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Nonnegative solutions of quasilinear elliptic problems with sublinear indefinite nonlinearity[1]

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ABSTRACT

We study the existence, nonexistence and multiplicity of nonnegative solutions for the quasilinear elliptic problem

$$\begin{cases} -\triangle_p u = a(x)u^q + \lambda b(x)u^r, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^N$, $\lambda > 0$ is a parameter, $\triangle_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the $p-$Laplace operator of u, $1 < p < N, 0 < q < p-1 < r \le p^* - 1, a(x), b(x)$ are bounded functions, the coefficient $b(x)$ is assumed to be nonnegative and $a(x)$ is allowed to change sign. The results of the semilinear equations are extended to the quasilinear problem.

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RESUMEN

Estudiamos la existencia, no existencia y multiplicidad de soluciones no negativas del problema elíptico cuasi-lineal

$$\begin{cases} -\triangle_p u = a(x)u^q + \lambda b(x)u^r, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

donde Ω es un dominio acotado en $\mathbf{R}^N$, $\lambda > 0$ es un parámetro, $\triangle_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ es el operador p−Laplaciano de u , $1 < p < N, 0 < q < p-1 < r \leq p^* - 1, a(x), b(x)$ son funciones acotadas, el coeficiente $b(x)$ se supone que es no negativo y $a(x)$ se le permite cambiar de signo. Los resultados de las ecuaciones semilineales se extienden a el problema cuasi-lineal.

Keywords and Phrases: Nonnegative solutions; quasilinear elliptic problems; sublinear indefinite nonlinearity; Existence and nonexistence.

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1 Introduction

Let us consider the problem

$$\begin{cases} -\triangle_p u = a(x)u^q + \lambda b(x)u^r, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{P_λ}$$

where $\Omega \subset \mathbf{R}^N$ is a smooth bounded domain, $\lambda > 0$, $1 < p < N, 0 < q < p-1 < r \leq p^* - 1$, $p^* = \frac{Np}{N-p}$, $b(x) \geq 0$, $a(x)$ change its sign, $\triangle_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the $p-$Laplace operator of u. Equations of the above form are mathematical models occuring in studies of the p-Laplace equation, generalized reaction-diffusion theory([7]), non-Newtonian fluid theory, and the turbulent flow of a gas in porous medium([8]). In the non-Newtonian fluid theory, the quantity p is characteristic of the medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids.

Recently, A.V.Lair and A.Mohammed in [11] considered the existence and nonexistence of positive entire large solutions of the semilinear elliptic equation

$$\triangle u = p(x)u^\alpha + q(x)u^\beta, \quad 0 < \alpha \leq \beta.$$

Francisco in [1] considered a sublinear indefinite nonlinearity problem of the form

$$\begin{cases} -\triangle u = a(x)u^q + \lambda b(x)u^p, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

where Ω is a smooth bounded domain in $\mathbf{R}^N$, $\lambda \in R$, $0 < q < 1 < p < r \leq 2^* - 1$, $b(x) \geq 0$, $a(x)$ change its sign. For more results we refer the reader to the works [12-15] and the references therein.

In recent years, the existence and uniqueness of the positive solutions for the single quasilinear elliptic equation with eigenvalue problems

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda f(u) = 0 & \text{in } \Omega, \\ u(x) = 0 \quad \partial\Omega, \end{cases} \tag{1.1}$$

with $\lambda > 0, p > 1, \Omega \subset \mathbf{R}^\mathbf{N}, N \geq 2$ have been studied by many authors, see [16-23] and the references therein. When f is strictly increasing on $\mathbf{R}^+, f(0) = 0, \lim_{s\to 0^+} f(s)/s^{p-1} = 0$ and $f(s) \leq \alpha_1 + \alpha_2 s^\mu, 0 < \mu < p-1, \alpha_1, \alpha_2 > 0$, it was shown in [16] that there exist at least two positive solutions for Eqs (1.1) when λ is sufficiently large. If $\lim_{s\to 0^+} \inf f(s)/s^{p-1} > 0, f(0) = 0$ and the monotonicity hypothesis $(f(s)/s^{p-1})' < 0$ holds for all $s > 0$. It was also shown in [17] that problem (1.1) has a unique positive large solution and at least one positive small solution when λ is large if f is nondecreasing; there exist $\alpha_1, \alpha_2 > 0$ such that $f(s) \leq \alpha_1 + \alpha_2 s^\beta, 0 < \beta < p-1; \lim_{s\to 0^+} \frac{f(s)}{s^{p-1}} = 0$, and there exist $T, Y > 0$ with $Y \geq T$ such that

$$(f(s)/s^{p-1})' > 0 \ \text{ for } \ s \in (0, T)$$

and

$$(f(s)/s^{p-1})' < 0 \text{ for } s > Y.$$

Yang and Xu in [10] established the existence for quasilinear elliptic equation

$$\begin{cases} -\triangle_p u = a(x)(u^m + \lambda u^n), \quad x \in \mathbf{R}^N \\ u > 0, \quad x \in \mathbf{R}^N \\ u \to 0, \quad |x| \to \infty \end{cases} \tag{1.2}$$

where $0 < m < p-1 < n$, they proved there exists a $\lambda^* > 0$ such that (1.2) has a positive solution for $0 < \lambda < \lambda^*$.

The quasilinear elliptic equations when $a(x) \equiv b(x) \equiv 1$ was considered in [2], although here under some restrictions on the p, q in the critical case $r = p^* - 1$. Problems of local "superlinearrity" and "sublinearity" for the $p-$ Laplace problem was considered in [3]. A class of quasilinear elliptic equations are study in [4]. For more results we refer the reader to the works [5-6] and the references therein.

Motivated by the results of the above papers. In this paper, we consider the quasilinear elliptic equations (P_λ). We modify the method developed Francisco Odair de Paiva in [1] and extend the results a quasilinear elliptic equation (P_λ), and complement results in [2-4, 10].

The paper is organized as follows. In section 2, we recall some facts that will be needed in the paper, and give the main results. In section 3, we give the proofs of the main results in this paper.

2 Main results and Preliminary

Let us first consider the following parameterized elliptic problems

$$\begin{cases} -\triangle_p u = a(x)u^q + \lambda b(x)u^r, & \text{in } \Omega \\ u \geq 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{Q_λ}$$

where Ω is a bounded domain in $\mathbf{R}^N$, $\lambda > 0$ is a parameter, $1 < p < N, 0 < q < p-1 < r \leq p^* - 1, a(x), b(x)$ are bounded functions, the coefficient $b(x)$ is assumed to be nonnegative and $a(x)$ is allowed to change sign. Because that $a(x)$ changes sign in Ω, so the Maximum principal is not applicable. Then, define

$$F_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{1}{q+1}\int_\Omega a(x)(u^+)^{q+1} - \frac{\lambda}{r+1}\int_\Omega b(x)(u^+)^{r+1}, u \in W_0^{1,p}(\Omega)$$

We know that $F_\lambda(u)$ is well define in $W_0^{1,p}(\Omega)$ and is of $C_0^1(\overline{\Omega})$

Definition 2.1. We call $u \in W_0^{1,p}(\Omega)$ is a weak solution of (Q_λ), if u is a critical points of $F_\lambda(u)$.

Throughout this paper, we always suppose that

(H_1) There exist $\lambda > 0$,a smooth subdomain $B_1 \in \Omega_a^+, m(x) \in L^\infty(B_1)$ with $m(x) \geq 0, m(x) \not\equiv 0, \mu > \lambda_1(B_1, m(x))$ such that

$$a(x)s^q + \lambda b(x)s^r \geq \mu m(x)s^{p-1}$$

for a.e.$x \in B_1$ and all $s \geq 0$; here $\lambda_1(B_1, m(x))$ denotes the principal eigenvalue of $-\triangle_p$ on $W_0^{1,p}(B_1)$ for the weight $m(x)$.

(H_2) For any $\lambda > 0$, there exists a smooth subdomain $B_2 \subset \Omega_a^+$,$s_1 > 0$ and $\theta_1 > \lambda_1(B_2)$, such that

$$a(x)s^q + \lambda b(x)s^r \geq \theta_1 s^{p-1}$$

for a.e. $x \in B_2$, and all $s \in [0, s_1]$; here $\lambda_1(B_2)$ denotes the principal eigenvalue of $-\triangle_p$ on $W_0^{1,p}(\Omega)$

(F_1) $a(x), b(x) \in L^\infty(\Omega)$, and

$$\Omega_a = \{x \in \Omega : a(x) \geq 0\}, \Omega_a^+ = \{x \in \Omega : a(x) > 0\}$$

$$\Omega_a^- = \{x \in \Omega : a(x) < 0\}, \Omega_b^+ = \{x \in \Omega : b(x) > 0\}$$

are nonempty;

(F_2) Ω_a^+ is open, $|\Omega_a^-| > 0$ and $\overline{\Omega_a^+} \bigcap \overline{\Omega_a^-} = \emptyset$;

(F_3) $\mathrm{int}(\Omega_b^+) \neq \emptyset$ and $b \geq 0$;

(F_4) $\Omega_a^+ \subset \Omega_b^+$ and $\overline{\Omega_a^+} \subset \Omega$;

(F_5) $\mathrm{int}(\Omega_a) = \bigcup_1^k U_i, U_i$ connected, and $U_i \bigcap \Omega_a^+ \neq \emptyset$.

As a consequence of assumption (F_5), by the Maximum principle, if u is a solution of (Q_λ) such that u is nontrivial in the components of Ω_a, then $u > 0$ in $\mathrm{int}(\Omega_a) \supset \Omega_a^+$.

Definition 2.2. If u is a weak solution of (Q_λ) and $u(x) > 0, a.e. x \in \Omega_a^+$, then $u \in W_0^{1,p}(\Omega)$ is a solution of (1.1).

Let

$$\lambda^* = \sup\{\lambda > 0; \ (1.1) \ \text{ has a solution}\}.$$

By a modification of the method given in [1], we obtain the following main results.

Theorem 2.1. Let $0 < q < p - 1 < r \leq p^* - 1$. Assume that $(F_1) - (F_5)$ hold, then there exists $\lambda^* \in (0, \infty)$ such that

(1) for all $\lambda \in (0, \lambda^*)$, problem (P_λ) has at least one weak solutions;

(2) for $\lambda = \lambda^*$, problem (P_λ) has at least one solution;

(3) for all $\lambda > \lambda^*$, problem (P_λ) has no solution.

Theorem 2.2. Let $0 < q < p-1 < r < p^*-1$. Assume that $(F_1)-(F_5)$ hold, then problem (P_λ) has at least two solutions for $0 < \lambda < \lambda^*$.

3 The proof of main results

Lemma 3.1. There is $\lambda_0 > 0$ such that for $0 < \lambda \le \lambda_0$, problem (P_λ) has a solution.

Proof. Let e be the unique positive solution of

$$\begin{cases} -\triangle_p e = 1, & \text{in } \Omega; \\ e = 0, & \text{on } \partial\Omega. \end{cases}$$

Since $0 < q < p-1 < r$, we can find $\lambda_0 > 0$ such that for all $0 < \lambda \le \lambda_0$ there exists $M = M(\lambda) > 0$ satisfying

$$M^{p-1} \ge M^q \|a\|_\infty \|e\|_\infty^q + \lambda M^r \|b\|_\infty \|e\|_\infty^r.$$

As a consequence, the function Me satisfies

$$-\triangle_p(Me) = M^{p-1} \ge M^q \|a\|_\infty \|e\|_\infty^q + \lambda M^r \|b\|_\infty \|e\|_\infty^r.$$

Hence Me is a supersolution of (P_λ). Then let $\overline{u} = Me$, we have that $\overline{u}$ is a supersolution for (Q_λ). Moreover 0 is a solution of (Q_λ), so let $\underline{u} = 0$ is a subsolution for (Q_λ). It follows form the sub-supersolution argument as in [5] or [6] that (Q_λ) has a nonnegative solution in $A = \{u \in W_0^{1,p} : 0 \le u(x) \le Me \text{ a.e.} x \in \Omega\}$. Then let $c = \inf_A F_\lambda$,

$$F_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{1}{q+1}\int_\Omega a(x)(u^+)^{q+1} - \frac{\lambda}{r+1}\int_\Omega b(x)(u^+)^{r+1}, u \in w_0^{1,p}(\Omega),$$

there exist $u_\lambda \in A$ such that $c = \inf_A F_\lambda(u_\lambda)$ and u_λ is a solution of (Q_λ). Also u_λ solves (P_λ) if $u_\lambda > 0$ a.e.$x \in \Omega_a^+$.

By contradiction, suppose that $u_\lambda \equiv 0$ a.e.$x \in \Omega_a^+$, let $\varphi \in C_c^\infty(\Omega_a^+)$ be nonnegative and nontrivial, then for sufficiently small $s > 0$, $u_\lambda + s\varphi \in A$

$$F_\lambda(u_\lambda + s\varphi) = F_\lambda(u_\lambda) + F_\lambda(s\varphi)$$

$$= F_\lambda(u_\lambda) + \frac{s^p}{p}\|\varphi\|^p - \frac{s^{q+1}}{q+1}\int_\Omega a(x)\varphi^{q+1} - \frac{\lambda s^{r+1}}{r+1}\int_\Omega b(x)\varphi^{r+1}$$

Then we have $F_\lambda(u_\lambda + s\varphi) < F_\lambda(u_\lambda)$, if $s > 0$ is small enough, however this contradicts that the infimum $c = \inf F_\lambda$ is achieve at u_λ . So $u_\lambda > 0$ a.e.$x \in \Omega_a^+$ and is a solution of (P_λ).

Lemma 3.2. (P_λ) has a solution for all $\lambda \in (0, \lambda^*)$.

Proof. Given $\lambda < \lambda^*$, let $u_{\overline{\lambda}}$ be a solution of $(P_{\overline{\lambda}})$, with $\lambda < \overline{\lambda} < \lambda^*$. Then

$$-\triangle_p u_{\overline{\lambda}} = a(x)u_{\overline{\lambda}}^q + \overline{\lambda}b(x)u_{\overline{\lambda}}^r \geq a(x)u_{\overline{\lambda}}^q + \lambda b(x)u_{\overline{\lambda}}^r,$$

which $u_{\overline{\lambda}}$ is a supersolution for (P_λ).

Consider $A = \{u \in W_0^{1,p} : 0 \leq u \leq u_{\overline{\lambda}}\}$, there exist $u_\lambda \in A$ such that $F_\lambda(u_\lambda) = \inf_A F_\lambda$, and u_λ is a solution of (Q_λ), as the proof of Lemma 3.1, u_λ is also the solution of (P_λ).

Lemma 3.3. Let $\lambda^* = \sup\{\lambda > 0 : (P_\lambda)$ has a solution$\}$, then $0 < \lambda^* < \infty$.

Proof. Under the assume (H_1), suppose that when $\lambda > 0$, (P_λ) has a solution $u_\lambda \in W_0^{1,p}(\Omega) \bigcap L^\infty(\Omega)$. Consider the eigenvalue problem with weight

$$\begin{cases} -\triangle_p v = \mu m(x)|v|^{p-2}, & \text{in } B_1; \\ v = 0, & \text{on } \partial B_1. \end{cases}$$

Since by (H_1), we have

$$\int_{B_1} |\bigtriangledown u_\lambda|^{p-2} \bigtriangledown u_\lambda \bigtriangledown \varphi = \int_{B_1} (a(x)u_\lambda^q + \lambda b(x)u_\lambda^r)\varphi \geq \mu \int_{B_1} m(x)u_\lambda^{p-1}\varphi$$

for all $\varphi \in C_c^\infty(\Omega), \varphi \geq 0$. This show that u_λ is an supersolution of (E_μ). Furthermore, $\epsilon\varphi_1$ is a subsolution of (E_μ), and $\epsilon\varphi_1 \leq u_\lambda$ for ϵ small enough.

$$\int_{B_1} |\bigtriangledown (\epsilon\varphi_1)|^{p-2} \bigtriangledown (\epsilon\varphi_1) \bigtriangledown \varphi = \lambda_1 \int_{B_1} m(x)(\epsilon\varphi_1)^{p-1}\varphi < \mu \int_{B_1} m(x)(\epsilon\varphi_1)^{p-1}\varphi$$

for $\varphi \in C_c^\infty(\Omega), \varphi \geq 0$; φ_1 is a positive eigenfunction associated to $\lambda_1(B_1, m(x))$. Then (E_λ) has a solution v with $\epsilon\varphi_1 \leq v \leq u_\lambda$,in particular $v \geq 0, v \not\equiv 0$. For above that μ is a principal eigenvalue of $-\triangle_p u$ on B for the weight $m(x)$. This is contradiction with $\mu > \lambda_1(B_1, m(x))$, and consequently $\lambda^* < +\infty$, moreover we can also obtain $\lambda^* > 0$ to the Lemma 4.1, so, $\lambda^* \in (0, \infty)$. Hence, when $\lambda > \lambda^*$, problem (P_λ) has no solution.

Lemma 3.4. For $\lambda = \lambda^*$, problem (P_λ) has at least one solution.

Proof. For the definition of λ^*, let λ_n be a sequence such that $\lambda_n \longrightarrow \lambda^*$ with $0 < \lambda_n < \lambda^*$,$\lambda_n$ increasing, let u_n be a solution of P_{λ_n} with $F_{\lambda_n}(u_n) < 0$ and $F'_{\lambda_n}(u_n) = 0$. We obtain

$$F_{\lambda_n}(u_n) + F'_{\lambda_n}(u_n) \cdot u_n \leq C\|u_n\|,$$

where

$$F_{\lambda_n}(u_n) = \frac{1}{p}\int_\Omega |\nabla u_n|^p - \frac{1}{q+1}\int_\Omega a(x)(u_n^+)^{q+1} - \frac{\lambda_n}{r+1}\int_\Omega b(x)(u_n^+)^{r+1},$$

$$F'_{\lambda_n}(u_n) \cdot u_n = \int_\Omega |\nabla u_n|^p - \int_\Omega a(x)(u_n^+)^{q+1} - \lambda_n \int_\Omega b(x)(u_n^+)^{r+1}$$

so by Theorem 1.2.1 of [9], we have

$$(\frac{1}{p} + 1)\|u_n\|^p \leq C\|u_n\|^{q+1} + c.$$

It shows that u_n is bounded in $W_0^{1,p}$, we have, for a subsequence, $u_n \longrightarrow u^*$ in $C^1(\overline{\Omega})$, hence u^* solves (Q_λ) in Ω. u^* is a solution of (P_λ)) if $u^* \not\equiv 0$ in Ω_a^+. Assume by contradiction $u^* \equiv 0$ in Ω_a^+. Under the assume (H_2), we have

$$\int_{B_2} |\bigtriangledown u_n|^{p-2} \bigtriangledown u_n \bigtriangledown \varphi = \int_{B_2} (a(x)u_n^q + \lambda_n b(x)u^r)\varphi \geq \theta_1 \int_{B_2} u_n^{p-1}\varphi$$

for n sufficiently large(so that $0 \leq u_n(x) \leq s_1$ on B_2,which is possible since $u_n \longrightarrow 0$ uniformly). So that u_n is a supersolution for the problem

$$\begin{cases} -\triangle_p v = \theta_1 |v|^{p-2}v, & \text{in } B_2; \\ v = 0, & \text{on } \partial B_2. \end{cases}$$

Moreover, since $\theta_1 > \lambda_1$, let $u_\varepsilon = \varepsilon\varphi_1$. We have

$$-\triangle_p (u_\varepsilon) = \lambda_1 u_\varepsilon^{p-1} < \theta_1 u_\varepsilon^{p-1}$$

and $\varepsilon\varphi_1 \leq u_n$ on B_2, for ($\varepsilon > 0$ sufficiently small). It shows that the existence of a solution v of (E_{θ_1}) with $\varepsilon\varphi_1 \leq v \leq u_n$. This is a contradiction with $\theta_1 > \lambda_1$ in assume (H_2). So, $u^* \not\equiv 0$ in Ω_a^+ and is a solution of (P_λ).

Proof of Theorem 2.2. From the Lemma 3.2, we have obtained u_λ is a local minimizer of $F_\lambda(u)$ and is a solution of (P_λ). In this section, we hope to find the second solution of the form $v = u_\lambda + u$, by the moutnain pass theorem,where u is a nonnegative solution of

$$\begin{cases} -\triangle_p (u_\lambda + u) = a(x)(u_\lambda + u^+)^q + \lambda b(x)(u_\lambda + u^+)^r, & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

$u \in W_0^{1,p}(\Omega)$, and $u \geq 0$. Then, $u_\lambda + u$ is a second solution of (P_λ). Define the associated functional

$$I_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla(u_\lambda + u)|^p - \int_\Omega H_\lambda(x, u)$$

$$H_\lambda(x, u) = G_\lambda(x, u_\lambda + u^+) - G_\lambda(x, u_\lambda) - g_\lambda(x, u_\lambda)u^+;$$

$$G_\lambda(x, u) = \int_\Omega g_\lambda(x, u)du; \;\; g_\lambda(x, u) = a(x)u^q + \lambda b(x)u^r.$$

Then, it follows that

$$I_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla(u_\lambda + u)|^p - \frac{1}{q+1}\int_\Omega a(x)[(u_\lambda + u^+)^{q+1} - u_\lambda^{q+1} - (q+1)u_\lambda^q u^+]$$

$$-\frac{\lambda}{r+1}\int_\Omega b(x)[(u_\lambda + u^+)^{r+1} - u_\lambda^{r+1} - (r+1)u_\lambda^r u^+]$$

(i) let $u^+ \in W_0^{1,p}(\Omega_a^+)$, and for $\|u^+\|$ sufficiently small, we have

$$I_\lambda(u) \geq \frac{1}{p}\int_\Omega |\nabla(u_\lambda + u)|^p - \frac{1}{p}\int |_\Omega \nabla(u_\lambda + u^+)|^p + \frac{1}{p}\int_\Omega |\nabla u_\lambda|^p + \int_\Omega g_\lambda(x, u_\lambda)u^+$$

then,

$$I_\lambda(u) \geq \frac{1}{p}\int_\Omega |\nabla u_\lambda|^p + \int_\Omega g_\lambda(x,u_\lambda)u^+ \geq \frac{1}{p}\int_\Omega |\nabla u_\lambda|^p = I_\lambda(0)$$

(ii) let $v_1 \in W_0^{1,p}(\Omega_b^+), v_1 \geq 0, v_1 \not\equiv 0$,such that $\int_\Omega b(x)v_1^{r+1} > 0$. We have, for large s

$$\begin{aligned} I_\lambda(sv_1) &= \frac{1}{p}\int_\Omega |\nabla(u_\lambda + sv_1)|^p - \frac{1}{q+1}\int_\Omega a(x)[(u_\lambda+sv_1)^{q+1} - u_\lambda^{q+1} - (q+1)u_\lambda^q sv_1] \\ &\quad -\frac{\lambda}{r+1}\int_\Omega b(x)[(u_\lambda+sv_1)^{r+1} - u_\lambda^{r+1} - (r+1)u_\lambda^r sv_1] \\ &= \frac{s^p}{p}\int_\Omega |\nabla(\frac{u_\lambda}{s} + v_1)|^p - \frac{s^{q+1}}{q+1}\int_\Omega a(x)[(\frac{u_\lambda}{s}+v_1)^{q+1} - (\frac{u_\lambda}{s})^{q+1} - \frac{(q+1)u_\lambda^q v_1}{s^q}] \\ &\quad -\frac{\lambda s^{r+1}}{r+1}\int_\Omega b(x)[(\frac{u_\lambda}{s}+v_1)^{r+1} - (\frac{u_\lambda}{s})^{r+1} - \frac{(r+1)u_\lambda^r v_1}{s^r}] \\ &= O(s^p) - \frac{\lambda s^{r+1}}{r+1}\int_\Omega b(x)v_1^{r+1} \longrightarrow -\infty \end{aligned}$$

as $s \longrightarrow \infty$.

(iii) We now prove $I_\lambda(u)$ satisfies the (PS) condition in $W_0^{1,p}(\Omega)$. Indeed, if u_k is a (PS) sequence, i.e. $I_\lambda(u_k) \longrightarrow c, I_\lambda'(u_k) \longrightarrow 0$. Then, for $p < \theta < r+1$,$\varepsilon_k \longrightarrow 0$, and some constant c, we have,

$$\theta I_\lambda(u_k) - I_\lambda'(u_k)\cdot u_k \leq c + \varepsilon_k\|u_k\|$$

where $\|u_k\|$ denotes the $W_0^{1,p}(\Omega)$ norm $(\int_\Omega |\nabla u|^p)^{\frac{1}{p}}$.

$$(\frac{\theta}{p} - 1)\|u_k\|^p \leq (\frac{\theta}{q+1} - 1)\int_\Omega a(x)u_k^{q+1} + \lambda(\frac{\theta}{r+1} - 1)\int_\Omega b(x)u_k^{r+1} + c + \varepsilon_k\|u_k\|$$

$$(\frac{\theta}{p} - 1)\|u_k\|^p + \lambda(1 - \frac{\theta}{r+1})\int_\Omega b(x)u_k^{r+1} \leq (\frac{\theta}{q+1} - 1)\int_\Omega a(x)u_k^{q+1} + c + \varepsilon_k\|u_k\|$$

By $a(x), b(x)$ is bounded in Ω, we obtain,

$$(\frac{\theta}{p} - 1)\|u_k\|^p + c_2\lambda(1 - \frac{\theta}{r+1})\|u_k\|^{r+1} \leq c_1(\frac{\theta}{q+1} - 1)\|u_k\|^{q+1} + c + \varepsilon_k\|u_k\|$$

since $q+1 < p < r+1$, this implies that the sequence (u_k) be bounded in $W_0^{1,p}(\Omega)$. Thus, from (i)-(iii), I_λ satisfies the assumptions of the mountain pass theorem,i.e.I_λ has a nontrivial critical point. This concludes the proof of Theorem 2.2.

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