

## Diagana Space and The Gas Absorption Model

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### ABSTRACT

Poorkarimi and Wiener established the existence of almost periodic solutions to a class of nonlinear hyperbolic partial differential equations with delay. Al-Islam then generalized the results of Poorkarimi and Wiener to the pseudo-almost periodic setting. In this paper, the results of Al-Islam will be extended to the space of weighted pseudo almost periodic functions, also known as Diagana Space. The class of nonlinear hyperbolic partial differential equations of Poorkarimi and Wiener represents a mathematical model for the dynamics of gas absorption.

### RESUMEN

Poorkarimi y Wiener establecieron la existencia de soluciones casi periódicas de una clase de ecuaciones diferenciales parciales hiperbólicas no lineales con retraso. Luego, Al-Islam generalizó los resultados de Poorkarimi y Wiener al caso seudo-cuasi periódico. En este artículo los resultados de Al-Islam se extenderán al espacio de funciones seudo-cuasi periódicas con peso, también conocido como espacio de Diagana. La clase de ecuaciones diferenciales parciales hiperbólicas no lineales de Poorkarimi y Wiener representa un modelo matemático de la dinámica de absorción de gas.

**Keywords and Phrases:** almost periodic solution, weighted pseudo-almost periodic, gas absorption.

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## 1 Introduction

Let  $L > 0$ . In Poorkarimi and Wiener [22], under some reasonable assumptions, the existence of both periodic and almost periodic solutions to the nonlinear hyperbolic second-order partial differential equation with delay given by

$$\begin{cases} u_{xt}(x, t) + a(x, t)u_x(x, t) = C(x, t, u(x, [t])) \\ u(0, t) = \varphi(t) \end{cases} \quad (1.1)$$

where  $a : [0, L] \times \mathbb{R} \mapsto \mathbb{R}$ ,  $C : [0, L] \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ , and  $\varphi : \mathbb{R} \mapsto \mathbb{R}$  are periodic (respectively, almost periodic) functions in the variable  $t$  and  $[t]$  denotes the greatest integer function:  $[t] := n$  for  $n \leq t < n + 1$  for an integer  $n$ , was established. Extensive use of similar assumptions as in [22] and [3] will be used to extend the above-mentioned existence results to the weighted pseudo-almost periodic setting. Eq.(1.1) is of great interest, being that it represents a mathematical model for the dynamics of gas absorption. Further details of the gas absorption model, Eq.(1.1), can also be seen in [22].

The existence of almost periodic, asymptotically almost periodic, almost automorphic [21], pseudo almost periodic [11], and more recently, weighted pseudo-almost periodic solutions to differential equations are among the most attractive topics in the qualitative theory of differential equations due to their applications in physics, mathematical biology, along with other areas of science and engineering.

The concept of pseudo almost periodicity was first introduced by Zhang [23, 25, 24] and generalizes the almost periodicity of Bohr. More details on the concept of pseudo almost periodicity as well as its applications to differential equations, functional differential, and partial differential equations can be easily found in the literature, especially in [1, 2, 4, 11, 7, 9, 10] and the references therein.

The more recent generalization of Zhang almost periodicity is the weighted pseudo almost periodicity of Diagana. The text that follows this introduction shows, as well as, compares the properties of the Zhang and Diagana almost periodic spaces. Following the comparisons of the Zhang and Diagana almost periodic spaces, the results contained in [3] will be generalized in the setting of Diagana space. For more details on Diagana space, the reader should refer to [6, 14, 20].

## 2 Weighted Pseudo Almost Periodic Functions

Let  $(BC(\mathbb{R}), \|\cdot\|_\infty)$  denote the Banach space of all bounded continuous functions  $\varphi : \mathbb{R} \mapsto \mathbb{R}$  endowed with the sup norm defined by

$$\|\varphi\|_\infty := \sup_{t \in \mathbb{R}} |\varphi(t)|.$$

**Definition 2.1.** [5, 11, 21] A continuous function  $g : \mathbb{R} \mapsto \mathbb{R}$  is called (Bohr) almost periodic if for each  $\varepsilon > 0$ , there exists  $l(\varepsilon) > 0$  such that every interval of length  $l(\varepsilon)$  contains a number  $\tau$  with the following property

$$|g(t + \tau) - g(t)| < \varepsilon$$

for each  $t \in \mathbb{R}$ .

The number  $\tau$  above is then called an  $\varepsilon$ -translation number of  $g$ , and the collection of those almost periodic functions will be denoted as  $AP(\mathbb{R})$ .

Although the concept of almost periodicity is a natural generalization of the classical periodicity, there are almost periodic functions that are not periodic. A classical example of an almost periodic function that is not periodic is the function defined by:

$$g(t) = \sin t + \sin(\sqrt{7}t) \text{ for each } t \in \mathbb{R}.$$

More details on properties of almost periodic functions can be found in the literature by Corduneanu [5], Diagana [11], N'Guérékata [21] and the references therein.

Let  $\mathbb{U}$  be the collection of all functions  $w$ , **weights**, such that  $w(t) > 0$  for almost each  $t \in \mathbb{R}$  and  $w \in L^1_{loc}(\mathbb{R})$ . Also, for each  $w \in \mathbb{U}$  and  $r > 0$ ,

$$\mu(r, w) := \int_{-r}^r w(t) dt.$$

From the collection of weights,  $\mathbb{U}$ , we define two subcollections of  $\mathbb{U}$  as:

$$\mathbb{U}_\infty := \{w \in \mathbb{U} : \lim_{r \rightarrow \infty} \mu(r, w) = \infty \text{ and } \liminf_{t \rightarrow \infty} w(t) > 0\},$$

$$\mathbb{U}_{\overline{w}} := \{w \in \mathbb{U}_\infty : w \text{ is bounded}\}$$

One can see that the subcollections defined above can be written as:  $\mathbb{U}_{\overline{w}} \subset \mathbb{U}_\infty \subset \mathbb{U}$  Define

$$PAP_0(\mathbb{R}) := \left\{ \phi \in BC(\mathbb{R}) : \lim_{r \rightarrow \infty} \frac{1}{2r} \int_{-r}^r |\phi(\sigma)| d\sigma = 0 \right\}.$$

**Definition 2.2.** [11, 23] A function  $f \in BC(\mathbb{R})$  is called pseudo almost periodic if it can be expressed as

$$f = g + \varphi,$$

where  $g \in AP(\mathbb{R})$  and  $\varphi \in PAP_0(\mathbb{R})$ .

The collection of such functions will be denoted by  $PAP(\mathbb{R})$ .

Note that the functions  $g$  and  $\varphi$  appearing in Definition 2.2 are respectively called the almost periodic and the ergodic perturbation components of  $f$ . Furthermore, the decomposition in Definition 2.2 is unique [23, 25, 24].

We now equip  $PAP(\mathbb{R})$  the collection of all pseudo almost periodic functions from  $\mathbb{R}$  into  $\mathbb{R}$  with the sup norm. It is not really hard to see that  $(PAP(\mathbb{R}), \|\cdot\|_\infty)$  is a closed subspace of  $BC(\mathbb{R})$  and hence is a Banach space.

An example of a pseudo almost periodic function is the function  $f$  defined by

$$f(t) = \sin t + \sin t\sqrt{2} + e^{-|t|}$$

for each  $t \in \mathbb{R}$ . The core of the construction of the weighted pseudo almost periodic space, is the enrichment of the space of ergodic perturbations,  $PAP_0(\mathbb{R})$ . That is, for  $w \in U_\infty$ , the **weighted ergodic space** is defined by:

$$PAP_0(\mathbb{R}, w) := \left\{ \phi \in BC(\mathbb{R}) : \lim_{r \rightarrow \infty} \frac{1}{\mu(r, w)} \int_{-r}^r |\phi(\sigma)|w(\sigma) d\sigma = 0 \right\}.$$

**Definition 2.3.** [14] A function  $f \in BC(\mathbb{R})$  is called weighted pseudo almost periodic if it can be expressed as

$$f = g + \varphi,$$

where  $g \in AP(\mathbb{R})$  and  $\varphi \in PAP_0(\mathbb{R}, w)$ . The collection of the functions defined above is **Diagana Space**, and will be denoted as  $D_{\overline{w}}(\mathbb{R})$ .

An example of a function  $f \in D_{\overline{w}}(\mathbb{R})$  is the function

$$f(t) = \cos t + \cos \sqrt{2}t + \frac{1}{1+t^2},$$

where  $w(t) = 1 + t^2$  for each  $t \in \mathbb{R}$ .

**Lemma 2.4.**  $AP(\mathbb{R}) \subset PAP(\mathbb{R}) \subset D_{\overline{w}}(\mathbb{R})$ .

In [19], it was shown that the decomposition  $f = g + \varphi$ , where  $g \in AP(\mathbb{R})$  and  $\varphi \in PAP_0(\mathbb{R}, w)$  is not unique. Hence, one cannot define  $D_{\overline{w}}(\mathbb{R})$ , equipped with the sup norm, to be a Banach Space, despite  $AP(\mathbb{R})$  and  $PAP_0(\mathbb{R}, w)$  being closed subspaces with respect to the sup norm. Therefore, with the possibility of being able to construct countably many decompositions of any weighted pseudo almost-periodic function, written as:  $\{g_n + \phi_n, n \in \mathbb{N}\}$ , had to be resolved to ensure the criterion of completeness for a Banach Space. To resolve this dilemma, in [20] another norm, which will be known as the **w-norm** in this writing, was constructed and defined as follows:

$$\|f\|_w := \inf_{n \in \mathbb{N}} (\|g_n\| + \|\phi_n\|) = \inf_{n \in \mathbb{N}} \left( \sup_{t \in \mathbb{R}} \|g_n(t)\| + \sup_{t \in \mathbb{R}} \|\phi_n(t)\| \right).$$

$\|\cdot\|_w$  is undoubtedly a norm on  $D_{\overline{w}}(\mathbb{R})$ .

**Theorem 2.5.**  $D_{\overline{w}}(\mathbb{R})$  is a Banach Space under the norm  $\|\cdot\|_w$ .

*Proof.* The proof of the theorem can be found in [20].

□

Let  $\mathbb{W}_\infty$  be the set of all functions  $w \in \mathbb{U}_\infty$  where there exists a measurable set  $K \subset \mathbb{R}$  such that for each  $\tau \in \mathbb{R}$ ,

$$\limsup_{|t| \rightarrow +\infty, t \in K} \frac{w(t + \tau)}{w(t)} := \inf_{m > 0} \left( \sup_{|t| > m, t \in K} \frac{w(t + \tau)}{w(t)} \right) < \infty$$

and

$$\lim_{r \rightarrow +\infty} \frac{\int_{K_r^\tau} w(t) dt}{\mu(r, w)} = 0,$$

where  $K_r^\tau = [-r, r] \setminus K + \tau$ .

**Lemma 2.6.** [17] *Let  $w \in \mathbb{W}_\infty$  and  $f \in D_{\overline{w}}(\mathbb{R})$  and if  $g$  is its almost periodic component, then*

$$g(\mathbb{R}) \subset \overline{f(\mathbb{R})}.$$

*Therefore,  $\|f\|_\infty \geq \|g\|_\infty \geq \inf_{t \in \mathbb{R}} |g(t)| \geq \inf_{t \in \mathbb{R}} |f(t)|$ .*

*Proof.* The proof of the lemma can be found in [17].

□

**Theorem 2.7.** *If  $(f_n)_{n \in \mathbb{N}} \subset D_{\overline{w}}(\mathbb{R})$  is a sequence which converges uniformly with respect to the  $w$ -norm to some  $f : \mathbb{R} \mapsto \mathbb{R}$ , then  $f$  is necessarily a weighted pseudo-almost periodic function.*

*Proof.* Write  $f_n = g_n + \varphi_n$  where  $(g_n)_n \subset AP(\mathbb{R})$  and  $(\varphi_n)_{n \in \mathbb{N}} \subset PAP_0(\mathbb{R}, w)$ . Suppose  $\|f_n - f\|_w \rightarrow 0$  as  $n \rightarrow \infty$  for some function  $f : \mathbb{R} \mapsto \mathbb{R}$ . Of course,  $f \in BC(\mathbb{R})$ , as a uniform limit of a sequence of bounded continuous functions. So to complete the proof, it needs to be shown that  $f \in D_{\overline{w}}(\mathbb{R})$ . For that, notice that by using Lemma 2.6, it follows that

$$\|g_n - g_m\|_w \leq \|f_n - f_m\|_w \text{ for all } n, m \in \mathbb{N}.$$

Now letting  $n, m \rightarrow \infty$  in the previous inequality it follows that

$$\lim_{n, m \rightarrow \infty} \|g_n - g_m\|_w \leq \lim_{n, m \rightarrow \infty} \|f_n - f_m\|_w = 0,$$

and hence  $(g_n)_{n \in \mathbb{N}} \subset AP(\mathbb{R})$  is a Cauchy sequence. Since  $(AP(\mathbb{R}), \|\cdot\|_\infty)$  is a Banach space, it follows that there exists  $g \in AP(\mathbb{R})$  such that  $\|g_n - g\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$ .

Now,  $f_n - g_n = \phi_n \rightarrow \varphi := f - g$  uniformly with respect to the  $w$ -norm as  $n \rightarrow \infty$ . Thus, writing

$$\varphi = (\varphi - \phi_n) + \phi_n,$$

it follows that:

$$\frac{1}{\mu(r, w)} \int_{-r}^r |\phi(\sigma)|w(\sigma) d\sigma \leq \|\phi_n - \phi\|_w + \frac{1}{\mu(r, w)} \int_{-r}^r |\phi_n(\sigma)|w(\sigma) d\sigma.$$

Let  $r \rightarrow \infty$  in the previous inequality, then

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{1}{\mu(r, w)} \int_{-r}^r |\phi(\sigma)|w(\sigma) d\sigma &\leq \|\phi_n - \phi\|_w + \lim_{r \rightarrow \infty} \frac{1}{\mu(r, w)} \int_{-r}^r |\phi_n(\sigma)|w(\sigma) d\sigma \\ &= \|\phi_n - \phi\|_w. \end{aligned}$$

Letting  $n \rightarrow \infty$  in the previous inequality, then

$$0 \leq \lim_{r \rightarrow \infty} \frac{1}{\mu(r, w)} \int_{-r}^r |\phi(\sigma)|w(\sigma) d\sigma \leq \lim_{n \rightarrow \infty} \|\phi_n - \phi\|_w = 0,$$

and hence

$$\lim_{r \rightarrow \infty} \frac{1}{\mu(r, w)} \int_{-r}^r |\phi(\sigma)|w(\sigma) d\sigma = 0.$$

Therefore,  $f = g + \varphi \in D_{\overline{w}}(\mathbb{R})$ . □

More details on properties of weighted pseudo-almost periodic functions can be found in the literature, especially in Diagana [14].

### 3 Existence of Weighted Pseudo Almost Periodic Solutions

Throughout the rest of the paper, it is assumed that the function  $a : [0, L] \times \mathbb{R} \mapsto \mathbb{R}$  satisfies the following:

$$\inf_{x \in [0, L], t \in \mathbb{R}} a(x, t) := m > 0.$$

Using the previous assumption, the initial value problem, Eq.(1.1), has a unique bounded solution, which can be explicitly given by:

$$u(x, t) = \varphi(t) + \int_0^x \int_{-\infty}^t \exp \left\{ - \int_{\tau}^t a(\xi, \theta) d\theta \right\} C(\xi, \tau, u(\xi, \lfloor \tau \rfloor)) d\tau d\xi \quad (3.1)$$

for each  $(x, t) \in [0, L] \times \mathbb{R}$ .

Before exploring the existence and uniqueness of a weighted pseudo-almost periodic solution to Eq.(1.1), consider the existence of a weighted pseudo-almost periodic solution to the first-order partial differential equation

$$\frac{\partial V}{\partial t}(x, t) + a(x, t)V(x, t) = f(x, t), \tag{3.2}$$

for each  $(x, t) \in [0, L] \times \mathbb{R}$ .

**Lemma 3.1.** *Assume  $t \mapsto f(x, t)$  is weighted pseudo-almost periodic uniformly in  $x \in [0, L]$ ,  $t \mapsto a(x, t)$  is almost periodic uniformly in  $x \in [0, L]$ . Then Eq. (4) has a unique weighted pseudo-almost periodic solution, which can be explicitly expressed by*

$$V(x, t) = \int_{-\infty}^t \exp \left\{ - \int_{\tau}^t a(x, \theta) d\theta \right\} f(x, \tau) d\tau. \tag{3.3}$$

Moreover,  $V$  satisfies the following a priori inequality

$$\|V\|_w \leq \frac{1}{m} \|f\|_w$$

where the  $w$ -norm is taken in both  $x \in [0, L]$  and  $t \in \mathbb{R}$ .

*Proof.* It is clear that the only bounded solution to Eq.(4) is given by Eq.(3.3). Now from the weighted pseudo almost periodicity of  $t \mapsto f(x, t)$  it follows that there exist two functions  $g$  and  $h$  with  $t \mapsto g(x, t) \in AP(\mathbb{R})$  for each  $x \in [0, L]$  and  $t \mapsto h(x, t) \in PAP_0(\mathbb{R}, w)$  for each  $x \in [0, L]$  such that  $f = g + h$ . Consequently,  $V(x, t) = V_g(x, t) + V_h(x, t)$  for  $x \in [0, L]$  and  $t \in \mathbb{R}$ , where

$$V_g(x, t) = \int_{-\infty}^t \exp \left\{ - \int_{\tau}^t a(x, \theta) d\theta \right\} g(x, \tau) d\tau, \text{ and}$$

$$V_h(x, t) = \int_{-\infty}^t \exp \left\{ - \int_{\tau}^t a(x, \theta) d\theta \right\} h(x, \tau) d\tau.$$

Thus to complete the proof we must show that  $t \mapsto V_g(x, t)$  belongs to  $AP(\mathbb{R})$  and that  $t \mapsto V_h(x, t)$  belongs to  $PAP_0(\mathbb{R}, w)$  uniformly in  $x \in [0, L]$ . The almost periodicity of  $t \mapsto V_g(x, t)$  ( $x \in [0, L]$ ) was established in [22]. Thus, it remains to show that  $t \mapsto V_h(x, t)$  belongs to  $PAP_0(\mathbb{R}, w)$  uniformly in  $x \in [0, L]$ .

Now for  $r > 0$ ,

$$\begin{aligned}
 & \frac{1}{\mu(r, w)} \int_{-r}^r \left\| \int_{-\infty}^t e^{-\int_{\tau}^t a(x, \theta) d\theta} h(x, \tau) w(\tau) d\tau \right\|_w dt \\
 \leq & \frac{1}{\mu(r, w)} \int_{-r}^r \int_{-\infty}^t e^{-\int_{\tau}^t a(x, \theta) d\theta} \|h(x, \tau)\|_w w(\tau) dt \\
 \leq & \frac{1}{\mu(r, w)} \int_{-r}^r \left[ \int_{-\infty}^t e^{-m(t-\tau)} \|h(x, \tau)\|_w d\tau \right] w(\tau) dt \\
 = & \frac{1}{\mu(r, w)} \int_{-r}^r \left[ \int_0^{+\infty} e^{-ms} \|h(x, t-s)\|_w w(s) ds \right] dt \\
 = & \int_0^{+\infty} e^{-ms} \left[ \frac{1}{\mu(r, w)} \int_{-r}^r \|h(x, t-s)\|_w w(t) dt \right] ds,
 \end{aligned}$$

by letting  $s = t - \tau$  ( $ds = -d\tau$ ).

For any  $w \in \mathbb{W}_\infty$ , it was shown in [17] that  $D_{\mathbb{W}}(\mathbb{R})$  is translation invariant with respect to the time variable  $t \in \mathbb{R}$ . Therefore, it follows that  $t \mapsto h(x, t - s)$  belongs to  $\text{PAP}_0(\mathbb{R}, w)$  uniformly in  $x \in [0, L]$ . That is,

$$\lim_{r \rightarrow \infty} \frac{1}{\mu(r, w)} \int_{-r}^r \|h(x, t - s)\|_w w(t) dt = 0$$

uniformly in  $x \in [0, L]$ . Using the Lebesgue Dominated Convergence Theorem completes the proof.  $\square$

**Theorem 3.2.** Assume  $t \mapsto a(x, t)$  is almost periodic and the functions  $t \mapsto \varphi(t)$ ,  $t \mapsto C(x, t, u(x, [t]))$  are weighted pseudo-almost periodic uniformly in  $x \in [0, L]$ . Additionally, assume  $C(x, t, u(x, [t]))$  satisfies the Lipschitz condition, that is, there exists  $K > 0$  such that

$$\|C(x, t, u(x, [t])) - C(x, t, V(x, [t]))\|_w \leq K \|u(x, [t]) - V(x, [t])\|_w$$

for all  $x \in [0, L]$  and  $t \in \mathbb{R}$ . Then Eq.(1.1) has a unique weighted pseudo-almost periodic solution.

*Proof.* Our proof follows along the same line as that given in [22] with the appropriate modifications. Indeed, for the first approximation, let  $u_0(x, t) \equiv 0$ . The next approximation is

$$u_1(x, t) = \varphi(t) + \int_0^x \int_{-\infty}^t \exp \left\{ -\int_{\tau}^t a(\xi, \theta) d\theta \right\} C(\xi, \tau, 0) d\tau d\xi.$$

Now since  $V(x, t) = \frac{\partial}{\partial x} u_1(x, t)$ , then from

$$\frac{\partial V}{\partial t}(x, t) + a(x, t)V(x, t) = C(x, t, 0)$$

and by Lemma 3.1 the weighted pseudo-almost periodicity of  $V(x, t)$  is obtained. Now

$$u_1(x, t) = \varphi(t) + \int_0^x V(\xi, t) d\xi.$$

and hence  $t \mapsto u_1(x, t)$  is pseudo almost periodic in  $t$  uniformly with respect to  $x \in [0, L]$  and

$$u_2(x, t) = \varphi(t) + \int_0^x \int_{-\infty}^t \exp \left\{ - \int_{\tau}^t a(\xi, \theta) d\theta \right\} C(\xi, \tau, u_1(\tau, \xi)) d\tau d\xi.$$

The relation  $\tilde{V}(x, t) = \frac{\partial}{\partial x} u_2(x, t)$  and the equation

$$\frac{\partial \tilde{V}}{\partial t}(x, t) + a(x, t) \tilde{V}(x, t) = C(x, t, u_1(x, t))$$

yields the weighted pseudo-almost periodicity of

$$u_2(x, t) = \varphi(t) + \int_0^x \tilde{V}(\xi, t) d\xi.$$

This shows that all successive approximations  $u_n(t)$  are weighted pseudo-almost periodic functions in  $t$  uniformly in  $x \in [0, L]$ . Therefore,

$$\lim_{n \rightarrow \infty} u_n(x, t) = u(x, t)$$

is a weighted pseudo-almost periodic function in  $t$  uniformly with respect to  $x \in [0, L]$ , by using Theorem 2.7. □

## 4 Example

To illustrate the main result of this paper(Theorem 3.2), consider the following nonlinear hyperbolic second-order partial differential equation

$$\begin{cases} u_{xt}(x, t) + (P(x) + \sin t)u_x(x, t) = H(t) \sin(u(x, [t])) \\ u(0, t) = 0 = \varphi(t) \end{cases} \tag{4.1}$$

where  $P(x) = \sum_{k=0}^n a_k x^k$  for  $x \in [0, 1]$  is a polynomial of degree  $n$  with real coefficients,

$$H(t) = \sin t + \sin t\sqrt{2} + w(t) \sin t$$

where

$$w(t) = \begin{cases} 1, & t \in [0, \infty), \\ e^{-t^2}, & t \in (-\infty, 0), \end{cases}$$

$$\begin{aligned} a(x, t) &= P(x) + \sin t, \\ C(x, t, u(x, t)) &= H(t) \sin(u(x, [t])), \\ \varphi(t) &= 0 \end{aligned}$$

for all  $x \in [0, 1]$  and  $t \in \mathbb{R}$ .

Suppose

$$\inf_{x \in [0, 1], t \in \mathbb{R}} a(x, t) = a_0 - 1,$$

where  $a_0 > 1$ . Then all the assumptions of Theorem 3.2 are fulfilled and therefore the next theorem holds.

**Theorem 4.1.** *Under previous assumptions, the hyperbolic partial differential equation Eq. (4.1) has a unique weighted pseudo-almost periodic solution.*

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