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A Common Fixed Point Theorem for Pairs of Mappings in Cone Metric Spaces

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ABSTRACT

In this paper, we present a common fixed point theorem in complete cone metric spaces which is a generalization of the theorem in [6]. This result also generalizes some theorems given in [4] and [9].

RESUMEN

En este artículo presentamos un teorema de punto de fijo común en espacios métricos cono completos, el cual es una generalización del teorema en [6]. También este resultado generaliza algunos teoremas en [4] y [9].

Keywords and Phrases: Cone metric space; Complete cone metric space; Fixed point

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1 Introduction

In [4], Guang and Xian reintroduced the concept of a cone metric space (known earlier as K-metric space, see [12]), replacing the set of real numbers by an ordered Banach space and proved some fixed point theorems for mapping satisfying various contractive conditions. Recently, Rezapour and Hamlbarani [9] generalized some results of [4] by omitting the assumption of normality in the results. Also many authors proved some fixed point theorems for contractive type mappings in cone metric spaces (see [1, 2, 3, 5, 7, 8, 10, 11]).

The main purpose of this paper is to present a common fixed point theorem for mappings in complete cone metric spaces.

2 Preliminaries

Throughout this paper, we denote by $\mathbf{N}$ the set of positive integers and by $\mathbf{R}$ the set of real numbers.

Definition 2.1. Let E be a real Banach space and P be a subset of E. P is called a cone if and only if:

(i) P is closed, nonempty and $P \neq \{0\}$,

(ii) $a, b \in \mathbf{R}$, $a, b \geq 0$, $x, y \in P$ implies $ax + by \in P$,

(iii) $x \in P$ and $-x \in P$ implies $x = 0$.

Given a cone $P \subseteq E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ if $x \leq y$ and $x \neq y$, and $x \ll y$ if $y - x \in \text{int}P$, where $\text{int}P$ is the interior of P.

The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq K\|y\|.$$

The least positive number satisfying the above is then called the normal constant of P.

Lemma 2.1. ([13]) *Let E be a real Banach space with a cone P. Then:*

(i) *If $x \leq y$ and $0 \leq a \leq b$, then $ax \leq by \quad$ for $x, y \in P$,*

(ii) *If $x \leq y$ and $u \leq v$, then $x + u \leq y + v$,*

(iii) *If* $x_n \leq y_n$ *for each* $n \in \mathbf{N}$, and $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ *then* $x \leq y$.

Lemma 2.2. ([10]) *If* P *is a cone,* $x \in P$, $\alpha \in \mathbf{R}$, $0 \leq \alpha < 1$, *and* $x \leq \alpha x$, *then* $x = 0$.

In the following definition, we suppose that E is a real Banach space, P is a cone in E with $\text{int}P \neq \emptyset$ and $\leq$ is partial ordering with respect to P.

Definition 2.2. Let X be a non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies:

(d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space. This definition is more general than that of a metric space.

Example 2.1. Let $E = \mathbf{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subset \mathbf{R}^2$, $X = \mathbf{R}^2$ and $d : X \times X \to E$ defined by

$$d(x, y) = d((x_1, x_2), (y_1, y_2)) = (\max\{|x_1 - y_1|, |x_2 - y_2|\}, \alpha \max\{|x_1 - y_1|, |x_2 - y_2|\}),$$

where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

3 Definitions and Lemmas

In this section we shall give some definitions and lemmas.

Definition 3.1.([4]) Let (X, d) be a cone metric space. A sequence $\{x_n\}$ in X is said to be:

(a) A convergent sequence if for every $c \in E$ with $0 \ll c$, there is $N \in \mathbf{N}$ such that for all $n \geq N$, $d(x_n, x) \ll c$ for some fixed x in X. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x, n \to \infty$.

(b) A Cauchy sequence if for every $c \in E$ with $0 \ll c$, there is $N \in \mathbf{N}$ such that for all $n, m \geq N$, $d(x_n, x_m) \ll c$.

A cone metric space (X, d) is said to be complete if every Cauchy sequence is convergent in X.

The following lemma was recently proved in ([3]), by omitting the normality condition.

Lemma 3.1. *Let* (X, d) *be a cone metric space. If* $\{x_n\}$ *is a convergent sequence in* X, *then the limit of* $\{x_n\}$ *is unique.*

The proof of the following lemma is straighforward and is omitted.

Lemma 3.2. *Let* (X, d) *be a cone metric space,* $\{x_n\}$ *be a sequence in* X*. If* $\{x_n\}$ *converges to* x *and* $\{x_{n_k}\}$ *is any subsequence of* $\{x_n\}$*, then* $\{x_{n_k}\}$ *converges to* x*.*

Lemma 3.3. ([10]) *Let* (X, d) *be a cone metric space and* $\{x_n\}$ *be a sequence in* X*. If there exists a sequence* $\{a_n\}$ *in* $\mathbf{R}$ *with* $a_n > 0$ *for all* $n \in \mathbf{N}$ *and* $\sum a_n < \infty$*, which satisfies* $d(x_{n+1}, x_n) \leq a_n M$ *for all* $n \in \mathbf{N}$ *and for some* $M \in E$ *with* $M \geq 0$*, then* $\{x_n\}$ *is a Cauchy sequence in* (X, d)*.*

Definition 3.2. ([11]) Let E and F be reel Banach spaces and P and Q be cones on E and F, respectively. Let (X, d) and (Y, ρ) be cone metric spaces, where $d : X \times X \to E$ and $\rho : Y \times Y \to F$. A function $f : X \to Y$ is said to be continuous at $x_0 \in X$, if for every $c \in F$ with $0 \ll c$, there exists $b \in E$ with $0 \ll b$ such that, $\rho(f(x), f(x_0)) \ll c$ whenever $x \in X$ and $d(x, x_0) \ll b$.

If f is continuous at every point of X, then it is said to be continuous on X.

Lemma 3.4. ([11]) *Let* (X, d) *and* (Y, ρ) *be cone metric spaces as in Definition 3.2. A function* $f : X \to Y$ *is continuous at a point* $x_0 \in X$ *if and only if whenever a sequence* $\{x_n\}$ *in* X *converges to* x_0*, the sequence* $\{f(x_n)\}$ *converges to* $f(x_0)$*.*

4 Main result

The following common fixed point theorem was proved in [6].

Theorem 4.1. *Let* (X, d) *be a complete metric space and let* f *and* g *be two continuous self-mappings of* X*. If there are positive numbers* $\alpha < 1$ *and* $\beta < 1$ *such that, for all* $x, y \in X$*,*

$$d(fgx, gy) \leq \alpha d(x, gy) \tag{1}$$

and

$$d(gfx, fy) \leq \beta d(x, fy), \tag{2}$$

then f *and* g *have a unique common fixed point.*

We now prove the following common fixed point theorem in complete cone metric spaces:

Theorem 4.2. *Let* (X, d) *be a complete cone metric space and* P *be a cone. Let* f *and* g *be self-mappings of* X *satisfying the following inequalities*

$$\begin{aligned} d(fgx, gx) &\leq ad(x, gx), \quad (3) \\ d(gfx, fx) &\leq bd(x, fx) \quad (4) \end{aligned}$$

for all x *in* X, *where* $a, b < 1$. *If either* f *or* g *is continuous, then* f *and* g *have a common fixed point.*

Proof. Let x_0 be an arbitrary point in X and define the sequence $\{x_n\}$ inductively by

$$x_{2n+1} = fx_{2n}, \quad x_{2n+2} = gx_{2n+1}$$

for $n = 0, 1, 2, \ldots$.

Note that if $x_n = x_{n+1}$ for some n, then x_n is a fixed point of f and g. Indeed, if $x_{2n} = x_{2n+1}$ for some $n \geq 0$, then x_{2n} is a fixed point of f. On the other hand, we have from inequality (4) that

$$\begin{aligned} d(x_{2n+2}, x_{2n+1}) &= d(gx_{2n+1}, fx_{2n}) = d(gfx_{2n}, fx_{2n}) \\ &\leq bd(x_{2n}, fx_{2n}) = bd(x_{2n}, x_{2n+1}) = 0 \end{aligned}$$

which implies $-d(x_{2n+1}, x_{2n+2}) \in P$. Also we have $d(x_{2n+1}, x_{2n+2}) \in P$. Hence $d(x_{2n+1}, x_{2n+2}) = 0$ and so $x_{2n+1} = x_{2n+2}$. Thus, x_{2n} is a common fixed point of f and g. If $x_{2n+1} = x_{2n+2}$ for some $n \geq 0$, similarly, by using inequality (3) leads to x_{2n+1} is a common fixed point of f and g.

Now we suppose that $x_n \neq x_{n+1}$ for all n. Using inequality (4), we have

$$\begin{aligned} d(x_{2n+2}, x_{2n+1}) &= d(gx_{2n+1}, fx_{2n}) = d(gfx_{2n}, fx_{2n}) \\ &\leq bd(x_{2n}, fx_{2n}) = bd(x_{2n}, x_{2n+1}). \end{aligned} \tag{5}$$

Similarly, using inequality (3) we have

$$\begin{aligned} d(x_{2n+1}, x_{2n}) &= d(fx_{2n}, gx_{2n-1}) = d(fgx_{2n-1}, gx_{2n-1}) \\ &\leq ad(x_{2n-1}, gx_{2n-1}) = ad(x_{2n-1}, x_{2n}). \end{aligned} \tag{6}$$

Suppose that $\alpha = \max\{a, b\}$. Then from inequalities (5) and (6) we have

$$d(x_{2n+1}, x_{2n+2}) \leq \alpha d(x_{2n}, x_{2n+1})$$

and

$$d(x_{2n}, x_{2n+1}) \leq \alpha d(x_{2n-1}, x_{2n}).$$

Thus, we obtain

$$d(x_{n+1}, x_{n+2}) \leq \alpha d(x_n, x_{n+1})$$

for $n = 0, 1, 2, \ldots$ and it follows that

$$d(x_n, x_{n+1}) \leq \alpha^n d(x_0, x_1).$$

for $n = 1, 2, 3 \ldots$. Since $\sum_{n=0}^{\infty} \alpha^n < \infty$, it follows from Lemma 3.3 that $\{x_n\}$ is a Cauchy sequence in the complete cone metric space (X, d) and so has a limit z in X.

Now we consider that f is continuous. Since $x_{2n+1} = fx_{2n}$, it follows from Lemma 3.4 that

$$z = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} fx_{2n} = fz$$

and so z is a fixed point of f.

Using inequality (4) we have

$$d(gz, z) = d(gfz, fz) \leq bd(z, fz) = bd(z, z) = 0$$

which implies $-d(gz, z) \in P$. Also we have $d(gz, z) \in P$. Hence $d(gz, z) = 0$ and so $gz = z$. We have therefore proved that z is a common fixed point of f and g.

Similarly, considering the continuity of g, it can be seen that f and g have a common fixed point and this completes the proof.

Putting $f = g$ and $k = \max\{a, b\}$ in Theorem 4.2, we get

Corollary 4.1. *Let (X, d) be a complete cone metric space and P be a cone. Let f be a self-mapping of X satisfying the following inequality*

$$d(f^2x, fx) \leq kd(fx, x) \tag{7}$$

for all x in X, where $k < 1$. If f is continuous, then f has a fixed point.

Putting $E = \mathbf{R}$, $P = \{x \in \mathbf{R} : x \geq 0\} \subset \mathbf{R}$ and $d : X \times X \to \mathbf{R}$ in Theorem 4.2 and Corollary 4.1, then we obtain the following corollaries.

Corollary 4.2. *Let (X, d) be a complete metric space and let f and g be self-mappings of X satisfying the following inequalities*

$$d(fgx, gx) \leq ad(x, gx), \tag{8}$$
$$d(gfx, fx) \leq bd(x, fx) \tag{9}$$

for all x in X, where $a, b < 1$. If either f or g is continuous, then f and g have a common fixed point.

Corollary 4.3. *Let (X, d) be a complete metric space and let f be a self-mapping of X satisfying the following inequality*

$$d(f^2x, fx) \leq kd(fx, x)$$

for all x in X, where $k < 1$. If f is continuous, then f has a fixed point.

We now illustrate Theorem 4.2 by the following example.

Example 4.1. Let $E = \mathbf{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subset \mathbf{R}^2$, $X = \mathbf{R}$ and the mapping $d : X \times X \to E$ defined by $d(x, y) = (\,|x - y|, |x - y|)$. Then (X, d) is a complete cone metric space.

Define the self-mappings $f, g : X \to X$ by

$$fx = \begin{cases} 0 & \text{if } x \leq 1 \\ x/4 & \text{if } x > 1 \end{cases}$$

and

$$gx = \frac{1}{4}x$$

for all x in X.

If $x \leq 1$, then we have

$$\begin{aligned} d(fgx, gx) &= d\Big(0, \frac{x}{4}\Big) = \Big(\frac{|x|}{4}, \frac{|x|}{4}\Big) \\ &= \frac{1}{3}\Big(\Big|x - \frac{x}{4}\Big|, \Big|x - \frac{x}{4}\Big|\Big) = d\Big(x, \frac{x}{4}\Big) = \frac{1}{3}d(x, gx) \end{aligned}$$

and

$$d(gfx, fx) = (0, 0) \leq \frac{1}{3}(|x|, |x|) = \frac{1}{3}d(x, 0) = \frac{1}{3}d(x, fx).$$

If $1 < x \leq 4$, then we have

$$\begin{aligned} d(fgx, gx) &= d\Big(0, \frac{x}{4}\Big) = \Big(\frac{|x|}{4}, \frac{|x|}{4}\Big) \\ &= \frac{1}{3}\Big(\Big|x - \frac{x}{4}\Big|, \Big|x - \frac{x}{4}\Big|\Big) = d\Big(x, \frac{x}{4}\Big) = \frac{1}{3}d(x, gx) \end{aligned}$$

and

$$\begin{aligned} d(gfx, fx) &= d\Big(\frac{x}{16}, \frac{x}{4}\Big) = \Big(\frac{3|x|}{16}, \frac{3|x|}{16}\Big) \\ &\leq \frac{1}{3}\Big(\Big|x - \frac{x}{4}\Big|, \Big|x - \frac{x}{4}\Big|\Big) = \frac{1}{3}d\Big(x, \frac{x}{4}\Big) = \frac{1}{3}d(x, fx). \end{aligned}$$

If $x > 4$, then we have

$$\begin{aligned} d(fgx, gx) &= d\Big(\frac{x}{16}, \frac{x}{4}\Big) = \Big(\frac{3|x|}{16}, \frac{3|x|}{16}\Big) \\ &\leq \frac{1}{3}\Big(\Big|x - \frac{x}{4}\Big|, \Big|x - \frac{x}{4}\Big|\Big) = \frac{1}{3}d\Big(x, \frac{x}{4}\Big) = \frac{1}{3}d(x, gx) \end{aligned}$$

and

$$\begin{aligned} d(gfx, fx) &= d\Big(\frac{x}{16}, \frac{x}{4}\Big) = \Big(\frac{3|x|}{16}, \frac{3|x|}{16}\Big) \\ &\leq \frac{1}{3}\Big(\Big|x - \frac{x}{4}\Big|, \Big|x - \frac{x}{4}\Big|\Big) = \frac{1}{3}d\Big(x, \frac{x}{4}\Big) = \frac{1}{3}d(x, fx). \end{aligned}$$

Thus, inequalities (3) and (4) are satisfied and also $x = 0$ is a common fixed point of f and g.

Remark 4.1. Inequalities (1) and (2) of Theorem 4.1 obviously imply inequalities (8) and (9) of Corollary 4.2. In general, inequalities (8) and (9) do not imply inequalities (1) and (2).

Example 4.2. Let $X = \mathbf{R}$ and $d(x, y) = |x - y|$. Define the self-mappings

$$f, g : X \to X \text{ by } fx = \frac{1}{2}x \text{ and } gx = \frac{1}{4}x$$

for all x in X. Then f and g satisfy inequalities (8) and (9). But, for $x = 0$ and $y \in \mathbf{R}$ $(y \neq 0)$ we get

$$\begin{aligned} d(fg(0), g(y)) &= \frac{1}{4}|y| \not\leq \alpha\frac{1}{4}|y| = \alpha d(0, g(y)), \\ d(gf(0), f(y)) &= \frac{1}{2}|y| \not\leq \beta\frac{1}{2}|y| = \beta d(0, f(y)) \end{aligned}$$

where $0 \leq \alpha, \beta < 1$. Therefore inequalities (1) and (2) are not satisfied.

The following theorems were proved in [4].

Theorem 4.3. *Let* (X, d) *be a complete cone metric space,* P *be a normal cone with normal constant* K*. Suppose the mapping* $T : X \to X$ *satisfies the contractive condition*

$$d(Tx, Ty) \leq kd(x, y), \tag{10}$$

for all $x, y \in X$*, where* $k \in [0, 1)$ *is a constant. Then* T *has a unique fixed point in* X*.*

Theorem 4.4. *Let* (X, d) *be a complete cone metric space,* P *be a normal cone with normal constant* K*. Suppose the mapping* $T : X \to X$ *satisfies the contractive condition*

$$d(Tx, Ty) \leq k(d(Tx, x) + d(Ty, y)), \tag{11}$$

for all $x, y \in X$*, where* $k \in [0, \frac{1}{2})$ *is a constant. Then,* T *has a unique fixed point in* X*.*

Theorem 4.5. *Let* (X, d) *be a complete cone metric space ,* P *be a normal cone with normal constant* K*. Suppose the mapping* $T : X \to X$ *satisfies the contractive condition*

$$d(Tx, Ty) \leq k(d(Tx, y) + d(x, Ty)), \tag{12}$$

for all $x, y \in X$*, where* $k \in [0, \frac{1}{2})$ *is a constant. Then,* T *has a unique fixed point in* X*.*

Note that Rezapour and Hamlbarani also proved these theorems by omitting the normality condition, see [9].

Remark 4.2. Inequalities (10), (11) and (12) obviously imply inequality (7) of Corollary 4.1. In general, this inequality do not imply inequalities (10), (11) and (12). Thus, it is obvious that

Corollary 4.1 that is a generalization of Theorem 4.3. If T is continuous in Theorem 4.4 and Theorem 4.5, then Corollary 4.1 is also a generalization of Theorem 4.4 and Theorem 4.5.

Example 4.3. Let $E = \mathbf{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subset \mathbf{R}^2$, $X = \mathbf{R}$ and the mapping $d : X \times X \to E$ defined by $d(x, y) = (\,|x - y|, |x - y|)$.

Define $f : X \to X$ by

$$fx = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } x > 0 \end{cases}$$

for all x in X. Then we have,

$$\begin{aligned} d(f^2x, fx) &= d(0,0) = (0,0) = kd(fx, x) \ \text{ for } x \leq 0 \text{ and} \\ d(f^2x, fx) &= d(x,x) = (0,0) = kd(fx, x) \ \text{ for } x > 0 \text{ where } k \in [0,1). \end{aligned}$$

Thus, inequality (7) is satisfied and also each $x \in [0, \infty)$ is a fixed point of f.

Now let $x > 0, y > 0$ and $x \neq y$. Then inequalities (10), (11) and (12) are not satisfied.

In fact, if inequality (10) hold for $x > 0$ and $y > 0$ $(x \neq y)$ where $0 \leq k < 1$, then we have

$$\begin{aligned} d(fx, fy) &= d(x, y) = (|x - y|, |x - y|) \\ &\leq kd(x, y) = k(|x - y|, |x - y|), \end{aligned}$$

and so $1 < k$. This is a contradiction because of $0 \leq k < 1$.

If inequality (11) hold for $x > 0$ and $y > 0$ $(x \neq y)$ where $0 \leq k < \frac{1}{2}$, then we have

$$\begin{aligned} d(fx, fy) &= d(x, y) = (|x - y|, |x - y|) \\ &\leq k(d(fx, x) + d(fy, y)) \\ &= k(d(x, x) + d(y, y)) = k(0, 0), \end{aligned}$$

and so this is a contradiction.

If inequality (12) hold for $x > 0$ and $y > 0$ $(x \neq y)$ where $0 \leq k < \frac{1}{2}$, then we have

$$\begin{aligned} d(fx, fy) &= d(x, y) = (|x - y|, |x - y|) \\ &\leq k(d(fx, y) + d(fy, x)) \\ &= k(d(x, y) + d(y, x)) = 2k(|x - y|, |x - y|), \end{aligned}$$

and so $\frac{1}{2} \leq k$. This is a contradiction because of $0 \leq k < \frac{1}{2}$.

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