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Euler's constant, new classes of sequences and estimates

Alina Sîntămărian
Department of Mathematics,
Technical University of Cluj-Napoca,
Str. Memorandumului nr. 28, 400114 Cluj-Napoca, Romania.
alina.sintamarian@math.utcluj.ro

ABSTRACT

We give two classes of sequences with the argument of the logarithmic term modified and also with some additional terms besides those in the definition sequence, and that converge quickly to $\gamma(a) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{a+k-1} - \ln\frac{a+n-1}{a}\right)$, where $a \in (0,+\infty)$. We present the pattern in forming these sequences, expressing the coefficients that appear with the Bernoulli numbers. Also, we obtain estimates containing best constants for $\sum_{k=1}^{n}\frac{1}{k} - \frac{1}{24(n+1/2)^2} - \ln\left(n+\frac{1}{2}-\frac{7}{960(n+1/2)^3}\right) - \gamma$ and $\gamma - \left(\sum_{k=1}^{n}\frac{1}{k} - \frac{1}{24(n+1/2)^2} + \frac{7}{960(n+1/2)^4} - \ln\left(n+\frac{1}{2}+\frac{31}{8064(n+1/2)^5}\right)\right)$, where $\gamma = \gamma(1)$ is the Euler's constant.

RESUMEN

Mostramos dos clases de secuencias con el argumento del término logarítmico modificado y también con algunos términos adicionales además de los definidos en la secuencia y que convergen rápidamente a $\gamma(a) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{a+k-1} - \ln\frac{a+n-1}{a}\right)$, donde $a \in (0,+\infty)$. Presentamos el patrón que forma las secuencias expresando los coeficientes que aparecen en los números de Bernoulli. Además, obtenemos estimaciones que contienen las mejores constantes para $\sum_{k=1}^{n}\frac{1}{k} - \frac{1}{24(n+1/2)^2} - \ln\left(n+\frac{1}{2}-\frac{7}{960(n+1/2)^3}\right) - \gamma$ y $\gamma - \left(\sum_{k=1}^{n}\frac{1}{k} - \frac{1}{24(n+1/2)^2} + \frac{7}{960(n+1/2)^4} - \ln\left(n+\frac{1}{2}+\frac{31}{8064(n+1/2)^5}\right)\right)$, donde $\gamma = \gamma(1)$ es la constante de Euler.

Keywords and Phrases: sequence, convergence, approximation, Euler's constant, Bernoulli number, estimate.

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1 Introduction

Let $H_n = \sum_{k=1}^{n} 1/k$ be the nth harmonic number and let $D_n = H_n - \ln n$. Euler's constant $\gamma = \lim_{n\to\infty} D_n$ is one of the most important constants in mathematics and is also the topic of many papers in the literature. This comes as a confirmation of what Leonhard Euler said about γ, namely that it is "worthy of serious consideration" ([10, pp. xx, 51]).

It is well-known (see [19], [20], [2], [5]) that

$$\frac{1}{2n + \frac{2\gamma-1}{1-\gamma}} \le D_n - \gamma < \frac{1}{2n + \frac{1}{3}}, \quad n \in \mathbb{N},$$

the numbers $\frac{2\gamma-1}{1-\gamma}$ and $\frac{1}{3}$ being the best constants with this property, i.e. $\frac{2\gamma-1}{1-\gamma}$ cannot be replaced by a smaller one and $\frac{1}{3}$ cannot be replaced by a larger one, so that the above-mentioned inequalities to hold for all $n \in \mathbb{N}$. Having in view that $\lim_{n\to\infty} n(D_n - \gamma) = 1/2$, one can say that the sequence $(D_n)_{n\in\mathbb{N}}$ converges to γ very slowly. In order to increase the speed of convergence to γ, D. W. DeTemple [7] modified the argument of the logarithmic term from D_n, considering the sequence $(R_n)_{n\in\mathbb{N}}$ defined by $R_n = H_n - \ln(n+1/2)$, and he proved that $\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}$, $n \in \mathbb{N}$. Sequences with higher rate of convergence to γ can be also obtained by subtracting a rational term from D_n: it is shown (see [21]) that $\lim_{n\to\infty} n^2(\gamma - D_n + 1/(2n)) = 1/12$. In our paper we shall try to combine these two methods, modifying the argument of the logarithmic term, and subtracting and adding terms in the definition sequence, to obtain quicker convergences to a generalization of Euler's constant. Also, we shall provide estimates regarding Euler's constant γ and this is the reason why further on we remind some of the estimates related to γ and containing best constants, that have been obtained in the literature:

$\frac{1}{24(n+a_1)^2} \le R_n - \gamma < \frac{1}{24(n+b_1)^2}, \quad n \in \mathbb{N} \quad$ ([3]);

$\frac{1}{12n^2+a_2} < \gamma - \left(D_n - \frac{1}{2n}\right) \le \frac{1}{12n^2+b_2}, \quad n \in \mathbb{N} \quad$ ([9]);

$\frac{7}{960(n+a_3)^4} \le \gamma - \left(H_n - \ln\left(n+\frac{1}{2}\right) - \frac{1}{24(n+1/2)^2}\right) < \frac{7}{960(n+b_3)^4}, \quad n \in \mathbb{N} \quad$ ([4]);

$\frac{17}{3840(n+a_4)^5} \le H_n - \ln\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3}\right) - \gamma < \frac{17}{3840(n+b_4)^5}, \quad n \in \mathbb{N} \quad$ ([6]),

with

$a_1 = 1/\sqrt{24\,(1-\gamma-\ln(3/2))} - 1$ and $b_1 = 1/2$;

$a_2 = 6/5$ and $b_2 = 2(7-12\gamma)/(2\gamma-1)$;

$a_3 = 1/\sqrt[4]{960/7\,(\ln(3/2)+\gamma-53/54)} - 1$ and $b_3 = 1/2$;

$a_4 = 1/\sqrt[5]{3840/17\,(1-\gamma-\ln(8783/5760))} - 1$ and $b_4 = 3305/12852$,

where a_i and b_i are the best constants with the property that the corresponding inequalities hold for all $n \in \mathbb{N}$, $i \in \{1,2,3,4\}$.

As we anticipated earlier, in the present paper we shall investigate a generalization of Euler's constant, namely the limit $\gamma(a)$ of the sequence $(y_n(a))_{n\in\mathbb{N}}$ defined by (see [11, p. 453], [16], [17],

[18])

$$y_n(a) = \sum_{k=1}^{n} \frac{1}{a+k-1} - \ln \frac{a+n-1}{a},$$

where $a \in (0, +\infty)$. Obviously, $\gamma(1) = \gamma$. Numerous results related to $\gamma(a)$ can be found, for example, in [16], [17], [18], [14], [12].

In Section 2 we give two classes of sequences with the argument of the logarithmic term modified and also with some additional terms besides those in the definition sequence $(y_n(a))_{n\in\mathbb{N}}$, and that converge quickly to $\gamma(a)$. We present the pattern in forming these sequences, expressing the coefficients that appear with the Bernoulli numbers. The two classes of sequences have as base the sequences $(\alpha_{n,2}(a))_{n\in\mathbb{N}}$ and $(\alpha_{n,3}(a))_{n\in\mathbb{N}}$ defined by

$$\begin{aligned}
\alpha_{n,2}(a) &= \sum_{k=1}^{n} \frac{1}{a+k-1} - \frac{1}{24\left(a+n-\frac{1}{2}\right)^2} - \ln\left(\frac{a+n-\frac{1}{2}}{a} - \frac{7}{960a\left(a+n-\frac{1}{2}\right)^3}\right), \\
\alpha_{n,3}(a) &= \sum_{k=1}^{n} \frac{1}{a+k-1} - \frac{1}{24\left(a+n-\frac{1}{2}\right)^2} + \frac{7}{960\left(a+n-\frac{1}{2}\right)^4} \\
&\quad - \ln\left(\frac{a+n-\frac{1}{2}}{a} + \frac{31}{8064a\left(a+n-\frac{1}{2}\right)^5}\right).
\end{aligned}$$

In Section 3 we prove estimates containing best constants for $\alpha_{n,2}(1) - \gamma$ and $\gamma - \alpha_{n,3}(1)$, $n \in \mathbb{N}$.

The following lemma, which we shall need in our proofs, was given by C. Mortici [13, Lemma] and is a consequence of the the Stolz-Cesàro Theorem, the $0/0$ case [8, Theorem B.2, p. 265].

Lemma 1.1. *Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence of real numbers and $x^* = \lim_{n\to\infty} x_n$. We suppose that there exists $\alpha \in \mathbb{R}$, $\alpha > 1$, such that*

$$\lim_{n\to\infty} n^{\alpha}(x_n - x_{n+1}) = l \in \overline{\mathbb{R}}.$$

Then there exists the limit

$$\lim_{n\to\infty} n^{\alpha-1}(x_n - x^*) = \frac{l}{\alpha-1}.$$

Also, recall that the digamma function ψ is the logarithmic derivative of the gamma function, i.e.

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad x \in (0, +\infty).$$

It is known that ([1, p. 258], [15, p. 337])

$$\psi(n+1) = -\gamma + H_n, \quad n \in \mathbb{N}. \tag{1}$$

From the recurrence formula ([1, p. 258])

$$\psi(x+1) = \psi(x) + \frac{1}{x}, \quad x \in (0, +\infty),$$

and the asymptotic formula ([1, p. 259])

$$\psi(x) \sim \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \frac{1}{240x^8} - \frac{1}{132x^{10}} + \cdots \quad (x \to \infty),$$

one obtains

$$\psi(x+1) \sim \ln x + \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \frac{1}{240x^8} - \frac{1}{132x^{10}} + \cdots \quad (x \to \infty). \tag{2}$$

2 Sequences that converge to $\gamma(a)$

Theorem 2.1. *Let $a \in (0, +\infty)$ and let $\gamma(a)$ be the limit of the sequence $(y_n(a))_{n\in\mathbb{N}}$ from Introduction. We consider the sequences $(\alpha_{n,2}(a))_{n\in\mathbb{N}}$ and $(\beta_{n,2}(a))_{n\in\mathbb{N}}$ defined by*

$$\begin{aligned}
\alpha_{n,2}(a) &= \sum_{k=1}^{n} \frac{1}{a+k-1} - \frac{1}{24\left(a+n-\frac{1}{2}\right)^2} - \ln\left(\frac{a+n-\frac{1}{2}}{a} - \frac{7}{960a\left(a+n-\frac{1}{2}\right)^3}\right),\\
\beta_{n,2}(a) &= \alpha_{n,2}(a) - \frac{31}{8064\left(a+n-\frac{1}{2}\right)^6}.
\end{aligned}$$

Then:

(i) $\displaystyle\lim_{n\to\infty} n^6(\alpha_{n,2}(a) - \gamma(a)) = \frac{31}{8064}$;

(ii) $\displaystyle\lim_{n\to\infty} n^8(\gamma(a) - \beta_{n,2}(a)) = \frac{7571}{1843200}$.

Proof. (i) Set $\varepsilon_n := \frac{1}{a+n}$, $n \in \mathbb{N}$. Since $\pm\frac{1}{2}\,\varepsilon_n \in (-1, 1)$, $-\frac{1}{2}\,\varepsilon_n - \frac{7}{960} \cdot \frac{\varepsilon_n^4}{\left(1-\frac{1}{2}\,\varepsilon_n\right)^3} \in (-1, 1]$ and $\frac{1}{2}\,\varepsilon_n - \frac{7}{960} \cdot \frac{\varepsilon_n^4}{\left(1+\frac{1}{2}\,\varepsilon_n\right)^3} \in (-1, 1]$, for every $n \in \mathbb{N}$, using the series expansion ([11, pp. 171–179]) we obtain

$$\begin{aligned}
&\alpha_{n,2}(a) - \alpha_{n+1,2}(a)\\
&= -\varepsilon_n - \frac{1}{24} \cdot \frac{\varepsilon_n^2}{\left(1-\frac{1}{2}\,\varepsilon_n\right)^2} + \frac{1}{24} \cdot \frac{\varepsilon_n^2}{\left(1+\frac{1}{2}\,\varepsilon_n\right)^2}\\
&\quad - \ln\left(1 - \frac{1}{2}\,\varepsilon_n - \frac{7}{960} \cdot \frac{\varepsilon_n^4}{\left(1-\frac{1}{2}\,\varepsilon_n\right)^3}\right) + \ln\left(1 + \frac{1}{2}\,\varepsilon_n - \frac{7}{960} \cdot \frac{\varepsilon_n^4}{\left(1+\frac{1}{2}\,\varepsilon_n\right)^3}\right)\\
&= \frac{31}{1344}\varepsilon_n^7 + \frac{4829}{230400}\varepsilon_n^9 + \frac{2913}{225280}\varepsilon_n^{11} + \frac{20456239}{2875392000}\varepsilon_n^{13} + O(\varepsilon_n^{15}).
\end{aligned}$$

It follows that

$$\lim_{n\to\infty} n^7(\alpha_{n,2}(a) - \alpha_{n+1,2}(a)) = \frac{31}{1344}.$$

Now, according to Lemma 1.1, we get

$$\lim_{n\to\infty} n^6(\alpha_{n,2}(a) - \gamma(a)) = \frac{31}{8064}.$$

(ii) We are able to write that

$$\begin{aligned}
&\beta_{n+1,2}(a) - \beta_{n,2}(a) \\
&= \alpha_{n+1,2}(a) - \alpha_{n,2}(a) - \frac{31}{8064} \cdot \frac{\varepsilon_n^6}{(1+\frac{1}{2}\varepsilon_n)^6} + \frac{31}{8064} \cdot \frac{\varepsilon_n^6}{(1-\frac{1}{2}\varepsilon_n)^6} \\
&= \frac{7571}{230400}\varepsilon_n^9 + \frac{10727}{225280}\varepsilon_n^{11} + O(\varepsilon_n^{13}).
\end{aligned}$$

Therefore

$$\lim_{n\to\infty} n^9(\beta_{n+1,2}(a) - \beta_{n,2}(a)) = \frac{7571}{230400},$$

and based on Lemma 1.1 we obtain

$$\lim_{n\to\infty} n^8(\gamma(a) - \beta_{n,2}(a)) = \frac{7571}{1843200}.$$

□

Also, considering the sequence in each of the following parts and using similar arguments as in Theorem 2.1, we get the indicated limit:

$$\delta_{n,2}(a) = \beta_{n,2}(a) + \frac{7571}{1843200\left(a+n-\frac{1}{2}\right)^8},\ n \in \mathbb{N},$$

$$\lim_{n\to\infty} n^{10}(\delta_{n,2}(a) - \gamma(a)) = \frac{511}{67584};$$

$$\eta_{n,2}(a) = \delta_{n,2}(a) - \frac{511}{67584\left(a+n-\frac{1}{2}\right)^{10}},\ n \in \mathbb{N},$$

$$\lim_{n\to\infty} n^{12}(\gamma(a) - \eta_{n,2}(a)) = \frac{5092085987}{241532928000};$$

$$\theta_{n,2}(a) = \eta_{n,2}(a) + \frac{5092085987}{241532928000\left(a+n-\frac{1}{2}\right)^{12}},\ n \in \mathbb{N},$$

$$\lim_{n\to\infty} n^{14}(\theta_{n,2}(a) - \gamma(a)) = \frac{8191}{98304};$$

$$\lambda_{n,2}(a) = \theta_{n,2}(a) - \frac{8191}{98304\left(a+n-\frac{1}{2}\right)^{14}},\ n \in \mathbb{N},$$

$$\lim_{n\to\infty} n^{16}(\gamma(a) - \lambda_{n,2}(a)) = \frac{25599939583183}{57755566080000};$$

$$\mu_{n,2}(a) = \lambda_{n,2}(a) + \frac{25599939583183}{57755566080000\left(a+n-\frac{1}{2}\right)^{16}},\ n \in \mathbb{N},$$

$$\lim_{n\to\infty} n^{18}(\mu_{n,2}(a) - \gamma(a)) = \frac{5749691557}{1882718208}.$$

We remark the pattern in forming the sequences from Theorem 2.1 and those mentioned above. For example, the general term of the sequence $(\mu_{n,2}(a))_{n\in\mathbb{N}}$ can be written in the form

$$\begin{aligned}\mu_{n,2}(a) &= \sum_{k=1}^{n}\frac{1}{a+k-1}-\frac{1}{2}\cdot\frac{B_2}{2}\cdot\frac{1}{\left(a+n-\frac{1}{2}\right)^2}\\ &\quad -\ln\left(\frac{a+n-\frac{1}{2}}{a}+\frac{2^3-1}{2^3}\cdot\frac{B_4}{4}\cdot\frac{1}{a\left(a+n-\frac{1}{2}\right)^3}\right)-\sum_{k=3}^{8}\frac{c_{k,2}}{\left(a+n-\frac{1}{2}\right)^{2k}},\end{aligned}$$

with

$$c_{k,2}=\begin{cases}\dfrac{2^{2k-1}-1}{2^{2k-1}}\cdot\dfrac{B_{2k}}{2k}, & \text{if } k=2p+1, p\in\mathbb{N},\\ \dfrac{2^{2k-1}-1}{2^{2k-1}}\cdot\dfrac{B_{2k}}{2k}+\dfrac{2}{k}\left(-\dfrac{2^3-1}{2^3}\cdot\dfrac{B_4}{4}\right)^{\frac{k}{2}}, & \text{if } k=2p+2, p\in\mathbb{N},\end{cases}$$

where B_{2k} is the 2kth Bernoulli number. Related to this remark, see also [16, Remark 3.4], [18, p. 71, Remark 2.1.3; pp. 100, 101, Remark 3.1.6].

For Euler's constant $\gamma = 0.5772156649\ldots$ we obtain, for example:

$\alpha_{2,2}(1) = 0.5772292855\ldots$; $\alpha_{3,2}(1) = 0.5772175963\ldots$;
$\beta_{2,2}(1) = 0.5772135395\ldots$; $\beta_{3,2}(1) = 0.5772155051\ldots$;
$\delta_{2,2}(1) = 0.5772162314\ldots$; $\delta_{3,2}(1) = 0.5772156875\ldots$;
$\eta_{2,2}(1) = 0.5772154386\ldots$; $\eta_{3,2}(1) = 0.5772156600\ldots$;
$\theta_{2,2}(1) = 0.5772157923\ldots$; $\theta_{3,2}(1) = 0.5772156663\ldots$;
$\lambda_{2,2}(1) = 0.5772155686\ldots$; $\lambda_{3,2}(1) = 0.5772156643\ldots$;
$\mu_{2,2}(1) = 0.5772157590\ldots$; $\mu_{3,2}(1) = 0.5772156651\ldots$.

As can be seen, $\lambda_{3,2}(1)$ is accurate to nine decimal places in approximating γ.

Theorem 2.2. *Let $a\in(0,+\infty)$ and let $\gamma(a)$ be the limit of the sequence $(y_n(a))_{n\in\mathbb{N}}$ from Introduction. We consider the sequences $(\alpha_{n,3}(a))_{n\in\mathbb{N}}$, $(\beta_{n,3}(a))_{n\in\mathbb{N}}$ and $(\delta_{n,3}(a))_{n\in\mathbb{N}}$ defined by*

$$\begin{aligned}\alpha_{n,3}(a) &= \sum_{k=1}^{n}\frac{1}{a+k-1}-\frac{1}{24\left(a+n-\frac{1}{2}\right)^2}+\frac{7}{960\left(a+n-\frac{1}{2}\right)^4}\\ &\quad -\ln\left(\frac{a+n-\frac{1}{2}}{a}+\frac{31}{8064a\left(a+n-\frac{1}{2}\right)^5}\right),\\ \beta_{n,3}(a) &= \alpha_{n,3}(a)+\frac{127}{30720\left(a+n-\frac{1}{2}\right)^8},\\ \delta_{n,3}(a) &= \beta_{n,3}(a)-\frac{511}{67584\left(a+n-\frac{1}{2}\right)^{10}}.\end{aligned}$$

Then:

(i) $\lim_{n\to\infty} n^8(\gamma(a)-\alpha_{n,3}(a)) = \dfrac{127}{30720}$;

(ii) $\lim_{n\to\infty} n^{10}(\beta_{n,3}(a)-\gamma(a)) = \dfrac{511}{67584}$;

(iii) $\lim_{n\to\infty} n^{12}(\gamma(a)-\delta_{n,3}(a)) = \dfrac{178161637}{8453652480}$.

Proof. (i) Set $\varepsilon_n := \frac{1}{a+n}$, $n \in \mathbb{N}$. Since $\pm\frac{1}{2}\varepsilon_n \in (-1,1)$, $\frac{1}{2}\varepsilon_n + \frac{31}{8064}\cdot\frac{\varepsilon_n^6}{(1+\frac{1}{2}\varepsilon_n)^5} \in (-1,1]$ and $-\frac{1}{2}\varepsilon_n + \frac{31}{8064}\cdot\frac{\varepsilon_n^6}{(1-\frac{1}{2}\varepsilon_n)^5} \in (-1,1]$, for every $n\in\mathbb{N}$, using the series expansion ([11, pp. 171–179]) we obtain

$$\begin{aligned}
&\alpha_{n+1,3}(a)-\alpha_{n,3}(a)\\
&= \varepsilon_n - \frac{1}{24}\cdot\frac{\varepsilon_n^2}{\left(1+\frac{1}{2}\varepsilon_n\right)^2} + \frac{1}{24}\cdot\frac{\varepsilon_n^2}{\left(1-\frac{1}{2}\varepsilon_n\right)^2} + \frac{7}{960}\cdot\frac{\varepsilon_n^4}{\left(1+\frac{1}{2}\varepsilon_n\right)^4} - \frac{7}{960}\cdot\frac{\varepsilon_n^4}{\left(1-\frac{1}{2}\varepsilon_n\right)^4}\\
&\quad -\ln\left(1+\frac{1}{2}\varepsilon_n + \frac{31}{8064}\cdot\frac{\varepsilon_n^6}{\left(1+\frac{1}{2}\varepsilon_n\right)^5}\right) + \ln\left(1-\frac{1}{2}\varepsilon_n + \frac{31}{8064}\cdot\frac{\varepsilon_n^6}{\left(1-\frac{1}{2}\varepsilon_n\right)^5}\right)\\
&= \frac{127}{3840}\varepsilon_n^9 + \frac{409}{8448}\varepsilon_n^{11} + \frac{5873471}{140894208}\varepsilon_n^{13} + \frac{2502391}{92897280}\varepsilon_n^{15}\\
&\quad + \frac{2826605}{210567168}\varepsilon_n^{17} + \frac{33340423721}{8302787297280}\varepsilon_n^{19} + O(\varepsilon_n^{21}).
\end{aligned}$$

Consequently,

$$\lim_{n\to\infty} n^9(\alpha_{n+1,3}(a)-\alpha_{n,3}(a)) = \frac{127}{3840},$$

and from this, based on Lemma 1.1, we get

$$\lim_{n\to\infty} n^8(\gamma(a)-\alpha_{n,3}(a)) = \frac{127}{30720}.$$

(ii) We have

$$\begin{aligned}
&\beta_{n,3}(a)-\beta_{n+1,3}(a)\\
&= \alpha_{n,3}(a)-\alpha_{n+1,3}(a) + \frac{127}{30720}\cdot\frac{\varepsilon_n^8}{(1-\frac{1}{2}\varepsilon_n)^8} - \frac{127}{30720}\cdot\frac{\varepsilon_n^8}{(1+\frac{1}{2}\varepsilon_n)^8}\\
&= \frac{2555}{33792}\,\varepsilon_n^{11} + \frac{114794663}{704471040}\,\varepsilon_n^{13} + \frac{18092183}{92897280}\varepsilon_n^{15} + \frac{36074257}{210567168}\varepsilon_n^{17} + O(\varepsilon_n^{19}).
\end{aligned}$$

Thus

$$\lim_{n\to\infty} n^{11}(\beta_{n,3}(a)-\beta_{n+1,3}(a)) = \frac{2555}{33792},$$

and applying Lemma 1.1, it follows that

$$\lim_{n\to\infty} n^{10}(\beta_{n,3}(a)-\gamma(a)) = \frac{511}{67584}.$$

(iii) We can write that

$$\begin{aligned}
&\delta_{n+1,3}(a)-\delta_{n,3}(a)\\
&=\beta_{n+1,3}(a)-\beta_{n,3}(a)-\frac{511}{67584}\cdot\frac{\varepsilon_n^{10}}{(1+\frac{1}{2}\,\varepsilon_n)^{10}}+\frac{511}{67584}\cdot\frac{\varepsilon_n^{10}}{(1-\frac{1}{2}\,\varepsilon_n)^{10}}\\
&=\frac{178161637}{704471040}\varepsilon_n^{13}+\frac{69794707}{92897280}\varepsilon_n^{15}+O(\varepsilon_n^{17}).
\end{aligned}$$

Hence

$$\lim_{n\to\infty} n^{13}(\delta_{n+1,3}(a)-\delta_{n,3}(a))=\frac{178161637}{704471040}.$$

This, along with Lemma 1.1, gives

$$\lim_{n\to\infty} n^{12}(\gamma(a)-\delta_{n,3}(a))=\frac{178161637}{8453652480}.$$

□

Also, considering the sequence in each of the following parts and using similar arguments as in Theorem 2.2, we get the indicated limit:

$$\eta_{n,3}(a)=\delta_{n,3}(a)+\frac{178161637}{8453652480\left(a+n-\frac{1}{2}\right)^{12}},\ n\in\mathbb{N},$$

$$\lim_{n\to\infty} n^{14}(\eta_{n,3}(a)-\gamma(a))=\frac{8191}{98304};$$

$$\theta_{n,3}(a)=\eta_{n,3}(a)-\frac{8191}{98304\left(a+n-\frac{1}{2}\right)^{14}},\ n\in\mathbb{N},$$

$$\lim_{n\to\infty} n^{16}(\gamma(a)-\theta_{n,3}(a))=\frac{118518239}{267386880};$$

$$\lambda_{n,3}(a)=\theta_{n,3}(a)+\frac{118518239}{267386880\left(a+n-\frac{1}{2}\right)^{16}},\ n\in\mathbb{N},$$

$$\lim_{n\to\infty} n^{18}(\lambda_{n,3}(a)-\gamma(a))=\frac{91282102592903}{29890034270208};$$

$$\mu_{n,3}(a)=\lambda_{n,3}(a)-\frac{91282102592903}{29890034270208\left(a+n-\frac{1}{2}\right)^{18}},\ n\in\mathbb{N},$$

$$\lim_{n\to\infty} n^{20}(\gamma(a)-\mu_{n,3}(a))=\frac{91546277357}{3460300800}.$$

We remark the pattern in forming the sequences from Theorem 2.2 and those mentioned above. For example, the general term of the sequence $(\mu_{n,3}(a))_{n\in\mathbb{N}}$ can be written in the form

$$\begin{aligned}
\mu_{n,3}(a)\ =&\sum_{k=1}^{n}\frac{1}{a+k-1}-\frac{1}{2}\cdot\frac{B_2}{2}\cdot\frac{1}{\left(a+n-\frac{1}{2}\right)^2}-\frac{2^3-1}{2^3}\cdot\frac{B_4}{4}\cdot\frac{1}{\left(a+n-\frac{1}{2}\right)^4}\\
&-\ln\left(\frac{a+n-\frac{1}{2}}{a}+\frac{2^5-1}{2^5}\cdot\frac{B_6}{6}\cdot\frac{1}{a\left(a+n-\frac{1}{2}\right)^5}\right)-\sum_{k=4}^{9}\frac{c_{k,3}}{\left(a+n-\frac{1}{2}\right)^{2k}},
\end{aligned}$$

with

$$c_{k,3} = \begin{cases} \dfrac{2^{2k-1}-1}{2^{2k-1}} \cdot \dfrac{B_{2k}}{2k}, & \text{if } k = 3p+1, p \in \mathbb{N}, \\ \dfrac{2^{2k-1}-1}{2^{2k-1}} \cdot \dfrac{B_{2k}}{2k}, & \text{if } k = 3p+2, p \in \mathbb{N}, \\ \dfrac{2^{2k-1}-1}{2^{2k-1}} \cdot \dfrac{B_{2k}}{2k} + \dfrac{3}{k}\left(-\dfrac{2^5-1}{2^5} \cdot \dfrac{B_6}{6}\right)^{\frac{k}{3}}, & \text{if } k = 3p+3, p \in \mathbb{N}, \end{cases}$$

where B_{2k} is the 2kth Bernoulli number. Related to this remark, see also [16, Remark 3.4], [18, p. 71, Remark 2.1.3; pp. 100, 101, Remark 3.1.6].

For Euler's constant $\gamma = 0.5772156649\ldots$ we obtain, for example:

$\alpha_{2,3}(1) = 0.5772135222\ldots;$ $\alpha_{3,3}(1) = 0.5772155039\ldots;$
$\beta_{2,3}(1) = 0.5772162315\ldots;$ $\beta_{3,3}(1) = 0.5772156875\ldots;$
$\delta_{2,3}(1) = 0.5772154387\ldots;$ $\delta_{3,3}(1) = 0.5772156601\ldots;$
$\eta_{2,3}(1) = 0.5772157923\ldots;$ $\eta_{3,3}(1) = 0.5772156663\ldots;$
$\theta_{2,3}(1) = 0.5772155686\ldots;$ $\theta_{3,3}(1) = 0.5772156643\ldots;$
$\lambda_{2,3}(1) = 0.5772157590\ldots;$ $\lambda_{3,3}(1) = 0.5772156651\ldots;$
$\mu_{2,3}(1) = 0.5772155491\ldots;$ $\mu_{3,3}(1) = 0.5772156647\ldots.$

As can be seen, $\theta_{3,3}(1)$ and $\mu_{3,3}(1)$ are accurate to nine decimal places in approximating γ.

Concluding, the following remark can be made. Let $a \in (0, +\infty)$ and $q \in \mathbb{N} \setminus \{1\}$. Let $n_0 = \min\left\{ n \in \mathbb{N} \,\middle|\, a + n - \frac{1}{2} + \frac{2^{2q-1}-1}{2^{2q-1}} \cdot \frac{B_{2q}}{2q} \cdot \frac{1}{\left(a+n-\frac{1}{2}\right)^{2q-1}} > 0 \right\}$. We consider the sequence $(\alpha_{n,q}(a))_{n \geq n_0}$ defined by

$$\begin{aligned} \alpha_{n,q}(a) \ &= \sum_{k=1}^{n} \frac{1}{a+k-1} - \sum_{k=1}^{q-1} \frac{2^{2k-1}-1}{2^{2k-1}} \cdot \frac{B_{2k}}{2k} \cdot \frac{1}{\left(a+n-\frac{1}{2}\right)^{2k}} \\ &\quad - \ln\left(\frac{a+n-\frac{1}{2}}{a} + \frac{2^{2q-1}-1}{2^{2q-1}} \cdot \frac{B_{2q}}{2q} \cdot \frac{1}{a\left(a+n-\frac{1}{2}\right)^{2q-1}} \right), \end{aligned}$$

for every $n \in \mathbb{N}$, $n \geq n_0$. Clearly, $\lim\limits_{n\to\infty} \alpha_{n,q}(a) = \gamma(a)$. Based on the sequence $(\alpha_{n,q}(a))_{n \geq n_0}$, a class of sequences convergent to $\gamma(a)$ can be considered, namely $\{(\alpha_{n,q,r}(a))_{n \geq n_0} | r \in \mathbb{N}, r \geq q+1\}$, where

$$\alpha_{n,q,r}(a) = \alpha_{n,q}(a) - \sum_{k=q+1}^{r} \frac{c_{k,q}}{\left(a+n-\frac{1}{2}\right)^{2k}},$$

for every $n \in \mathbb{N}$, $n \geq n_0$, with

$$c_{k,q} = \begin{cases} \dfrac{2^{2k-1}-1}{2^{2k-1}} \cdot \dfrac{B_{2k}}{2k}, & \text{if } k \in \{qp+1, qp+2, \ldots, qp+q-1\}, p \in \mathbb{N}, \\ \dfrac{2^{2k-1}-1}{2^{2k-1}} \cdot \dfrac{B_{2k}}{2k} + \dfrac{q}{k}\left(-\dfrac{2^{2q-1}-1}{2^{2q-1}} \cdot \dfrac{B_{2q}}{2q}\right)^{\frac{k}{q}}, & \text{if } k = qp+q, p \in \mathbb{N}. \end{cases}$$

In this section we have obtained that the sequence $(\alpha_{n,q}(a))_{n\in\mathbb{N}}$ converges to $\gamma(a)$ as $n^{-(2q+2)}$ and that the sequence $(\alpha_{n,q,r}(a))_{n\in\mathbb{N}}$ converges to $\gamma(a)$ as $n^{-(2r+2)}$, for $q \in \{2,3\}$ and $r \in \{q+1, q+2, q+3, q+4, q+5, q+6\}$.

3 Best bounds

Let $(\alpha_n)_{n\in\mathbb{N}}$ be the sequence defined by $\alpha_n = \alpha_{n,2}(1)$. In part (i) of Theorem 2.1 we have proved that

$$\lim_{n\to\infty} n^6(\alpha_n - \gamma) = \frac{31}{8064}. \tag{3}$$

Proposition 3.1. *We have $\gamma < \alpha_{n+1} < \alpha_n$, for every $n \in \mathbb{N}$.*

Proof. We have

$$\alpha_{n+1} - \alpha_n = \frac{1}{n+1} - \frac{1}{24\left(n+\frac{3}{2}\right)^2} + \frac{1}{24\left(n+\frac{1}{2}\right)^2} - \ln\frac{960\left(n+\frac{3}{2}\right)^4 - 7}{\left(n+\frac{3}{2}\right)^3} + \ln\frac{960\left(n+\frac{1}{2}\right)^4 - 7}{\left(n+\frac{1}{2}\right)^3}.$$

Considering the function $h : [1,+\infty) \to \mathbb{R}$, defined by

$$h(x) = \frac{1}{x+1} - \frac{1}{24\left(x+\frac{3}{2}\right)^2} + \frac{1}{24\left(x+\frac{1}{2}\right)^2} - \ln\frac{960\left(x+\frac{3}{2}\right)^4 - 7}{\left(x+\frac{3}{2}\right)^3} + \ln\frac{960\left(x+\frac{1}{2}\right)^4 - 7}{\left(x+\frac{1}{2}\right)^3},$$

and differentiating it, we obtain that

$$\begin{aligned}
h'(x) &= -\frac{1}{(x+1)^2} + \frac{1}{12\left(x+\frac{3}{2}\right)^3} - \frac{1}{12\left(x+\frac{1}{2}\right)^3} \\
&\quad - \frac{3840\left(x+\frac{3}{2}\right)^3}{960\left(x+\frac{3}{2}\right)^4 - 7} + \frac{3}{x+\frac{3}{2}} + \frac{3840\left(x+\frac{1}{2}\right)^3}{960\left(x+\frac{1}{2}\right)^4 - 7} - \frac{3}{x+\frac{1}{2}} \\
&= (28569600x^8 + 228556800x^7 + 783330048x^6 + 1500185088x^5 + 1754428416x^4 \\
&\quad + 1282873344x^3 + 573399572x^2 + 143606824x + 15502847) \\
&\quad /[3(x+1)^2(2x+1)^3(2x+3)^3(960x^4 + 1920x^3 + 1440x^2 + 480x + 53) \\
&\quad \times(960x^4 + 5760x^3 + 12960x^2 + 12960x + 4853)] > 0,
\end{aligned}$$

for every $x \in [1,+\infty)$. It follows that the function h is strictly increasing on $[1,+\infty)$. Also, one can observe that $\lim_{x\to\infty} h(x) = 0$. These imply that $h(x) < 0$, for every $x \in [1,+\infty)$. Therefore $\alpha_{n+1} - \alpha_n < 0$, for every $n \in \mathbb{N}$, i.e. the sequence $(\alpha_n)_{n\in\mathbb{N}}$ is strictly decreasing. Because $\lim_{n\to\infty} \alpha_n = \gamma$, we conclude that $\gamma < \alpha_{n+1} < \alpha_n$, for every $n \in \mathbb{N}$. $\square$

Now we give our first main result of this section.

Theorem 3.2. *Let* $c = \sqrt[6]{\frac{31}{8064\left(\frac{53}{54} - \ln\frac{4853}{3240} - \gamma\right)}}$. *We have*

$$\frac{31}{8064(n+c-1)^6} \leq \alpha_n - \gamma < \frac{31}{8064\left(n+\frac{1}{2}\right)^6},$$

for every $n \in \mathbb{N}$. *Moreover, the constants* $c-1$ *and* $\frac{1}{2}$ *are the best possible with this property.*

Proof. Note that h is the function from the proof of Proposition 3.1. Let $(u_n)_{n\in\mathbb{N}}$ be the sequence defined by

$$u_n = \alpha_n - \frac{31}{8064(n+c-1)^6}.$$

We have

$$u_{n+1} - u_n = \alpha_{n+1} - \alpha_n - \frac{31}{8064(n+c)^6} + \frac{31}{8064(n+c-1)^6}.$$

We consider the function $f : [1, +\infty) \to \mathbb{R}$ defined by

$$f(x) = h(x) - \frac{31}{8064(x+c)^6} + \frac{31}{8064(x+c-1)^6}.$$

Differentiating, we get that

$$\begin{aligned} f'(x) &= h'(x) + \frac{31}{1344(x+c)^7} - \frac{31}{1344(x+c-1)^7} \\ &= [(x-2)\sum_{k=0}^{20} a_k x^k + a]/[1344(x+1)^2(2x+1)^3(2x+3)^3 \\ &\quad \times (960x^4 + 1920x^3 + 1440x^2 + 480x + 53) \\ &\quad \times (960x^4 + 5760x^3 + 12960x^2 + 12960x + 4853)(x+c)^7(x+c-1)^7]. \end{aligned}$$

One can verify that $a_i > 0$, $i \in \{0, 1, \ldots, 20\}$ and $a > 0$. It follows that $f'(x) > 0$, for every $x \in [2, +\infty)$. Hence, the function f is strictly increasing on $[2, +\infty)$. Also, one can see that $\lim_{x\to\infty} f(x) = 0$. From these we obtain that $f(x) < 0$, for every $x \in [2, +\infty)$. So, $u_{n+1} - u_n < 0$, for every $n \geq 2$, i.e. the sequence $(u_n)_{n\geq 2}$ is strictly decreasing. Having in view that $\lim_{n\to\infty} u_n = \gamma$, we are able to write that $\gamma < u_n$, for every $n \geq 2$. Consequently,

$$\frac{31}{8064(n+c-1)^6} \leq \alpha_n - \gamma,$$

for every $n \in \mathbb{N}$, and the constant $c-1$ is the best possible with this property (the equality holds only when $n = 1$).

Let $(v_n)_{n\in\mathbb{N}}$ be the sequence defined by

$$v_n = \alpha_n - \frac{31}{8064\left(n+\frac{1}{2}\right)^6}.$$

Then

$$\nu_{n+1} - \nu_n = \alpha_{n+1} - \alpha_n - \frac{31}{8064\left(n+\frac{3}{2}\right)^6} + \frac{31}{8064\left(n+\frac{1}{2}\right)^6}.$$

Differentiating the function $g : [1, +\infty) \to \mathbb{R}$, defined by

$$g(x) = h(x) - \frac{31}{8064\left(x+\frac{3}{2}\right)^6} + \frac{31}{8064\left(x+\frac{1}{2}\right)^6},$$

we obtain that

$$\begin{aligned} g'(x) &= h'(x) + \frac{31}{1344\left(x+\frac{3}{2}\right)^7} - \frac{31}{1344\left(x+\frac{1}{2}\right)^7} \\ &= -(93776707584x^{14} + 1312873906176x^{13} + 8441879298048x^{12} \\ &\quad +33033108455424x^{11} + 87842644390912x^{10} + 167855050098688x^9 \\ &\quad +237559279782912x^8 + 252802412814336x^7 + 203105932312256x^6 \\ &\quad +122459215673472x^5 + 54452798252624x^4 + 17269355301696x^3 \\ &\quad +3675508601216x^2 + 465873090688x + 26069935939) \\ &\quad /[21(x+1)^2(2x+1)^7(2x+3)^7(960x^4 + 1920x^3 + 1440x^2 + 480x + 53) \\ &\quad \times(960x^4 + 5760x^3 + 12960x^2 + 12960x + 4853)]. \end{aligned}$$

Thus $g'(x) < 0$, for every $x \in [1, +\infty)$. Hereby, the function g is strictly decreasing on $[1, +\infty)$. Clearly, $\lim_{x\to\infty} g(x) = 0$. These yield $g(x) > 0$, for every $x \in [1, +\infty)$. Then $\nu_{n+1} - \nu_n > 0$, for every $n \in \mathbb{N}$, which means that the sequence $(\nu_n)_{n\in\mathbb{N}}$ is strictly increasing. Since $\lim_{n\to\infty} \nu_n = \gamma$, it follows that $\nu_n < \gamma$, for every $n \in \mathbb{N}$. We can therefore write that

$$\alpha_n - \gamma < \frac{31}{8064\left(n+\frac{1}{2}\right)^6}, \tag{4}$$

for every $n \in \mathbb{N}$. It remains to prove that the constant $\frac{1}{2}$ is the best possible with the property that the above inequality (4) holds for every $n \in \mathbb{N}$, and this can be achieved as follows. We have just proved that

$$\sqrt[6]{\frac{31}{8064(\alpha_n - \gamma)}} - n > \frac{1}{2}, \quad n \in \mathbb{N}. \tag{5}$$

Using (1) and (2), we get that

$$\begin{aligned}
\alpha_n - \gamma &= H_n - \frac{1}{24\left(n+\frac{1}{2}\right)^2} - \ln\left(n + \frac{1}{2} - \frac{7}{960\left(n+\frac{1}{2}\right)^3}\right) - \gamma \\
&= \psi(n+1) - \frac{1}{24\left(n+\frac{1}{2}\right)^2} - \ln\left(n + \frac{1}{2} - \frac{7}{960\left(n+\frac{1}{2}\right)^3}\right) \\
&= \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{252n^6} + \frac{1}{240n^8} + O\left(\frac{1}{n^{10}}\right) \\
&\quad - \frac{1}{24n^2\left(1+\frac{1}{2n}\right)^2} - \ln\left(1 + \frac{1}{2n} - \frac{7}{960n^4\left(1+\frac{1}{2n}\right)^3}\right) \\
&= \frac{31}{8064n^6} - \frac{31}{2688n^7} + O\left(\frac{1}{n^8}\right).
\end{aligned} \tag{6}$$

Let $A_n = \sqrt[6]{\frac{31}{8064n^6(\alpha_n-\gamma)}}$, $n \in \mathbb{N}$. Clearly, $\lim_{n\to\infty} A_n = 1$, having in view (3). Then, based on (6), we have

$$\begin{aligned}
\sqrt[6]{\frac{31}{8064(\alpha_n-\gamma)}} - n = n(A_n - 1) &= \frac{n}{\sum_{k=0}^{5} A_n^k}\left(\frac{1}{\frac{8064}{31}n^6(\alpha_n-\gamma)} - 1\right) \\
&= \frac{n}{\sum_{k=0}^{5} A_n^k}\left(\frac{1}{1 - \frac{3}{n} + O\left(\frac{1}{n^2}\right)} - 1\right) \\
&= \frac{1}{\sum_{k=0}^{5} A_n^k} \cdot \frac{3 + O\left(\frac{1}{n}\right)}{1 - \frac{3}{n} + O\left(\frac{1}{n^2}\right)} \to \frac{1}{6}\cdot 3 = \frac{1}{2} \quad (n\to\infty).
\end{aligned} \tag{7}$$

Indeed, from (5) and (7) we obtain that $\frac{1}{2}$ is the best constant with the property that inequality (4) holds for every $n \in \mathbb{N}$, and now the proof is complete. □

Let $(\tilde{\alpha}_n)_{n\in\mathbb{N}}$ be the sequence defined by $\tilde{\alpha}_n = \alpha_{n,3}(1)$. In part (i) of Theorem 2.2 we have proved that

$$\lim_{n\to\infty} n^8(\gamma - \tilde{\alpha}_n) = \frac{127}{30720}. \tag{8}$$

Proposition 3.3. *We have $\tilde{\alpha}_n < \tilde{\alpha}_{n+1} < \gamma$, for every $n \in \mathbb{N}$.*

Proof. We have

$$\begin{aligned}
\tilde{\alpha}_{n+1} - \tilde{\alpha}_n &= \frac{1}{n+1} - \frac{1}{24\left(n+\frac{3}{2}\right)^2} + \frac{1}{24\left(n+\frac{1}{2}\right)^2} + \frac{7}{960\left(n+\frac{3}{2}\right)^4} - \frac{7}{960\left(n+\frac{1}{2}\right)^4} \\
&\quad - \ln\frac{8064\left(n+\frac{3}{2}\right)^6 + 31}{\left(n+\frac{3}{2}\right)^5} + \ln\frac{8064\left(n+\frac{1}{2}\right)^6 + 31}{\left(n+\frac{1}{2}\right)^5}.
\end{aligned}$$

Considering the function $\tilde{h}:[1,+\infty)\to\mathbb{R}$, defined by

$$\begin{aligned}\tilde{h}(x) &= \frac{1}{x+1}-\frac{1}{24\left(x+\frac{3}{2}\right)^2}+\frac{1}{24\left(x+\frac{1}{2}\right)^2}+\frac{7}{960\left(x+\frac{3}{2}\right)^4}-\frac{7}{960\left(x+\frac{1}{2}\right)^4}\\ &\quad -\ln\frac{8064\left(x+\frac{3}{2}\right)^6+31}{\left(x+\frac{3}{2}\right)^5}+\ln\frac{8064\left(x+\frac{1}{2}\right)^6+31}{\left(x+\frac{1}{2}\right)^5},\end{aligned}$$

and differentiating it, we obtain that

$$\begin{aligned}\tilde{h}'(x) &= -\frac{1}{(x+1)^2}+\frac{1}{12\left(x+\frac{3}{2}\right)^3}-\frac{1}{12\left(x+\frac{1}{2}\right)^3}-\frac{7}{240\left(x+\frac{3}{2}\right)^5}+\frac{7}{240\left(x+\frac{1}{2}\right)^5}\\ &\quad -\frac{48384\left(x+\frac{3}{2}\right)^5}{8064\left(x+\frac{3}{2}\right)^6+31}+\frac{5}{x+\frac{3}{2}}+\frac{48384\left(x+\frac{1}{2}\right)^5}{8064\left(x+\frac{1}{2}\right)^6+31}-\frac{5}{x+\frac{1}{2}}\\ &= -(297308454912x^{14}+4162318368768x^{13}+26769400971264x^{12}\\ &\quad +104792256479232x^{11}+278852137150464x^{10}+533369371889664x^{9}\\ &\quad +755912773435392x^{8}+806006485057536x^{7}+649383667564032x^{6}\\ &\quad +393129697342464x^{5}+175861357984144x^{4}+56282483209792x^{3}\\ &\quad +12150419739472x^{2}+1576326994464x+91873672505)\\ &\quad /[15(x+1)^2(2x+1)^5(2x+3)^5\\ &\quad \times(8064x^6+24192x^5+30240x^4+20160x^3+7560x^2+1512x+157)\\ &\quad \times(8064x^6+72576x^5+272160x^4+544320x^3\\ &\qquad +612360x^2+367416x+91885)]<0,\end{aligned}$$

for every $x\in[1,+\infty)$. It follows that the function $\tilde{h}$ is strictly decreasing on $[1,+\infty)$. Also, one can observe that $\lim_{x\to\infty}\tilde{h}(x)=0$. These imply that $\tilde{h}(x)>0$, for every $x\in[1,+\infty)$. Therefore $\tilde{\alpha}_{n+1}-\tilde{\alpha}_n>0$, for every $n\in\mathbb{N}$, i.e. the sequence $(\tilde{\alpha}_n)_{n\in\mathbb{N}}$ is strictly increasing. Because $\lim_{n\to\infty}\tilde{\alpha}_n=\gamma$, we conclude that $\tilde{\alpha}_n<\tilde{\alpha}_{n+1}<\gamma$, for every $n\in\mathbb{N}$. $\square$

Now we give our second main result of this section.

Theorem 3.4. *Let* $\tilde{c}=\sqrt[8]{\frac{127}{30720\left(\gamma-\frac{4777}{4860}-\ln\frac{91885}{61236}\right)}}$. *We have*

$$\frac{127}{30720(n+\tilde{c}-1)^8}\le\gamma-\tilde{\alpha}_n<\frac{127}{30720\left(n+\frac{1}{2}\right)^8},$$

for every $n\in\mathbb{N}$. *Moreover, the constants* $\tilde{c}-1$ *and* $\frac{1}{2}$ *are the best possible with this property.*

Proof. Note that $\tilde{h}$ is the function from the proof of Proposition 3.3. Let $(\tilde{u}_n)_{n\in\mathbb{N}}$ be the sequence defined by

$$\tilde{u}_n=\tilde{\alpha}_n+\frac{127}{30720(n+\tilde{c}-1)^8}.$$

We have

$$\tilde{u}_{n+1} - \tilde{u}_n = \tilde{\alpha}_{n+1} - \tilde{\alpha}_n + \frac{127}{30720(n+\tilde{c})^8} - \frac{127}{30720(n+\tilde{c}-1)^8}.$$

We consider the function $\tilde{f} : [1, +\infty) \to \mathbb{R}$ defined by

$$\tilde{f}(x) = \tilde{h}(x) + \frac{127}{30720(x+\tilde{c})^8} - \frac{127}{30720(x+\tilde{c}-1)^8}.$$

Differentiating, we get that

$$\begin{aligned}
\tilde{f}'(x) &= \tilde{h}'(x) - \frac{127}{3840(x+\tilde{c})^9} + \frac{127}{3840(x+\tilde{c}-1)^9} \\
&= -[(x-2)\sum_{k=0}^{30} \tilde{a}_k x^k + \tilde{a}]/[3840(x+1)^2(2x+1)^5(2x+3)^5 \\
&\quad \times (8064x^6 + 24192x^5 + 30240x^4 + 20160x^3 + 7560x^2 + 1512x + 157) \\
&\quad \times (8064x^6 + 72576x^5 + 272160x^4 + 544320x^3 + 612360x^2 \\
&\quad\ + 367416x + 91885)(x+\tilde{c})^9(x+\tilde{c}-1)^9].
\end{aligned}$$

One can verify that $\tilde{a}_i > 0$, $i \in \{0, 1, \ldots, 30\}$ and $\tilde{a} > 0$. It follows that $\tilde{f}'(x) < 0$, for every $x \in [2, +\infty)$. Hence, the function $\tilde{f}$ is strictly decreasing on $[2, +\infty)$. Also, one can see that $\lim_{x\to\infty} \tilde{f}(x) = 0$. From these we obtain that $\tilde{f}(x) > 0$, for every $x \in [2, +\infty)$. So, $\tilde{u}_{n+1} - \tilde{u}_n > 0$, for every $n \geq 2$, i.e. the sequence $(\tilde{u}_n)_{n\geq 2}$ is strictly increasing. Having in view that $\lim_{n\to\infty} \tilde{u}_n = \gamma$, we are able to write that $\tilde{u}_n < \gamma$, for every $n \geq 2$. Consequently,

$$\frac{127}{30720(n+\tilde{c}-1)^8} \leq \gamma - \tilde{\alpha}_n,$$

for every $n \in \mathbb{N}$, and the constant $\tilde{c} - 1$ is the best possible with this property (the equality holds only when $n = 1$).

Let $(\tilde{v}_n)_{n\in\mathbb{N}}$ be the sequence defined by

$$\tilde{v}_n = \tilde{\alpha}_n + \frac{127}{30720\left(n+\frac{1}{2}\right)^8}.$$

Then

$$\tilde{v}_{n+1} - \tilde{v}_n = \tilde{\alpha}_{n+1} - \tilde{\alpha}_n + \frac{127}{30720\left(n+\frac{3}{2}\right)^8} - \frac{127}{30720\left(n+\frac{1}{2}\right)^8}.$$

Differentiating the function $\tilde{g} : [1, +\infty) \to \mathbb{R}$, defined by

$$\tilde{g}(x) = \tilde{h}(x) + \frac{127}{30720\left(x+\frac{3}{2}\right)^8} - \frac{127}{30720\left(x+\frac{1}{2}\right)^8},$$

we obtain that

$$\begin{aligned}
\tilde{g}'(x) &= \tilde{h}'(x) - \frac{127}{3840\left(x+\frac{3}{2}\right)^9} + \frac{127}{3840\left(x+\frac{1}{2}\right)^9} \\
&= (212667885158400x^{20} + 4253357703168000x^{19} + 40151062789226496x^{18} \\
&\quad +237836352044924928x^{17} + 991344446628691968x^{16} + 3090222937974767616x^{15} \\
&\quad +7473501796931665920x^{14} + 14356208148056506368x^{13} + 22242021354079121408x^{12} \\
&\quad +28060052688951263232x^{11} + 28976739296839394304x^{10} + 24531604126817085440x^9 \\
&\quad +16993882015446322432x^8 + 9580270969643116544x^7 + 4353524933287391360x^6 \\
&\quad +1571495149494432512x^5 + 440878222795013392x^4 + 93004870693412928x^3 \\
&\quad +13980891645509980x^2 + 1352763769145912x + 64676820697555) \\
&\quad /[15(x+1)^2(2x+1)^9(2x+3)^9 \\
&\quad \times(8064x^6 + 24192x^5 + 30240x^4 + 20160x^3 + 7560x^2 + 1512x + 157) \\
&\quad \times(8064x^6 + 72576x^5 + 272160x^4 + 544320x^3 + 612360x^2 + 367416x + 91885)].
\end{aligned}$$

Thus $\tilde{g}'(x) > 0$, for every $x \in [1, +\infty)$. Hereby, the function $\tilde{g}$ is strictly increasing on $[1, +\infty)$. Clearly, $\lim_{x\to\infty} \tilde{g}(x) = 0$. These yield $\tilde{g}(x) < 0$, for every $x \in [1, +\infty)$. Then $\tilde{v}_{n+1} - \tilde{v}_n < 0$, for every $n \in \mathbb{N}$, which means that the sequence $(\tilde{v}_n)_{n\in\mathbb{N}}$ is strictly decreasing. Since $\lim_{n\to\infty} \tilde{v}_n = \gamma$, it follows that $\gamma < \tilde{v}_n$, for every $n \in \mathbb{N}$. We can therefore write that

$$\gamma - \tilde{\alpha}_n < \frac{127}{30720\left(n+\frac{1}{2}\right)^8}, \tag{9}$$

for every $n \in \mathbb{N}$. It remains to prove that the constant $\frac{1}{2}$ is the best possible with the property that the above inequality (9) holds for every $n \in \mathbb{N}$, and this can be achieved as follows. We have just proved that

$$\sqrt[8]{\frac{127}{30720(\gamma - \tilde{\alpha}_n)}} - n > \frac{1}{2}, \quad n \in \mathbb{N}. \tag{10}$$

Using (1) and (2), we get that

$$\begin{aligned}
\gamma - \tilde{\alpha}_n &= \gamma - H_n + \frac{1}{24\left(n+\frac{1}{2}\right)^2} - \frac{7}{960\left(n+\frac{1}{2}\right)^4} + \ln\left(n + \frac{1}{2} + \frac{31}{8064\left(n+\frac{1}{2}\right)^5}\right) \\
&= -\psi(n+1) + \frac{1}{24\left(n+\frac{1}{2}\right)^2} - \frac{7}{960\left(n+\frac{1}{2}\right)^4} + \ln\left(n + \frac{1}{2} + \frac{31}{8064\left(n+\frac{1}{2}\right)^5}\right) \\
&= -\frac{1}{2n} + \frac{1}{12n^2} - \frac{1}{120n^4} + \frac{1}{252n^6} - \frac{1}{240n^8} + \frac{1}{132n^{10}} + O\left(\frac{1}{n^{12}}\right) \\
&\quad + \frac{1}{24n^2\left(1+\frac{1}{2n}\right)^2} - \frac{7}{960n^4\left(1+\frac{1}{2n}\right)^4} + \ln\left(1 + \frac{1}{2n} + \frac{31}{8064n^6\left(1+\frac{1}{2n}\right)^5}\right) \\
&= \frac{127}{30720n^8} - \frac{127}{7680n^9} + O\left(\frac{1}{n^{10}}\right).
\end{aligned} \tag{11}$$

Let $\tilde{A}_n = \sqrt[8]{\frac{127}{30720n^8(\gamma-\tilde{\alpha}_n)}}$, $n \in \mathbb{N}$. Clearly, $\lim_{n\to\infty} \tilde{A}_n = 1$, having in view (8). Then, based on (11), we have

$$\sqrt[8]{\frac{127}{30720(\gamma-\tilde{\alpha}_n)}} - n = n(\tilde{A}_n - 1) = \frac{n}{\sum_{k=0}^{7} \tilde{A}_n^k}\left(\frac{1}{\frac{30720}{127}n^8(\gamma-\tilde{\alpha}_n)} - 1\right)$$
$$= \frac{n}{\sum_{k=0}^{7} \tilde{A}_n^k}\left(\frac{1}{1-\frac{4}{n}+O\left(\frac{1}{n^2}\right)} - 1\right)$$
$$= \frac{1}{\sum_{k=0}^{7} \tilde{A}_n^k} \cdot \frac{4+O\left(\frac{1}{n}\right)}{1-\frac{4}{n}+O\left(\frac{1}{n^2}\right)} \to \frac{1}{8}\cdot 4 = \frac{1}{2} \quad (n\to\infty). \tag{12}$$

Combining (10) and (12) we obtain that $\frac{1}{2}$ is the best constant with the property that inequality (9) holds for every $n \in \mathbb{N}$, and now the proof is complete. □

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