\begin{aligned} X - \bigcup_{\lambda \in \Lambda_0} V_\lambda &= \left(\bigcup_{\lambda \in \Lambda} V_\lambda \cup I\right) - \bigcup_{\lambda \in \Lambda_0} V_\lambda = \bigcup_{\lambda \notin \Lambda_0} V_\lambda \cup I \\ &\subset \bigcup_{\lambda \notin \Lambda_0} G_{x(\lambda)} \cup I \subset (X - A) \cup I = X - (A - I), \end{aligned}$$

which implies $A - I \subset \bigcup_{\lambda \in \Lambda_0} V_\lambda$ and hence $A \subset \bigcup_{\lambda \in \Lambda_0} V_\lambda \cup I$. This shows that A is αS-paracompact (mod $\mathcal{I}$). $\square$

Theorem 3.3. Let $(X, \tau, \mathcal{I})$ be a space. Then, the following properties hold:

(1) If A is an open αS-paracompact (mod $\mathcal{I}$) subset of $(X, \tau, \mathcal{I})$, then A is S-paracompact (mod $\mathcal{I}$).

(2) If A is a clopen subset of $(X, \tau, \mathcal{I})$, then A is αS-paracompact (mod $\mathcal{I}$) if and only if it is S-paracompact (mod $\mathcal{I}$).

Proof. (1) Let A be an open αS-paracompact (mod $\mathcal{I}$) subset of $(X, \tau, \mathcal{I})$. Let $\mathcal{U} = \{U_\mu : \mu \in \Delta\}$ be a τ_A-open cover of A. Since A is τ-open, we have $\mathcal{U}$ is a τ-open cover of A and hence, there exist $I \in \mathcal{I}$ and a τ-locally finite collection $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ of τ-semi-open sets which refines $\mathcal{U}$ such that $A \subset \bigcup\{V_\lambda : \lambda \in \Lambda\} \cup I$. It follows that $A \subset \bigcup\{V_\lambda \cap A : \lambda \in \Lambda\} \cup (I \cap A)$ and so, the collection $\mathcal{V}_A = \{V_\lambda \cap A : \lambda \in \Lambda\}$ is a τ_A-locally finite τ_A-semi-open refinement of $\mathcal{U}$ and is an $\mathcal{I}_A$-cover of A. Therefore, A is S-paracompact (mod $\mathcal{I}$).
(2) If A is a clopen and αS-paracompact (mod $\mathcal{I}$) subset of $(X, \tau, \mathcal{I})$, then from (1) we obtain that A is is S-paracompact (mod $\mathcal{I}$). Conversely, let $\mathcal{U} = \{U_\mu : \mu \in \Delta\}$ be a τ-open cover of A. The collection $\mathcal{V} = \{A \cap U_\mu : \mu \in \Delta\}$ is a τ_A-open cover of the S-paracompact (mod $\mathcal{I}$) subspace $(A, \tau_A, \mathcal{I}_A)$ and hence, there exist $I_A \in \mathcal{I}_A$ and a τ_A-locally finite τ_A-semi-open refinement $\mathcal{W} = \{W_\lambda : \lambda \in \Lambda\}$ of $\mathcal{V}$ such that $A = \bigcup\{W_\lambda : \lambda \in \Lambda\} \cup I_A$. It is easy to see that $\mathcal{W}$ refines $\mathcal{U}$ and by Lemma 2.1(3), we have that $W_\lambda \in SO(X, \tau)$ for each $\lambda \in \Lambda$. To show $\mathcal{W} = \{W_\lambda : \lambda \in \Lambda\}$ is τ-locally finite, let $x \in X$. Si $x \in A$, then there exists $O_x \in \tau_A \subset \tau$ containing x such that O_x intersects at most finitely many members of $\mathcal{W}$. Otherwise $X \setminus A$ is a τ-open set containing x which intersects no member of $\mathcal{W}$. Therefore, $\mathcal{W}$ is τ-locally finite and such that

$A = \bigcup\{W_\lambda : \lambda \in \Lambda\} \cup I_\lambda \subset \bigcup\{W_\lambda : \lambda \in \Lambda\} \cup I$ for some $I \in \mathcal{I}$. Thus, A is αS-paracompact (mod $\mathcal{I}$). □

As a consequence of Theorem 3.3, we obtain the following result.

Corollary 3.3. Every clopen subspace of a S-paracompact (mod $\mathcal{I}$) space is S-paracompact (mod $\mathcal{I}$).

Lemma 3.1. Let A be a subset of a space $(X, \tau, \mathcal{I})$. If every open cover of A has a locally finite closed refinement $\mathcal{V}$ such that $A \subset \bigcup\{V : V \in \mathcal{V}\} \cup I$ for some $I \in \mathcal{I}$, then $\mathcal{V}$ has a locally finite open refinement $\mathcal{W}$ such that $A \subset \bigcup\{W : W \in \mathcal{W}\} \cup I$.

Proof. Let $\mathcal{U}$ be an open cover of A. By hypothesis, there exist $I \in \mathcal{I}$ and a locally finite closed refinement $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ of $\mathcal{U}$ such that $A \subset \bigcup\{V_\lambda : \lambda \in \Lambda\} \cup I$. For each $x \in A$, there exists an open set G_x containing x such that G_x intersects at most finitely many members of $\mathcal{V}$. Note that the collection $\mathcal{G} = \{G_x : x \in A\}$ is an open cover of A and again by hypothesis, there exist $J \in \mathcal{I}$ and a locally finite closed refinement $\mathcal{H} = \{H_\mu : \mu \in \Delta\}$ of $\mathcal{G}$ such that $A \subset \bigcup\{H_\mu : \mu \in \Delta\} \cup J$. Now, as $\{H_\mu : H_\mu \cap V_\lambda = \emptyset\} \subset \mathcal{H}$, then the collection $\{H_\mu : H_\mu \cap V_\lambda = \emptyset\}$ is locally finite and $\bigcup\{H_\mu : H_\mu \cap V_\lambda = \emptyset\} = \bigcup\{\mathrm{Cl}(H_\mu) : H_\mu \cap V_\lambda = \emptyset\} = \mathrm{Cl}(\bigcup\{H_\mu : H_\mu \cap V_\lambda = \emptyset\})$, it follows that $O_\lambda = X - \bigcup\{H_\mu : H_\mu \cap V_\lambda = \emptyset\}$ is an open set and $V_\lambda \subset O_\lambda$, for each $\lambda \in \Lambda$. For each $\mu \in \Delta$ and $\lambda \in \Lambda$, we have

$$H_\mu \cap O_\lambda \neq \emptyset \Longleftrightarrow H_\mu \cap V_\lambda \neq \emptyset. \qquad (*)$$

Since $\mathcal{V}$ refines $\mathcal{U}$, for every $\lambda \in \Lambda$ there exists $U(\lambda) \in \mathcal{U}$ such that $V_\lambda \subset U(\lambda)$. Put $W_\lambda = O_\lambda \cap U(\lambda)$, then the collection $\mathcal{W} = \{W_\lambda : \lambda \in \Lambda\}$ is an open refinement of $\mathcal{U}$. Furthermore, if $x \in A$ there exists an open set D_x such that D_x intersects at most finitely many members of $\mathcal{H}$, it follows from $(*)$ that $\mathcal{W}$ is locally finite. Also, $A \subset \bigcup\{V_\lambda : \lambda \in \Lambda\} \cup I \subset \bigcup\{O_\lambda \cap U(\lambda) : \lambda \in \Lambda\} \cup I = A \subset \bigcup\{W_\lambda : \lambda \in \Lambda\} \cup I$. □

The following theorem shows that, in the presence of the axiom of regularity, the notions of α-paracompact (mod $\mathcal{I}$) and αS-paracompact (mod $\mathcal{I}$) subsets are equivalent.

Theorem 3.4. Let $\mathcal{I}$ be an ideal on a regular space (X, τ) and A be a subset of X. Then, A is α-paracompact (mod $\mathcal{I}$) if and only if it is αS-paracompact (mod $\mathcal{I}$).

Proof. Necessity is obvious from the definitions. To show sufficiency, assume A is an αS-paracompact (mod $\mathcal{I}$) subset of $(X, \tau, \mathcal{I})$ and let $\mathcal{U} = \{U_\mu : \mu \in \Delta\}$ be an open cover of A. For each $x \in A$, there exists $\mu(x) \in \Delta$ such that $x \in U_{\mu(x)}$ and since $(X, \tau, \mathcal{I})$ is a regular space, there exists an open set V_x such that $x \in V_x \subset \mathrm{Cl}(V_x) \subset U_{\mu(x)}$. Thus, $\mathcal{V} = \{V_x : x \in A\}$ is an open cover of A and because A is αS-paracompact (mod $\mathcal{I}$), there exist $I \in \mathcal{I}$ and a locally finite semi-open refinement $\mathcal{W} = \{W_\lambda : \lambda \in \Lambda\}$ of $\mathcal{V}$ such that $A \subset \bigcup\{W_\lambda : \lambda \in \Lambda\} \cup I$. Since $\mathcal{W}$ refines $\mathcal{V}$, then for each $\lambda \in \Lambda$ there exists $x(\lambda) \in X$ such that $W_\lambda \subset V_{x(\lambda)}$ and so, $W_\lambda \subset \mathrm{Cl}(W_\lambda) \subset \mathrm{Cl}(V_{x(\lambda)}) \subset U_{\mu(x(\lambda))}$. Obviously the collection $\{\mathrm{Cl}(W_\lambda) : \lambda \in \Lambda\}$ is a locally finite closed refinement of $\mathcal{U}$ such that

$A \subset \bigcup\{\mathrm{Cl}(W_\lambda) : \lambda \in \Lambda\} \cup I$. By Lemma 3.1, the open cover $\mathcal{U}$ of A has a locally finite open refinement $\mathcal{H}$ such that $A \subset \bigcup\{H : H \in \mathcal{H}\} \cup I$. Therefore, A is an α-paracompact (mod $\mathcal{I}$) subset of $(X, \tau, \mathcal{I})$. □

Proposition 3.3. If A is an αS-paracompact (mod $\mathcal{I}$) subset of a space $(X, \tau, \mathcal{I})$ and B is a subset of X with $\partial(B) \in \mathcal{I}$, then $A \cap \mathrm{Cl}(B)$ is αS-paracompact (mod $\mathcal{I}$).

Proof. Let $\mathcal{U}$ be an open cover of $A \cap \mathrm{Cl}(B)$. Then $\mathcal{U}' = \mathcal{U} \cup \{X - \mathrm{Cl}(B)\}$ is an open cover of A and so, there exist $I \in \mathcal{I}$ and a locally finite semi-open refinement $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ of $\mathcal{U}'$ such that $A \subset \bigcup\{V_\lambda : \lambda \in \Lambda\} \cup I$. Then, $\partial(\mathrm{Cl}(B)) \subset \partial(B) \in \mathcal{I}$ and

$$A \cap \mathrm{Cl}(B) \subset \bigcup_{\lambda \in \Lambda} V_\lambda \cap \mathrm{Int}(\mathrm{Cl}(B)) \cup J,$$

where $J = [(\bigcup\{V_\lambda : \lambda \in \Lambda\}) \cap \partial(\mathrm{Cl}(B))] \cup (I \cap \mathrm{Cl}(B)) \in \mathcal{I}$. Thus, the collection $\mathcal{V}' = \{V_\lambda \cap \mathrm{Int}(\mathrm{Cl}(B)) : \lambda \in \Lambda\}$ is a locally finite semi-open refinement of $\mathcal{U}$ such that $A \cap \mathrm{Cl}(B) \subset \bigcup\{V : V \in \mathcal{V}'\} \cup J$. Therefore, $A \cap \mathrm{Cl}(B)$ is αS-paracompact (mod $\mathcal{I}$). □

The following result follows from Proposition 3.3 and the fact that the topological frontier of a semi-open (resp. semi-closed) set is nowhere dense.

Corollary 3.4. If A is an αS-paracompact (mod $\mathcal{N}$) subset of a space $(X, \tau, \mathcal{I})$ and B is either semi-open or semi-closed, then $A \cap \mathrm{Cl}(B)$ is αS-paracompact (mod $\mathcal{N}$).

Remark 3.2. *If* $\{V_\lambda : \lambda \in \Lambda\}$ *is a locally finite collection of subsets of a space* (X, τ)*, then the collection* $\{\partial(V_\lambda) : \lambda \in \Lambda\}$ *is locally finite.*

According to [7], if $\mathcal{I}$ is an ideal on a space (X, τ) and $\mathfrak{F}$ is the collection of all closed sets of (X, τ), then the collection $\{A \subset X : \mathrm{Cl}(A) \in \mathcal{I}\}$ is an ideal contained in $\mathcal{I}$. The ideal generated by the collection of whole closed sets in $\mathcal{I}$ is denoted by $\langle \mathcal{I} \cap \mathfrak{F} \rangle$. It is clear that $\langle \mathcal{I} \cap \mathfrak{F} \rangle = \{A \subset X : \mathrm{Cl}(A) \in \mathcal{I}\}$.

Proposition 3.4. Let A be a subset of a space $(X, \tau, \mathcal{I})$. If A is αS-paracompact (mod $\langle \mathcal{I} \cap \mathfrak{F} \rangle$) and $\mathcal{N} \subset \mathcal{I}$, then $\mathrm{Cl}(A)$ is αS-paracompact (mod $\mathcal{I}$).

Proof. Let $\mathcal{U}$ be an open cover of $\mathrm{Cl}(A)$. By hypothesis, there exist $I_A \in \langle \mathcal{I} \cap \mathfrak{F} \rangle$ and a locally finite collection $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ of semi-open sets such that $\mathcal{V}$ refines $\mathcal{U}$ and $A \subset \bigcup\{V_\lambda : \lambda \in \Lambda\} \cup I_A$. Then,

$$\mathrm{Cl}(A) \subset \bigcup_{\lambda \in \Lambda} \mathrm{Cl}(V_\lambda) \cup \mathrm{Cl}(I_A) = \left(\bigcup_{\lambda \in \Lambda} V_\lambda\right) \cup \left(\bigcup_{\lambda \in \Lambda} \partial(V_\lambda)\right) \cup \mathrm{Cl}(I_A).$$

By Remark 3.2, the collection $\{\partial(V_\lambda) : \lambda \in \Lambda\}$ is locally finite and $\partial(V_\lambda) \in \mathcal{N}$ for each $\lambda \in \Lambda$. Thus, by [6, Lemma 2.1], we have $\bigcup\{\partial(V_\lambda) : \lambda \in \Lambda\} \in \mathcal{N} \subset \mathcal{I}$. Put $I = \bigcup\{\partial(V_\lambda) : \lambda \in \Lambda\} \cup \mathrm{Cl}(I_A)$, then $I \in \mathcal{I}$ and $\mathrm{Cl}(A) \subset \bigcup_{\lambda \in \Lambda} V_\lambda \cup I$. Therefore, $\mathrm{Cl}(A)$ is αS-paracompact (mod $\mathcal{I}$). □

Since $\mathcal{N}$ is the ideal of nowhere dense subsets of (X, τ), $A \in \mathcal{N}$ if and only if $\mathrm{Cl}(A) \in \mathcal{N}$. In the case that $\mathcal{I} = \mathcal{N}$, then $\langle \mathcal{I} \cap \mathfrak{F} \rangle = \mathcal{N}$. The following corollary is a direct consequence of Proposition 3.4.

Corollary 3.5. If A is an αS-paracompact (mod $\mathcal{N}$) subset of a space $(X, \tau, \mathcal{I})$, then $\mathrm{Cl}(A)$ is αS-paracompact (mod $\mathcal{N}$).

Lemma 3.2. *[7] If* $\{A_\lambda : \lambda \in \Lambda\}$ *is a locally finite collection of meager sets of a space* (X, τ)*, then* $\bigcup\{A_\lambda : \lambda \in \Lambda\}$ *is meager.*

Theorem 3.5. *If* $\{A_\lambda : \lambda \in \Lambda\}$ *is a locally finite collection of* αS*-paracompact (mod* $\mathcal{M}$*) subsets of a space* (X, τ)*, then* $\bigcup\{A_\lambda : \lambda \in \Lambda\}$ *is* αS*-paracompact (mod* $\mathcal{M}$*).*

Proof. Let $\mathcal{U}$ be an open cover of $\bigcup\{A_\lambda : \lambda \in \Lambda\}$ and put $\mathcal{U}_\lambda = \{U \in \mathcal{U} : U \cap A_\lambda \neq \emptyset\}$ for each $\lambda \in \Lambda$. By the hypothesis, there exist $M_\lambda \in \mathcal{M}$ and a locally finite collection $\mathcal{V}_\lambda$ of semi-open sets such that $\mathcal{V}_\lambda$ refines $\mathcal{U}_\lambda$ and $A_\lambda \subset \bigcup\{V : V \in \mathcal{V}_\lambda\} \cup M_\lambda$. Then, we have

$$A_\lambda \subset \bigcup_{V \in \mathcal{V}_\lambda} (V \cap \mathrm{Int}(\mathrm{Cl}(A_\lambda))) \cup \bigcup_{V \in \mathcal{V}_\lambda} (V \cap \partial(\mathrm{Cl}(A_\lambda))) \cup M_\lambda.$$

For each $V \in \mathcal{V}_\lambda$ and each $\lambda \in \Lambda$, $V \cap \partial(\mathrm{Cl}(A_\lambda))$ is nowhere dense and the collection $\{V \cap \partial(\mathrm{Cl}(A_\lambda)) : V \in \mathcal{V}_\lambda, \lambda \in \Lambda\}$ is locally finite, so by [6, Lemma 2.1], the union of all elements of $\{V \cap \partial(\mathrm{Cl}(A_\lambda)) : V \in \mathcal{V}_\lambda, \lambda \in \Lambda\}$ is a nowhere dense set. By Lemma 3.2, we obtain $\bigcup\{M_\lambda : \lambda \in \Lambda\} \in \mathcal{M}$ and

$$M = \bigcup_{\lambda \in \Lambda} \bigcup_{V \in \mathcal{V}_\lambda} V \cap \partial(\mathrm{Cl}(A_\lambda)) \cup \bigcup_{\lambda \in \Lambda} M_\lambda \in \mathcal{M}.$$

Now, the collection $\{V \cap \mathrm{Int}(\mathrm{Cl}(A_\lambda)) : V \in \mathcal{V}_\lambda, \lambda \in \Lambda\}$ of semi-open sets is locally finite and refines $\mathcal{U}$ and also

$$\bigcup_{\lambda \in \Lambda} A_\lambda \subset \bigcup_{\lambda \in \Lambda} \bigcup_{V \in \mathcal{V}_\lambda} V \cap \mathrm{Int}(\mathrm{Cl}(A_\lambda)) \cup M.$$

Therefore, $\bigcup\{A_\lambda : \lambda \in \Lambda\}$ is αS-paracompact (mod $\mathcal{M}$). □

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