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L^p local uncertainty inequality for the Sturm-Liouville transform

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ABSTRACT

In this paper, we give analogues of local uncertainty inequality for the Sturm-Liouville transform on $[0, \infty[$. A generalization of Donoho-Stark's uncertainty principle is obtained for this transform.

RESUMEN

En este artículo entregamos resultados análogos de una desigualdad de incertidumbre local de la transformada Sturm-Liouville en $[0, \infty[$. Una generalización del principio de incertidumbre de Donoho-Stark se obtiene de esta transformación.

Keywords and Phrases: Sturm-Liouville transform; local uncertainty principle; Donoho-Stark's uncertainty principle.

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1 Introduction

We consider the second-order differential operator defined on $]0,\infty[$ by

$$\Delta u := u'' + \frac{A'}{A}u' + \rho^2 u,$$

where A is a nonnegative function satisfying certain conditions and ρ is a nonnegative real number. This operator plays an important role in analysis. For example, many special functions (orthogonal polynomials) are eigenfunctions of an operator of Δ type. The radial part of the Beltrami-Laplacian in a symmetric space is also of Δ type. Many aspects of such operators have been studied [2, 10, 17, 18, 19]. In particular, the two references [2, 17] investigate standard constructions of harmonic analysis, such as translation operators, convolution product, and Fourier transform, in connection with Δ.

Many uncertainty principles have already been proved for the Sturm-Liouville operarator Δ, namely by Rösler and Voit [14] who established an uncertainty principle for Hankel transforms. Bouattour and Trimèche [1] proved a Beurling's theorem for the Sturm-Liouville transform. Daher et al. [3, 4, 5] give some related versions of the uncertainty principle for the Sturm-Liouville transform (Hardy's theorem and Miyachi's theorem). Ma [9] proved a Heisenberg uncertainty principle for the Sturm-Liouville transform.

Building on the ideas of Faris [7] and Price [12, 13], we show a local uncertainty principle for the Sturm-Liouville transform $\mathcal{F}$. More precisely, we will show the following result. If $1 < p \leq 2$, $q = p/(p-1)$ and $0 < a < (2\alpha+2)/q$, there is a constant $K(a)$ such that for every $f \in L^p(\mu)$ and every measurable subset $E \subset [0,\infty[$ such that $0 < \nu(E) < \infty$,

$$\left(\int_E |\mathcal{F}(f)(\lambda)|^q d\nu(\lambda)\right)^{1/q} \leq K(a)\Big(\nu(E)\Big)^{\frac{a}{2\alpha+2}} \|x^a f\|_{L^p(\mu)}, \tag{1.1}$$

where μ is the measure given by $d\mu(x) := A(x)dx$, and ν is the Plancherel measure associated to $\mathcal{F}$. (For more details see the next section.)
This inequality generalizes the local uncertainty principle for the Hankel transform given by Ghobber et al. [8] and Omri [11].

We shall use the local uncertainty principle (1.1); and building on the techniques of Donoho and Stark [6], we show a continuous-time principles for the L^p theory, when $1 < p \leq 2$.

This paper is organized as follows. In Section 2 we list some basic properties of the Sturm-Liouville transform $\mathcal{F}$ (Plancherel theorem, inversion formula,...). In Section 3 we show a local uncertainty principle for the Sturm-Liouville $\mathcal{F}$. The Section 4 is devoted to Donoho-Stark's uncertainty principle for the Sturm-Liouville transform $\mathcal{F}$ in the L^p theory, when $1 < p \leq 2$.

2 The Sturm-Liouville transform $\mathcal{F}$

We consider the second-order differential operator Δ defined on $]0,\infty[$ by

$$\Delta u := u'' + \frac{A'}{A}u' + \rho^2 u,$$

where ρ is a nonnegative real number and

$$A(x) := x^{2\alpha+1}B(x), \quad \alpha > -1/2,$$

for B a positive, even, infinitely differentiable function on $\mathbb{R}$ such that $B(0) = 1$. Moreover we assume that A and B satisfy the following conditions:

(i) A is increasing and $\lim\limits_{x\to\infty} A(x) = \infty$.

(ii) $\frac{A'}{A}$ is decreasing and $\lim\limits_{x\to\infty} \frac{A'(x)}{A(x)} = 2\rho$.

(iii) There exists a constant $\delta > 0$ such that

$$\frac{A'(x)}{A(x)} = 2\rho + D(x)\exp(-\delta x) \quad \text{if } \rho > 0,$$

$$\frac{A'(x)}{A(x)} = \frac{2\alpha+1}{x} + D(x)\exp(-\delta x) \quad \text{if } \rho = 0,$$

where D is an infinitely differentiable function on $]0,\infty[$, bounded and with bounded derivatives on all intervals $[x_0,\infty[$, for $x_0 > 0$. This operator was studied in [2, 10, 17], and the following results have been established:

(I) For all $\lambda \in \mathbb{C}$, the equation

$$\begin{cases} \Delta u = -\lambda^2 u \\ u(0) = 1, \ \ u'(0) = 0 \end{cases}$$

admits a unique solution, denoted by φ_λ, with the following properties:

- for $x \geq 0$, the function $\lambda \to \varphi_\lambda(x)$ is analytic on $\mathbb{C}$;
- for $\lambda \in \mathbb{C}$, the function $x \to \varphi_\lambda(x)$ is even and infinitely differentiable on $\mathbb{R}$;
- for all $\lambda, x \in \mathbb{R}$,

$$|\varphi_\lambda(x)| \leq 1. \tag{2.1}$$

(II) For nonzero $\lambda \in \mathbb{C}$, the equation $\Delta u = -\lambda^2 u$ has a solution Φ_λ satisfying

$$\Phi_\lambda(x) = \frac{1}{\sqrt{A(x)}}\exp(i\lambda x)V(x,\lambda),$$

with $\lim_{x\to\infty} V(x,\lambda) = 1$. Consequently there exists a function (spectral function)

$$\lambda \mapsto c(\lambda),$$

such that

$$\varphi_\lambda = c(\lambda)\Phi_\lambda + c(-\lambda)\Phi_{-\lambda} \quad \text{for nonzero } \lambda \in \mathbb{C}.$$

Moreover there exist positive constants k_1, k_2 and k such that

$$k_1|\lambda|^{2\alpha+1} \le |c(\lambda)|^{-2} \le k_2|\lambda|^{2\alpha+1}$$

for all λ such that $\mathrm{Im}\lambda \le 0$ and $|\lambda| \ge k$.

Notation 2.1. We denote by

• μ the measure defined on $[0, \infty[$ by $d\mu(x) := A(x)dx$; and by $L^p(\mu)$, $1 \le p \le \infty$, the space of measurable functions f on $[0, \infty[$, such that

$$\|f\|_{L^p(\mu)} := \left(\int_0^\infty |f(x)|^p d\mu(x)\right)^{1/p} < \infty, \quad 1 \le p < \infty,$$
$$\|f\|_{L^\infty(\mu)} := \operatorname*{ess\ sup}_{x\in[0,\infty[} |f(x)| < \infty;$$

• ν the measure defined on $[0, \infty[$ by $d\nu(\lambda) := \dfrac{d\lambda}{2\pi|c(\lambda)|^2}$; and by $L^p(\nu)$, $1 \le p \le \infty$, the space of measurable functions f on $[0, \infty[$, such that $\|f\|_{L^p(\nu)} < \infty$.

The Fourier transform associated with the operator Δ is defined on $L^1(\mu)$ by

$$\mathcal{F}(f)(\lambda) := \int_0^\infty \varphi_\lambda(x)f(x)d\mu(x) \quad \text{for } \lambda \in \mathbb{R}.$$

Some of the properties of the Fourier transform $\mathcal{F}$ are collected bellow (see [2, 10, 17, 18]).

Theorem 2.2. (i) $L^1 - L^\infty$-boundedness. *For all* $f \in L^1(\mu)$, $\mathcal{F}(f) \in L^\infty(\nu)$ *and*

$$\|\mathcal{F}(f)\|_{L^\infty(\nu)} \le \|f\|_{L^1(\mu)}. \tag{2.2}$$

(ii) Inversion theorem. *Let* $f \in L^1(\mu)$*, such that* $\mathcal{F}(f) \in L^1(\nu)$. *Then*

$$f(x) = \int_0^\infty \varphi_\lambda(x)\mathcal{F}(f)(\lambda)d\nu(\lambda), \quad \text{a.e. } x \in [0, \infty[. \tag{2.3}$$

(iii) Plancherel theorem. *The Fourier transform* $\mathcal{F}$ *extends uniquely to an isometric isomorphism of* $L^2(\mu)$ *onto* $L^2(\nu)$. *In particular,*

$$\|f\|_{L^2(\mu)} = \|\mathcal{F}(f)\|_{L^2(\nu)}. \tag{2.4}$$

Using relations (2.2) and (2.4) with Marcinkiewicz's interpolation theorem [15, 16], we deduce that for every $1 \le p \le 2$, and for every $f \in L^p(\mu)$, the function $\mathcal{F}(f)$ belongs to the space $L^q(\nu)$, $q = p/(p-1)$, and

$$\|\mathcal{F}(f)\|_{L^q(\nu)} \le \|f\|_{L^p(\mu)}. \tag{2.5}$$

3 L^p local uncertainty inequality

This section is devoted to establish a local uncertainty principle for the Sturm-Liouville transform $\mathcal{F}$, more precisely, we will show the following theorem.

Theorem 3.1. *If* $1 < p \leq 2$, $q = p/(p-1)$ *and* $0 < a < (2\alpha+2)/q$, *then for all* $f \in L^p(\mu)$ *and all measurable subset* $E \subset [0,\infty[$ *such that* $0 < \nu(E) < \infty$,

$$\left(\int_E |\mathcal{F}(f)(\lambda)|^q d\nu(\lambda)\right)^{1/q} \leq K(a)\Big(\nu(E)\Big)^{\frac{a}{2\alpha+2}} \|x^a f\|_{L^p(\mu)},$$

$$K(a) = \Big(qa\Big)^{-\frac{qa}{2\alpha+2}}\Big(2\alpha+2-qa\Big)^{\frac{(q-1)a}{2\alpha+2}}\left[1+\frac{qa}{2\alpha+2-qa}\Big(\sup_{x\in[0,r_0]} B(x)\Big)^{1/q}\right],$$

where

$$r_0 = \Big(qa\Big)^{\frac{q}{2\alpha+2}}\Big(2\alpha+2-qa\Big)^{\frac{1-q}{2\alpha+2}}\Big(\nu(E)\Big)^{-\frac{1}{2\alpha+2}}.$$

Proof. For $r > 0$, denote by χ_E, $\chi_{[0,r[}$ and $\chi_{[r,\infty[}$ the characteristic functions.
Let $f \in L^p(\mu)$, $1 < p \leq 2$ and let $q = p/(p-1)$. By Minkowski's inequality, for all $r > 0$,

$$\begin{aligned}\|\mathcal{F}(f)\chi_E\|_{L^q(\nu)} &\leq \|\mathcal{F}(f\chi_{[0,r[})\chi_E\|_{L^q(\nu)} + \|\mathcal{F}(f\chi_{[r,\infty[})\chi_E\|_{L^q(\nu)} \\ &\leq \Big(\nu(E)\Big)^{1/q}\|\mathcal{F}(f\chi_{[0,r[})\|_{L^\infty(\nu)} + \|\mathcal{F}(f\chi_{[r,\infty[})\|_{L^q(\nu)};\end{aligned}$$

hence it follows from (2.2) and (2.5) that

$$\|\mathcal{F}(f)\chi_E\|_{L^q(\nu)} \leq \Big(\nu(E)\Big)^{1/q}\|f\chi_{[0,r[}\|_{L^1(\mu)} + \|f\chi_{[r,\infty[}\|_{L^p(\mu)}. \tag{3.1}$$

On the other hand, by Hölder's inequality,

$$\|f\chi_{[0,r[}\|_{L^1(\mu)} \leq \|x^{-a}\chi_{[0,r[}\|_{L^q(\mu)}\|x^a f\|_{L^p(\mu)}.$$

By hypothesis $a < (2\alpha+2)/q$,

$$\|x^{-a}\chi_{[0,r[}\|_{L^q(\mu)} \leq \frac{r^{-a+(2\alpha+2)/q}}{(2\alpha+2-qa)^{1/q}}\Big(\sup_{x\in[0,r]} B(x)\Big)^{1/q},$$

and therefore,

$$\|f\chi_{[0,r[}\|_{L^1(\mu)} \leq \frac{r^{-a+(2\alpha+2)/q}}{(2\alpha+2-qa)^{1/q}}\Big(\sup_{x\in[0,r]} B(x)\Big)^{1/q}\|x^a f\|_{L^p(\mu)}. \tag{3.2}$$

Moreover,

$$\|f\chi_{[r,\infty[}\|_{L^p(\mu)} \leq \|x^{-a}\chi_{[r,\infty[}\|_{L^\infty(\mu)}\|x^a f\|_{L^p(\mu)} \leq r^{-a}\|x^a f\|_{L^p(\mu)}. \tag{3.3}$$

Combining the relations (3.1), (3.2) and (3.3), we deduce that

$$\|\mathcal{F}(f)\chi_E\|_{L^q(\nu)} \leq \left[r^{-a} + \Big(\nu(E)\Big)^{1/q}\frac{r^{-a+(2\alpha+2)/q}}{(2\alpha+2-qa)^{1/q}}\Big(\sup_{x\in[0,r]} B(x)\Big)^{1/q}\right]\|x^a f\|_{L^p(\mu)}.$$

We choose $r = r_0 = \Big(qa\Big)^{\frac{q}{2\alpha+2}}\Big(2\alpha+2-qa\Big)^{\frac{1-q}{2\alpha+2}}\Big(\nu(E)\Big)^{-\frac{1}{2\alpha+2}}$, we obtain the desired inequality. □

Remark 3.2. (i) The Local uncertainty principle for the Sturm-Liouville transform $\mathcal{F}$ generalizes the local uncertainty principle for the Hankel transform (see [8, 11]).

(ii) If $1 < p \le 2$ and $0 < a < (2\alpha+2)/q$, where $q = p/(p-1)$, then for every $f \in L^p(\mu)$,

$$\sup_{E\subset[0,\infty[,\ 0<\nu(E)<\infty}\left[\Big(\nu(E)\Big)^{-\frac{a}{2\alpha+2}}\|\mathcal{F}(f)\chi_E\|_{L^q(\nu)}\right] \le K(a)\|x^a f\|_{L^p(\mu)}.$$

The left hand side is known to be an equivalent norm of $\mathcal{F}(f)$ in the Lorentz-space $L^{p_a,q}(\nu)$, where

$$p_a = \frac{q(2\alpha+2)}{2\alpha+2-qa}.$$

4 L^p Donoho-Stark uncertainty principle

Let T and E be measurable subsets of $[0,\infty[$. We introduce the time-limiting operator P_T by

$$P_T f := f\chi_T, \tag{4.1}$$

and, we introduce the partial sum operator S_E by

$$\mathcal{F}(S_E f) = \mathcal{F}(f)\chi_E. \tag{4.2}$$

Lemma 4.1. *If* $\nu(E) < \infty$ *and* $f \in L^p(\mu)$, $1 \le p \le 2$,

$$S_E f(x) = \int_E \varphi_\lambda(x)\mathcal{F}(f)(\lambda)d\nu(\lambda).$$

Proof. Let $f \in L^p(\mu)$, $1 \le p \le 2$ and let $q = p/(p-1)$. Then by (2.1), Hölder's inequality and (2.5),

$$\begin{aligned}\|\mathcal{F}(f)\chi_E\|_{L^1(\nu)} &= \int_E |\mathcal{F}(f)(\lambda)|d\nu(\lambda)\\ &\le \Big(\nu(E)\Big)^{1/p}\|\mathcal{F}(f)\|_{L^q(\nu)}\\ &\le \Big(\nu(E)\Big)^{1/p}\|f\|_{L^p(\mu)},\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{F}(f)\chi_E\|_{L^2(\nu)} &= \left(\int_E |\mathcal{F}(f)(\lambda)|^2 d\nu(\lambda)\right)^{1/2}\\ &\le \Big(\nu(E)\Big)^{\frac{q-2}{2q}}\|\mathcal{F}(f)\|_{L^q(\nu)}\\ &\le \Big(\nu(E)\Big)^{\frac{q-2}{2q}}\|f\|_{L^p(\mu)}.\end{aligned}$$

Thus $\mathcal{F}(f)\chi_E \in L^1(\mu) \cap L^2(\mu)$ and by (4.2),

$$S_E f = \mathcal{F}^{-1}\Big(\mathcal{F}(f)\chi_E\Big).$$

This combined with (2.3) gives the result. $\square$

Let T and E be measurable subsets of $[0,\infty[$. We say that a function $f \in L^p(\mu)$, $1 \leq p \leq 2$, is ε-concentrated to T in $L^p(\mu)$-norm, if there is a measurable function $g(t)$ vanishing outside T such that $\|f-g\|_{L^p(\mu)} \leq \varepsilon\|f\|_{L^p(\mu)}$. Similarly, we say that $\mathcal{F}(f)$ is ε-concentrated to E in $L^q(\nu)$-norm, $q = p/(p-1)$, if there is a function $h(\lambda)$ vanishing outside E with $\|\mathcal{F}(f)-h\|_{L^q(\nu)} \leq \varepsilon\|\mathcal{F}(f)\|_{L^q(\nu)}$.

If f is ε_T-concentrated to T in $L^p(\mu)$-norm (g being the vanishing function) then by (4.1),

$$\|f - P_T f\|_{L^p(\mu)} = \left(\int_{[0,\infty[\setminus T} |f(t)|^p d\mu(t)\right)^{1/p} \leq \|f-g\|_{L^p(\mu)} \leq \varepsilon_T\|f\|_{L^p(\mu)} \tag{4.3}$$

and therefore f is ε_T-concentrated to T in $L^p(\mu)$-norm if and only if

$$\|f - P_T f\|_{L^p(\mu)} \leq \varepsilon_T\|f\|_{L^p(\mu)}.$$

From (4.2) it follows as for P_T that $\mathcal{F}(f)$ is ε_E-concentrated to E in $L^q(\nu)$-norm, $q = p/(p-1)$, if and only if

$$\|\mathcal{F}(f) - \mathcal{F}(S_E f)\|_{L^q(\nu)} \leq \varepsilon_E\|\mathcal{F}(f)\|_{L^q(\mu)}. \tag{4.4}$$

Let $B_p(E)$, $1 \leq p \leq 2$, be the set of functions $f \in L^p(\mu)$ that are bandlimited to E (i.e. $f \in B_p(E)$ implies $S_E f = f$).

The spaces $B_p(E)$ satisfy the following property.

Lemma 4.2. *Let T and E be measurable subsets of $[0,\infty[$ such that $0 < \nu(E) < \infty$. For $f \in B_p(E)$, $1 < p \leq 2$ and $0 < a < (2\alpha+2)(1-\frac{1}{p})$,*

$$\|P_T f\|_{L^p(\mu)} \leq K(a)\Big(\mu(T)\Big)^{1/p}\Big(\nu(E)\Big)^{\frac{1}{p}+\frac{a}{2\alpha+2}}\|x^a f\|_{L^p(\mu)},$$

where $K(a)$ is the constant given by Theorem 3.1.

Proof. If $\mu(T) = \infty$, the inequality is clear. Assume that $\mu(T) < \infty$. For $f \in B_p(E)$, $1 < p \leq 2$, from Lemma 4.1,

$$f(t) = \int_E \varphi_\lambda(t)\mathcal{F}(f)(\lambda)d\nu(\lambda),$$

and by (2.1), Hölder's inequality and Theorem 3.1,

$$|f(t)| \leq \Big(\nu(E)\Big)^{1/p}\left(\int_E |\mathcal{F}(f)(\lambda)|^q d\nu(\lambda)\right)^{1/q} \leq K(a)\Big(\nu(E)\Big)^{\frac{1}{p}+\frac{a}{2\alpha+2}}\|x^a f\|_{L^p(\mu)},$$

where $q = p/(p-1)$. Hence,

$$\|P_T f\|_{L^p(\mu)} = \left(\int_T |f(t)|^p d\mu(t)\right)^{1/p} \leq K(a)\Big(\mu(T)\Big)^{1/p}\Big(\nu(E)\Big)^{\frac{1}{p}+\frac{a}{2\alpha+2}}\|x^a f\|_{L^p(\mu)},$$

which yields the result. □

It is useful to have uncertainty principle for the $L^p(\mu)$-norm.

Theorem 4.3. *Let* T *and* E *be measurable subsets of* $[0,\infty[$ *such that* $0 < \nu(E) < \infty$*; and let* $f \in B_p(E)$*,* $1 < p \leq 2$ *and* $0 < a < (2\alpha+2)(1-\frac{1}{p})$*. If* f *is* ε_T*-concentrated to* T*, then*

$$\|f\|_{L^p(\mu)} \leq \frac{K(a)}{1-\varepsilon_T}\Big(\mu(T)\Big)^{1/p}\Big(\nu(E)\Big)^{\frac{1}{p}+\frac{a}{2\alpha+2}}\|x^a f\|_{L^p(\mu)}.$$

Proof. Let $f \in B_p(E)$, $1 < p \leq 2$. Since f is ε_T-concentrated to T in $L^p(\mu)$-norm, then by (4.3) and Lemma 4.2,

$$\begin{aligned}\|f\|_{L^p(\mu)} &\leq \varepsilon_T\|f\|_{L^p(\mu)} + \|P_T f\|_{L^p(\mu)}\\ &\leq \varepsilon_T\|f\|_{L^p(\mu)} + K(a)\Big(\mu(T)\Big)^{1/p}\Big(\nu(E)\Big)^{\frac{1}{p}+\frac{a}{2\alpha+2}}\|x^a f\|_{L^p(\mu)}.\end{aligned}$$

Thus,

$$(1-\varepsilon_T)\|f\|_{L^p(\mu)} \leq K(a)\Big(\mu(T)\Big)^{1/p}\Big(\nu(E)\Big)^{\frac{1}{p}+\frac{a}{2\alpha+2}}\|x^a f\|_{L^p(\mu)},$$

which gives the result. □

Another uncertainty principle for the $L^p(\mu)$ theory is obtained.

Theorem 4.4. *Let* E *be measurable subset of* $[0,\infty[$ *such that* $0 < \nu(E) < \infty$*; and let* $f \in L^p(\mu)$*,* $1 < p \leq 2$ *and* $0 < a < (2\alpha+2)(1-\frac{1}{p})$*. If* $\mathcal{F}(f)$ *is* ε_E*-concentrated to* E *in* $L^q(\nu)$*-norm,* $q = p/(p-1)$*, then*

$$\|\mathcal{F}(f)\|_{L^q(\nu)} \leq \frac{K(a)}{1-\varepsilon_E}\Big(\nu(E)\Big)^{\frac{a}{2\alpha+2}}\|x^a f\|_{L^p(\mu)}.$$

Proof. Let $f \in L^p(\mu)$, $1 < p \leq 2$. Since $\mathcal{F}(f)$ is ε_E-concentrated to E in $L^q(\nu)$-norm, $q = p/(p-1)$, then by (4.4) and Theorem 3.1,

$$\begin{aligned}\|\mathcal{F}(f)\|_{L^q(\nu)} &\leq \varepsilon_E\|\mathcal{F}(f)\|_{L^q(\nu)} + \left(\int_E |\mathcal{F}(f)(\lambda)|^q d\nu(\lambda)\right)^{1/q}\\ &\leq \varepsilon_E\|\mathcal{F}(f)\|_{L^q(\nu)} + K(a)\Big(\nu(E)\Big)^{\frac{a}{2\alpha+2}}\|x^a f\|_{L^p(\mu)}.\end{aligned}$$

Thus,

$$(1-\varepsilon_E)\|\mathcal{F}(f)\|_{L^q(\nu)} \leq K(a)\Big(\nu(E)\Big)^{\frac{a}{2\alpha+2}}\|x^a f\|_{L^p(\mu)},$$

which proves the result. □

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