CUBO A Mathematical Journal
Vol.15, N$^{\underline{o}}$02, (111–119). June 2013

Existence and uniqueness solution of a class of quasilinear parabolic boundary control problems

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ABSTRACT

This paper presents an optimal control of processes described by a quasilinear parabolic systems with controls in the coefficients of equation, in the boundary condition and in the right side of this equation. Theorems concarning the existence and uniqueness for the solution of the cosidering problem are invistigated.

RESUMEN

Este artículo presenta un control óptimo de procesos descritos por un sistema parabólico cuasilineal con control en los coeficientes de la ecuación, en la condición de frontera y en el lado derecho de esta ecuación. Se investigan los teoremas relacionados con la existencia y unicidad para la solución del problema considerado

Keywords and Phrases: Optimal control, Quasilinear Parabolic Equation, Existence and Uniqueness Theorems.

2010 AMS Mathematics Subject Classification: 49J20, 49K20, 49M29, 49M30

1 Introduction

Optimal control problems for partial differential equations are currently of much interest. A larage amount of the theoretical concept which governed by quasilinear parabolic equations [1-5] has been investigated in the field of optimal control problems. These problems have dealt with the processes of hydro- and gasdynamics, heatphysics, filtration, the physics of plasma and others [6-8]. The study and determination of the optimal regimes of heat conduction processes at a long interval of the change of temperture gives rise to optimal control problems with respect to a quasilinear equation of parabolic type. In this work, we consider a constrained optimal control problem with respect to a quasilinear parabolic equation with controls in the coefficients of the equation. The existence and uniqueness of the optimal control problem is proved.

2 Statement of the problem

Let D is a bounded domain of the N-dimensional Euclidean space E_N; Γ be the boundary of D, assumed to be sufficiently smooth; ν is the exterior unit normal of Γ; $T > 0$ be a fixed time ; $\Omega = D \times (0, T]$; $S = \Gamma \times (0, T]$.

Now we consider a class of optimal control problems governed by the following quasilinear parabolic system.

$$L(v)y(x,t) = f(x,t,v_2), (x,t) \in \Omega,$$

$$y(x,0) = \phi(x), x \in D,$$

$$\textstyle\sum_{i=1}^{n} \lambda_i(y, v_0) \frac{\partial y}{\partial x_i} \cos(\nu, x_i)|_S = g(\zeta, t), (x,t) \in S \tag{1}$$

where $\phi \in L_2(D), g(\zeta, t) \in L_2(S)$ are given functions and the differential operator L takes the following form:

$$L(v)z(x,t) = \frac{\partial z}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}[\lambda_i(z, v_0)\frac{\partial z}{\partial x_i}] + \sum_{i=1}^{n} B_i(z, v_1)\frac{\partial z}{\partial x_i} \tag{2}$$

$y(x,t), v = (v_0, v_1, v_2)$ are the state and the controls rspectively for the system (1).

Furthermore, we consider the functional of the form

$$J_\beta(v) = \int_S [y(\zeta,t) - f_0(\zeta,t)]^2 d\zeta dt + \beta \sum_{m=0}^{2} \|v_m - \omega_m\|_{L_2}^2, \tag{3}$$

which is to minimized under condition (1) and additional restricitions

$$\nu_0 \le \lambda_i(y, v_0) \le \mu_0, \nu_1 \le B_i(y, v_1) \le \mu_1, r_1 \le y(x,t) \le r_2, i = \overline{1,n} \tag{4}$$

over the class

$$V = \{v = (v_0, v_1, v_2) : v_m = (v_{0m}, v_{1m}, \cdots, v_{im}, \cdots) \in l_2, \|v_m\|_{l_2} \leq R_m, m = \overline{0,2}\}$$

and $f_0(\zeta, t) \in L_2(S)$ is a given function and $\beta \geq 0, \nu_j, \mu_j, j = 1, 2, r_1, r_2, R_m > 0$ are positive numbers, $\omega_m = (\omega_{0m}, \omega_{1m}, \cdots, \omega_{im}, \cdots) \in l_2, m = \overline{0,2}$ are given numbers.

Throughout this paper, we adopt the following assumptions.

Assumption 2.1: V is closed and bonded subset of l_2.

Assumption 2.2: The functions $B_i(y, v_1), \lambda_i(y, v_0), i = \overline{1,n}$ are continuous on $(y, v) \in [r_1, r_2] \times l_2$ have continuous derivatives in y at $\forall (y, v) \in [r_1, r_2] \times l_2$ and $\frac{\partial B_i}{\partial y}$, $\frac{\partial \lambda_i}{\partial y}, i = \overline{1,n}$are bounded.

Assumption 2.3: The function $f(x, t, v_2)$ is given function continuous in v_2 on l_2 for almost all $(x, t) \in \Omega$, bounded and measurable in x, t on Ω $\forall v_2 \in l_2$.

Assumption 2.4: The functions $B_i(y, v_1), \lambda_i(y, v_0), i = \overline{1,n}, f(x, t, v_2)$ satisfy a Lipschitz condition for v_1, v_0, v_2 ,then

$$|B_i(y(x, t), v_1 + \delta v_1) - B_i(y(x, t), v_1)| \leq S_0(x, t)\|\delta v_1\|_{l_2}, i = \overline{1,n}$$

$$|\lambda_i(y(x, t), v_0 + \delta v_0) - \lambda_i(y(x, t), v_0)| \leq S_1(x, t)\|\delta v_0\|_{l_2}, i = \overline{1,n}$$

$$|f(x, t, v_2 + \delta v_2) - f(x, t, v_2)| \leq S_2(x, t)\|\delta v_2\|_{l_2}$$

for almost all $(x, t) \in \Omega, \forall y \in [r_1, r_2], \forall v_m, v_m + \delta v_m \in l_2$ such that $\|v_m\|_{l_2}, \|v_m + \delta v_m\|_{l_2} \leq R_m$ where $S_m(x, t) \in L_\infty, m = \overline{0,2}$.

Assumption 2.5: The first derivatives of the functions $B_i(y, v_0), \lambda_i(y, v_0), i = \overline{1,n}$ and $f(x, t, v_2)$ with respect to v are continuous functions in $[r_1, r_2] \times l_2$ and for any $v_m \in l_2$ such that $\|v_m\|_{l_2} \leq R_m, m = \overline{0,2}$.

Definition 2.1: The problem of finding the function $y = y(x, t) \in V_2^{0,1}(\Omega)$ from condition (1)-(2) at given $v \in V$ is called the reduced problem.

Definition 2.2: A function $y = y(x, t) \in V_2^{1,0}(\Omega)$ is said to be a solution of the problem

(1)-(2), if for all $\eta = \eta(x,t) \in W_2^{1,1}(\Omega)$ the equation

$$\int_\Omega [-y\frac{\partial \eta}{\partial t} + \sum_{i=1}^n \lambda_i(y,v_0)\frac{\partial y}{\partial x_i}\frac{\partial \eta}{\partial x_i} - \sum_{i=1}^n B_i(y,v_1)(\frac{\partial y}{\partial x_i})\eta(x,t)$$

$$-f(x,t,v_2)\eta(x,t)]dxdt = \int_D \phi(x)\eta(x,0)dx + \int_S g(\zeta,t)\eta(\zeta,t)d\zeta dt, \tag{5}$$

is valid and $\eta(x,T) = 0$.

It is proved in [8] that, under the foregoing assumptions, a reduced problem (1)-(2) has a unique solution and $|\frac{\partial y}{\partial x_i}| \le C_1, i = \overline{1,n}$ almost at all $(x,t) \in \Omega, \forall v \in V$, where C_1 is a certain constant.

3 The Existence Theorem

Optimal control problems of the coefficients of differential equations do not always have solution [9]. Examples in [10] and elswhere of problems of the type (1)-(4) having no solution for $\beta = 0$. A problem of minimization of a functional is said to be unstable, when a minimizing sequare does not converge to an element minimizing the functional [6].

To begin with, we need

Theorem 3.1 Under the above assumptions for every solution of the reduced problem (1)-(2) the following estimate is valid:

$$\|\delta y\|_{V_2^{1,0}(\Omega)} \le C_2[\|\sqrt{\sum_{i=1}^n (\Delta\lambda_i \frac{\partial y}{\partial x_i})^2}\|_{L_2(\Omega)} + \|\Delta f - \sum_{i=1}^n \Delta B_i \frac{\partial y}{\partial x_i}\|_{L_2(\Omega)}], \tag{6}$$

where $\delta y(x,t) = y(x,t;v+\delta v) - y(x,t;v), \delta y(x,t) \in W_2^{1,1}(\Omega)$, $\Delta\lambda_i = \lambda_i(u,v_0+\delta v_0) - \lambda_i(u,v_0)$,$\Delta B_i = B_i(u,v_1+\delta v_1) - B_i(u,v_1)$, $\Delta f = f(x,t,v_2+\delta v_2) - f(x,t,v_2)$ and $C_2 \ge 0$ is a constant not dependent on $\delta v = (\delta v_0, \delta v_1, \delta v_2), \delta v_m \in l_2, m = \overline{0,2}$.

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Set $\delta y(x,t) = y(x,t,v+\delta v) - y(x,t;v), y = y(x,t;v), \overline{y} = y(x,t;v+\delta v)$. From (5) it follows that

$$\int_\Omega [-\delta y\frac{\partial \eta}{\partial t} + \sum_{i=1}^n \overline{\lambda_i}\frac{\partial \delta y}{\partial x_i}\frac{\partial \eta}{\partial x_i} + \sum_{i=1}^n \frac{\partial \lambda_i(y+\theta_{1i},v_0+\delta v_0)}{\partial y}\frac{\partial y}{\partial x_i}\frac{\partial \eta}{\partial x_i}\delta y$$

$$+\sum_{i=1}^n \Delta\lambda_i\frac{\partial y}{\partial x_i}\frac{\partial \eta}{\partial x_i} + \sum_{i=1}^n \overline{B_i}\frac{\partial \delta y}{\partial x_i}\eta + \sum_{i=1}^n \Delta B_i(\frac{\partial y}{\partial x_i})\eta$$

$$-\sum_{i=1}^n \frac{\partial B_i(y+\theta_{2i},v_1+\delta v_1)}{\partial y}\frac{\partial y}{\partial x_i}\delta y\eta - \Delta f\eta]dxdt = 0 \tag{7}$$

for all $\eta = \eta(x,t) \in W_2^{1,1}(\Omega)$ and $\eta(x,T) = 0$.

Here $\theta_{1i}, \theta_{2i} \in (0,1), i = \overline{1,n}$ is some number, $\overline{\lambda_i} \equiv \lambda_i(y+\delta y, \nu_0 + \delta\nu_0)$,$\Delta\lambda_i \equiv \lambda_i(y, \nu_0 + \delta\nu_0) - \lambda_i(y,\nu_0)$, $\overline{B_i} \equiv B_i(y+\delta y, \nu_1 + \delta\nu_1)$,$\Delta B_i \equiv B_i(y, \nu_1+\delta\nu_1) - \lambda_i(y,\nu_1), i = \overline{1,n}, i = \overline{1,n}$, $\Delta f \equiv f(x,t,\nu_2+\delta\nu_2) - f(x,t,\nu_2)$.

Let $\eta_h(x,t) = \frac{1}{h}\int_{t-h}^{t}\overline{\eta}(x,\tau)d\tau, 0 < h < \tau$ where $\overline{\eta} = \delta y(x,t)$ at $(x,t) \in \Omega_{t_1}$, zero at $t > t_1 (t_1 \le T - h)$ and $\Omega_{t_1} = D \times (0,t_1]$. In identity (5) put $\eta(x,t)$ instead of $\eta_h(x,t)$, and following the method in [11,p. 166-168] we obtain

$$\frac{1}{2}\int_D (\delta y)^2 dx + \int_{\Omega_{t_1}} [\sum_{i=1}^n \overline{\lambda_i}(\frac{\partial \delta y}{\partial x_i})^2 + \sum_{i=1}^n \frac{\partial\lambda_i(y+\theta_{1i}, \nu_0+\delta\nu_0)}{\partial y}\frac{\partial y}{\partial x_i}\frac{\partial\delta y}{\partial x_i}\delta y] dxdt$$

$$+\int_{\Omega_{t_1}}\sum_{i=1}^n \Delta\lambda_i \frac{\partial y}{\partial x_i}\frac{\partial \delta}{\partial x_i} dxdt + \sum_{i=1}^n \frac{\partial B_i(y+\theta_{2i},\nu_1+\delta\nu_1)}{\partial y}\frac{\partial y}{\partial x_i}(\delta y)^2 dxdt$$

$$+\int_{\Omega_{t_1}}\sum_{i=1}^n \overline{B_i}\frac{\partial\delta y}{\partial x_i}\delta y + \int_{\Omega_{t_1}}\sum_{i=1}^n \Delta B_i(\frac{\partial y}{\partial x_i})\delta y dxdt - \int_{\Omega_{t_1}}\Delta f \delta y dxdt = 0 \tag{8}$$

Hence,from the above assumptions and applying Cauchy Bunyakoviskii inequality, we obtain

$$\frac{1}{2}\int_D (\delta y(x,t_1))^2 dx + \nu_0 \int_{\Omega_{t_1}}\sum_{i=1}^n |\frac{\partial\delta y}{\partial x_i}|^2 dxdt$$

$$\le (C_3 + C_4)(\int_{\Omega_{t_1}}\sum_{i=1}^n |\frac{\partial \delta y}{\partial x_i}|^2 dxdt)^{\frac{1}{2}}(\int_{\Omega_{t_1}}(\delta y(x,t))^2 dxdt)^{\frac{1}{2}}$$

$$+\{\int_{\Omega_{t_1}}\sum_{i=1}^n |\Delta\lambda_i \frac{\partial y}{\partial x_i}|^2 dxdt\}^{\frac{1}{2}}(\int_{\Omega_{t_1}}\sum_{i=1}^n |\frac{\partial\delta y}{\partial x_i}|^2 dxdt)^{\frac{1}{2}} + C_5\int_{\Omega_{t_1}}(\delta y(x,t))^2 dxdt$$

$$-\int_0^{t_1}\{\int_D |\Delta f - \sum_{i=1}^n \Delta B_i(\frac{\partial y}{\partial x_i})|^{\frac{1}{2}} dx (\int_D (\delta y)^2 dx)^{\frac{1}{2}}\}dt, \tag{9}$$

where C_3, C_4, C_5 are positive constants not depending on $\delta\nu$.

Applying Cauchy's inequality with ε and combine similar terms, then multiply both sides by two, we obtain

$$\|\delta y(x,t_1)\|^2_{L_2(D)} + \frac{\nu_0}{2}\|\sum_{i=1}^n \frac{\partial\delta y}{\partial x_i}\|^2_{L_2(\Omega_{t_1})} \le C_6\|\delta y(x,t)\|^2_{L_2(\Omega_{t_1})}$$

$$+2\{\int_{\Omega_{t_1}}\sum_{i=1}^n |\Delta\lambda_i \frac{\partial y}{\partial x_i}|^2 dxdt\}^{\frac{1}{2}}\|\sum_{i=1}^n \frac{\partial\delta y}{\partial x_i}\|^2_{L_2(\Omega_{t_1})}$$

$$+2\max_{0\le\tau\le t_1}\|\delta y(x,\tau\|_{L_2(D)}\int_0^{t_1}\{\int_D |\Delta f - \sum_{i=1}^n \Delta B_i(\frac{\partial y}{\partial x_i})|^2 dx\}^{\frac{1}{2}} dt \tag{10}$$

Now we replace

$$y(t_1) = \max_{0\le\tau\le t_1}\|\delta y(x,\tau\|_{L_2(D)}, \|\delta y(x,t)\|^2_{L_2(\Omega_{t_1})} = t_1(y(t_1))^2.$$

This gives us the inequality

$$\|\delta y(x,t_1)\|^2_{L_2(D)} + \frac{\nu_0}{2}\|\sum_{i=1}^n \frac{\partial \delta y}{\partial x_i}\|^2_{L_2(\Omega_{t_1})} \le C_6 t_1 (y(t_1))^2$$

$$+2\{\int_{\Omega_{t_1}} \sum_{i=1}^n |\Delta\lambda_i \frac{\partial y}{\partial x_i}|^2 dxdt\}^{\frac{1}{2}} \| \sum_{i=1}^n \frac{\partial \delta y}{\partial x_i}\|^2_{L_2(\Omega_{t_1})}$$

$$+2y(t_1)\int_0^{t_1}\{\int_D |\Delta f - \sum_{i=1}^n \Delta B_i(\frac{\partial y}{\partial x_i})|^2 dx\}^{\frac{1}{2}} dt \equiv j(t_1). \tag{11}$$

From this follows the two inequalities

$$(y(t_1))^2 \le j(t_1) \tag{12}$$

and

$$\|\sum_{i=1}^n \frac{\partial \delta y}{\partial x_i}\|^2_{L_2(\Omega_{t_1})} \le \frac{2}{\nu_0} j(t_1) \tag{13}$$

We take the square root of both sides of (12) and (13), add together the resulting inequalities and then majorize the right-hand side in the same way in [12] (pp. 117-118) and this proves the estimate (6). This completes the proof of the theorm.

Corollary 3.1 Under the above assumptions, the right part of estimate (6) converges to zero at $\sum_{m=0}^2 \|\delta v_m\|_{l_2} \to 0$, therefore

$$\|\delta y\|_{V_2^{1,0}(\Omega)} \to 0 \quad \text{at} \quad \sum_{m=0}^2 \|\delta v_m\|_{l_2} \to 0. \tag{14}$$

Hence from the theorem on trace [13] we get

$$\|\delta y\|_{L_2(\Omega)} \to 0, \|\delta y\|_{L_2(S)} \to 0 \quad \text{at} \quad \sum_{m=0}^2 \|\delta v_m\|_{l_2} \to 0. \tag{15}$$

Now we consider the functional $J_0(v) = \int_S [y(\zeta,t) - f_0(\zeta,t)]^2 d\zeta dt$.

Theorem 3.2 The functional $J_0(v)$ is continuous on V.

proof

Let $\delta v = (\delta v_0, \delta v_1, \delta v_2), \delta v_m \in l_2, m = \overline{0,2}$ be an increment of control on an element $v \in V$ such that $v + \delta v \in V$. For the increment of $J_0(v)$ we have

$$\Delta J_0(v) = J_0(v+\delta v) - J_0(v) = 2\int_S [y(\zeta,t) - f_0(\zeta,t)]\delta y(\zeta,t) d\zeta dt + \int_S [\delta y(\zeta,t)]^2 d\zeta dt \tag{16}$$

Applying the Cauchy-Bunyakovskii inequality, we obtain

$$|\Delta J_0(\nu)| \leq 2\|y(\zeta,t) - f_0(\zeta,t)\|_{L_2(S)}\|\delta y(\zeta,t)\|_{L_2(S)} + \|\delta y(\zeta,t)\|^2_{L_2(S)} \tag{17}$$

An application of the Corollary 3.1 completes the proof.

Theorem 3.3 For any $\beta \geq 0$ the problem (1)-(4) has a least one solution.

proof
The set of V is closed and bounded in l_2. Since $J_0(\nu)$ is continuous on V by Theorem 3.2, so is

$$J_\beta(\nu) = J_0(\nu) + \beta \sum_{m=0}^{2} \|\nu_m - w_m\|^2_{l_2}. \tag{18}$$

Then from the Weierstrass theorem [14] it follows that the problem (1)-(4) has a least one solution. This completes the proof of the theorm.

4 The Uniqueness Theorem

According to the above discussions, we ca easily obtain a theorem concerning solution uniqueness for the considering optimal control problem (1)-(4).

Theorem 4.1 There exists a dense set K of l_2 such that for any $\omega_m \in K, m = \overline{0,2}$ the problem (1)-(4) for $\beta > 0$ has a unique solution.

proof The functional $J_0(\nu)$ is bounded below, and the foreging establishes that it is continues on V. Furthermore, l_2 is uniformaly convex [12]. It thus follows from a theorm in [16] that the space l_2 contains an everywhere-dense subset K such that the problem (1)-(4) has a unque solution when $\omega_m \in K, m = \overline{0,2}$ and $\beta > 0$. This completes the proof of the theorm.

5 Conclusion

We have investigated a constrained optimal control problems governed by quasilinear parabolic equations with controls in the coefficients of the equation. The existence and uniqueness of the optimal control problem is proved.

6 Acknowledgment

The authers gratefully acknowledgment the referee, who made useful suggestions and remarks which helped to improve the paper.

Received: September 2011. Accepted: September 2012.

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