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Weak Solutions of Fractional Order Pettis Integral Inclusions with Multiple Time Delay in Banach Spaces

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ABSTRACT

We study the existence of weak solutions for nonlinear integral inclusion with multiple time delay. The main result of the paper is based on the fixed point theorem of Mönch type and the technique of measure of weak noncompactness.

RESUMEN

Estudiamos la existencia de soluciones débiles de la inclusión integral no lineal con retardos temporales múltiples. El resultado principal del artículo se basa en el Teorema de Punto Fijo de tipo Mönch y la técnica de medida de la no-compacidad débil

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1 Introduction

Fractional differential equations have been of great interest recently. It is due to the development of the theory of fractional calculus itself and by application of such constructions in various fields of science and engineering such as control theory, physics, mechanics, electrochemistry, porous media, etc. There are many works discussing the solvability of nonlinear fractional differential equations and inclusions, see the monographs of Abbas *et al.* [2], Kilbas *et al.* [14], Lakshmikantham *et al.* [15], Podlubny [18], Tarasov [20], the papers of Agarwal *et al.* [3, 4, 5], Benchohra *et al.* [7, 8], Kilbas and Marzan [13], Salem [19], Vityuk and Golushkov [21], and the references therein.

In [12], R. W. Ibrahim and H. A. Jalab studied the existence of solutions of the following fractional integral inclusion

$$u(t)-\sum_{i=1}^{m} b_i(t)u(t-\tau_i)\in I^{\alpha}F(t,u(t));\ \text{if } t\in[0,T], \tag{1}$$

where $\tau_i < t \in [0,T], b_i:[0,T]\to\mathbb{R},\ i=1\dots,n$ are continuous functions, and $F:[0,T]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a given multivalued map.

In [1], Abbas and Benchohra considered the following fractional integral equation with delay

$$u(x,y)=\sum_{i=1}^{m} g_i(x,y)u(x-\xi_i,y-\mu_i)+I_\theta^r f(x,y,u(x,y));\ \text{if } (x,y)\in J:=[0,a]\times[0,b], \tag{2}$$

$$u(x,y)=\Phi(x,y);\ \text{if } (x,y)\in\tilde{J}:=[-\xi,a]\times[-\mu,b]\backslash(0,a]\times(0,b], \tag{3}$$

where $a,b>0,\ \theta=(0,0),\ \xi_i,\mu_i\geq 0;\ i=1\dots,m,\ \xi=\max_{i=1\dots,m}\{\xi_i\},\ \mu=\max_{i=1\dots,m}\{\mu_i\},\ I_\theta^r$ is the left-sided mixed Riemann-Liouville integral of order $r=(r_1,r_2)\in(0,\infty)\times(0,\infty),\ f:J\times\mathbb{R}^n\to\mathbb{R}^n,\ g_i:J\to\mathbb{R};\ i=1\dots m$ are given continuous functions, and $\Phi:\tilde{J}\to\mathbb{R}^n$ is a given continuous function such that

$$\Phi(x,0)=\sum_{i=1}^{m} g_i(x,0)\Phi(x-\xi_i,-\mu_i);\ x\in[0,a],$$

and

$$\Phi(0,y)=\sum_{i=1}^{m} g_i(0,y)\Phi(-\xi_i,y-\mu_i);\ y\in[0,b].$$

Motivated by the above papers, in this paper, we consider the following fractional integral inclusion with multiple time delay:

$$u(x,y)-\sum_{i=1}^{m} g_i(x,y)u(x-\xi_i,y-\mu_i)\in I_\theta^{\alpha}F(x,y,u(x,y));\quad (x,y)\in J_a\times J_b. \tag{4}$$

$$u(x,y)=\Psi(x,y);\quad (x,y)\in\tilde{J}=[-\xi,a]\times[-\mu,b]\backslash(0,a]\times(0,b], \tag{5}$$

where $J_a = [0, a]$, $J_b = [0, b]$ for $a, b > 0$, $\theta = (0, 0)$, $\xi = \max_{i=1\dots m}\{\xi_i\}$, $\mu = \max_{i=1\dots m}\{\mu_i\}$, I_θ^α is the left sided mixed Pettis integral of order α, $\alpha = (\alpha_1, \alpha_2) \in (0, \infty) \times (0, \infty)$, $F : J_a \times J_b \times E \to \mathcal{P}(E)$ is a multivalued map ($\mathcal{P}(E)$ is the family of all nonempty subsets of E), $g_i : J_a \times J_b \to \mathbb{R}; i = 1, \dots m$ are given continuous functions, and $\Psi : \tilde{J} \to E$ is a given continuous function such that

$$\Psi(0, y) = \sum_{i=1}^{m} g_i(0, y)\Psi(-\xi_i, y - \mu_i);\ y \in [0, b],$$

and

$$\Psi(x, 0) = \sum_{i=1}^{m} g_i(x, 0)\Psi(x - \xi_i, -\mu_i);\ x \in [0, a].$$

E is a Banach space with norm $\|.\|$. Our result is based on fixed point theorem of Mönch type and the technique of measure of weak noncompactness. Let us mention that other tools like the nonlinear alternative of Leray-Schauder type, the Banach fixed point theorem and Schauder's fixed point theorem, such have been used to analyze the above problem in the scalar case [1, 2]. The present results complement and extend those considered in the scalar case.

2 Preliminaries

In this section, we introduce the notation, definitions, and preliminary facts that will be used in the remainder of this survey paper. Let $\mathbb{R}$ denote the real line and let $J_a = [0, a]$ and $J_b = [0, b]$ be two closed and bounded intervals in $\mathbb{R}$ for some real numbers $a > 0$ and $b > 0$. Throughout the paper, E is a Banach space with norm $\|.\|$ and dual E^*. Also $(E, w) = (E, \sigma(E, E^*))$ denotes the space E with its weak topology. We take $C(J_a \times J_b, E)$ to be the Banach space of continuous functions $u : J_a \times J_b \to E$, with the usual supremum norm

$$\|u\|_\infty = \sup\{\|u(x, y)\|, \quad (x, y) \in J_a \times J_b\}.$$

Definition 2.1. *[17] The function $x : J_a \times J_b \to E$ is said to be Pettis integrable on $J_a \times J_b$ if and only if there is an element $x_{I\times J} \in E$ corresponding to each $I \times J \subset J_a \times J_b$ (I and J are measurable), such that $\varphi(x_{I\times J}) = \int_I \int_J \varphi(x(s, t))dsdt$ for all $\varphi \in E^*$ where the integral on the right is assumed to exist in the sense of Lebesgue (by definition, $x_{I\times J} = \int_I \int_J x(s, t)dsdt$).*

We let $L^1(J_a \times J_b, E)$ denote the Banach space of measurable functions $u : J_a \times J_b \to E$ that are Pettis integrable, equipped with the norm

$$\|u\|_{L^1} = \int_0^a \int_0^b \|u(x, y)\|dxdy.$$

Let *P(E)* is the family of all nonempty subsets of E.

A multivalued map $G : E \to$ *P(E)* has convex (closed) valued if $G(x)$ is convex (closed) for all $x \in E$. We say that G is bounded on bounded sets if $G(B)$ is bounded in E for each bounded set

B of E (i.e. $\sup_{x\in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$). The map G is upper semicontinuous (u.s.c) on E if for each $x_0 \in E$, the set $G(x_0)$ is a nonempty closed subset of E and for each open set N of E containing $G(x_0)$ there exists an open neighborhood M of x_0 such that $G(M) \subseteq N$. The mapping G has a fixed point if there exists $x \in E$ such that $x \in G(x)$.

In what follows $P_{cl}(E) = \{Y \in P(E) : Y \text{ is closed}\}$, $P_b(E) = \{Y \in P(E) : Y \text{ is bounded}\}$, $P_{cp}(E) = \{Y \in P(E) : Y \text{ is compact}\}$, and $P_{cp,cv}(E) = \{Y \in P(E) : Y \text{ is compact and convex}\}$.

A multivalued map $G : J_a \times J_b \to P_{cl}(E)$ is said to be measurable if for each $\omega \in E$ the function

$$(x, y) \to d(\omega, G(x, y)) = \inf\{|\omega - \upsilon| : \upsilon \in G(x, y)\}$$

is measurable. For more details on multivalued maps see the books of Aubin and Cellina [6], Deimling [10].

Definition 2.2. *A function $h : E \to E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to weakly convergent sequence in E (ie for any $(x_n)_n$ in E with $x_n \to x$ in (E, w), $h(x_n) \to h((x))$ in (E, w)).*

Definition 2.3. *A function $F : Q \to P_{cl,cv}(Q)$ has weakly sequentially closed graph if for any sequence $(x_n, y_n) \in Q \times Q$, where $y_n \in F(x_n)$ for $n \in \{1, 2, ...\}$,and where both $x_n \to x$ in (E, ω) and $y_n \to y$ in (E, ω) then $y \in F(x)$.*

Proposition 2.4. *[17, 11] If $x(.)$ is Pettis integrable and $h(.)$ is a measurable and essentially bounded real-valued function, then $x(.)h(.)$ is Pettis integrable.*

Definition 2.5. *[9] Let E be a Banach space, Ω_E the bounded subsets of E and B_1 the unit ball of E. The De Blasi measure of weak noncompactness is the map $\beta : \Omega_E \to [0, \infty)$ defined by*
$\beta(X) = \inf\{\epsilon > 0 :$ *there exists a weakly compact subset* Ω *of* $E : X \subset \epsilon B_1 + \Omega\}$

Properties: De Blasi measure of noncompactness satisfies some properties

(a) $A \subset B \Rightarrow \beta(A) \leq \beta(B)$,

(b) $\beta(A) = 0 \Leftrightarrow A$ is relatively compact,

(c) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$,

(d) $\beta(\overline{A}^{\omega}) = \beta(A)$, ($\overline{A}^{\omega}$ denotes the weak closure of A),

(e) $\beta(A + B) \leq \beta(A) + \beta(B)$,

(f) $\beta(\lambda A) = |\lambda|\beta(A)$,

(g) $\beta(conv(A)) = \beta(A)$,

(h) $\beta(\cup_{|\lambda|\leq h}\lambda A = h\beta(A)$.

The following result follows directly from the Hahn-Banach theorem.

Proposition 2.6. *Let E be a normed space with $x_0 \neq 0$ then there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$.*

For a given set V of functions $v : J_a \times J_b \to E$ let us denote by

$$V(x,y) = \{v(x,y), v \in V\}, \ (x,y) \in J_a \times J_b$$

and

$$V(J_a \times J_b) = \{v(x,y) : v \in V, \ (x,y) \in J_a \times J_b\}.$$

For completeness, we recall the definition of the fractional Pettis-integral of order $\alpha > 0$. Let $\alpha_1, \ \alpha_2 > 0$ and $\alpha = (\alpha_1, \alpha_2)$. For $h \in L^1(J_a \times J_b, E)$, the expression

$$(I_0^\alpha h)(x,y) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1 - 1}(y-t)^{\alpha_2 - 1} h(s,t) dsdt$$

where the sign "$\int$" denotes the Pettis integral and $\Gamma(.)$ is the Euler gamma function, is called the left sided mixed Pettis integral of order α.

For our purpose we will need the following fixed point theorem.

Theorem 2.7. *[16] Let E be a Banach space with Q a nonempty, bounded, closed, convex and equicontinuous subset of metrizable locally convex vector space $C(J,E)$ such that $0 \in Q$. Suppose that $T : Q \to P_{cl,cv}(Q)$ has weakly-sequentially closed graph. If the implication*

$$\overline{V} = \overline{conv}(\{0\} \cup T(V)) \Rightarrow V \ \ \textit{is relatively weakly compact}, \tag{6}$$

holds for every subset $V \subset Q$, then the operator T has a fixed point.

3 Main Results

we first define what we mean by solution of the problem (4)-(5).

Definition 3.1. *A function $u \in C(J,E)$ is said to be solution of problem (4)-(5) if there exists a function $v \in L^1(J_a \times J_b, E)$ with $v(x,y) \in F(x,y,u(x,y))$ and such that*

$$u(x,y) = \sum_{i=1}^{m} g_i(x,y)u(x-\xi_i, y-\mu_i) + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1} v(s,t) dsdt$$

and the function u satisfies condition (5) on $\tilde{J}$.

For any $u \in C(J_a \times J_b, E)$, we define the set

$$S_{F,u} = \{v \in L^1(J_a \times J_b, E), v(x,y) \in F(x,y,u(x,y)), (x,y) \in J_a \times J_b\}$$

This is known as the set of *selection function.* Set

$$G = \max_{i=1\dots m} \{ \sup_{(x,y) \in J_a \times J_b} |g_i(x,y)| \}.$$

We are now in the position to state and prove our existence result for the problem (4)-(5). We first list the following hypotheses.

(H1) $F : J_a \times J_b \times E \to P_{cp,cl,cv}(E)$, has weakly sequentially closed graph.

(H2) For each $u \in C(J_a \times J_b, E)$, there exists a measurable function $v : J_a \times J_b \to E$ with $v(x,y) \in F(x,y,u(x,y))$ a.e. on $J_a \times J_b$ and v is Pettis integrable on $J_a \times J_b$.

(H3) There exists $p \in L^\infty(J_a \times J_b, \mathbb{R}_+)$ such that

$$\|F(x,y,u)\|_P = \sup\{\|v\| : v \in F(x,y,u)\} \le p(x,y),$$

for $(x,y) \in J_a \times J_b$ and each $u \in E$.

(H4) There exists a number $R > 0$ such that

$$\frac{p^* a^{\alpha_1} b^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)(1-mG)} < R, \tag{7}$$

where $p^* = \|p\|_\infty$.

(H5) Let $r_0 > 0$ be arbitrary (but fixed). For any $\epsilon > 0$ and for any subset $X \subset B_{r_0}$, there exists a closed subset $I_\epsilon \subset J_a \times J_b$ such that $\mu(J_a \times J_b \backslash I_\epsilon) < \epsilon$ and

$$\beta(F(T \times X)) \le \sup_{(x,y)\in T} p(x,y)\beta(X),$$

for each closed subset T of I_ϵ, where μ denotes the Lebesgue measure in $\mathbb{R}^2$.

The main result in this paper reads as follows.

Theorem 3.2. *Assume that assumptions (H1)-(H5) hold. If*

$$mG + \frac{p^* a^{\alpha_1} b^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)} < 1, \tag{8}$$

then problem(4)-(5) has at least one solution on J.

Proof. To transform problem (4)-(5) into a fixed point problem, we define a multivalued map $\Omega : C(J,E) \to P_{cl}(C(J,E))$ as

$$\Omega(u) = \{h \in C(J,E) \text{ such that}$$

$$h(x,y) = \begin{cases} \Psi(x,y) & \text{if } (x,y) \in \tilde{J}, \\ \sum_{i=1}^m g_i(x,y)u(x-\xi_i, y-\mu_i) & \text{if } \ \upsilon \in S_{F,u}, \\ +\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}\upsilon(s,t)dsdt; & (x,y) \in J_a \times J_b. \end{cases}$$

where $\Psi(\cdot,\cdot)$ is the function defined by (5). Now, we prove that Ω satisfies all the assumptions of the Theorem 2.7 and thus Ω has a fixed point which is a solution of problem (4)-(5). □

First notice that, for all $u \in C(J, E)$, there exists a Pettis integral $\upsilon : J_a \times J_b \to E$ such that $\upsilon(x,y) \in F(x,y,u(x,y))$ for a.e. $(x,y) \in J_a \times J_b$ (Assumption (H2)) then $\varphi(\upsilon(x,y)) \in L^1(J_a \times J_b)$ for any $\varphi \in E^*$. From the definition of the integral of fractional order we have

$$\begin{aligned} I^\alpha \varphi(\upsilon(x,y,)) &= \int_0^x \int_0^y \frac{(x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \varphi(\upsilon(s,t))dsdt \\ &= \int_0^x \int_0^y \varphi\left(\frac{(x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \upsilon(s,t)\right) dsdt \end{aligned}$$

exists for almost every $(x,y) \in J_a \times J_b$ and is an element from $L^1(J_a \times J_b)$, that is, for almost every $(x,y) \in J_a \times J_b$, $s \in (0,x)$, $t \in (0,y)$ the measurable function

$$\varphi\left(\frac{(x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \upsilon(s,t)\right) = \frac{(x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \varphi(\upsilon(s,t))$$

is Lebesgue integrable, hence the function $(s,t) \to \frac{(x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \upsilon(s,t)$ is Pettis integrable on $J_a \times J_b$, and thus the operator Ω is well defined.

Let $R > 0$ and consider the set

$$\begin{aligned} Q \;=\; & \{u \in C(J,E) : \|u\|_\infty \leq R \\ & \text{and } \|u(x_2,y_2) - u(x_1,y_1)\| \leq R \sum_{i=1}^m |g_i(x_2,y_2) - g_i(x_1,y_1)| \\ & + \frac{p^*}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)} [x_2^{\alpha_1} y_2^{\alpha_2} - x_1^{\alpha_1} y_1^{\alpha_2}]; \text{ for } (x_1,y_1),(x_2,y_2) \in J_a \times J_b\} \end{aligned}$$

Clearly, the subset Q is closed, bounded, convex and equicontinuous subset of a metrisable locally convex vector space $C(J,E)$. The remainder of the proof will be given in four steps.

Step 1: $\Omega(u)$ is convex for each $u \in Q$.

For that, let $0 < \lambda < 1$, $h_1, h_2 \in \Omega(u)$, obviously if $(x,y) \in \tilde{J}$ then $\lambda h_1(x,y) + (1-\lambda)h_2(x,y) \in \Omega(u)$. Now if $(x,y) \in J_a \times J_b$, then there exists $\upsilon_1, \upsilon_2 \in S_{F,u}$ such that

$$h_i(x,y) = \sum_{i=1}^m g_i(x,y)u(x-\xi_i, y-\mu_i) + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha 2-1} \upsilon_i(s,t)dsdt;$$

$i = 1, 2$, Then for each $(x,y) \in J_a \times J_b$ we have

$$\begin{aligned} (\lambda h_1 + (1-\lambda)h_2)(x,y) &= \sum_{i=1}^m g_i(x,y)u(x-\xi_i, y-\mu_i) \\ + \quad \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} & \int_0^x \int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}(\lambda \upsilon_1(s,t) + (1-\lambda)\upsilon_2(s,t))dsdt. \end{aligned}$$

Since $S_{F,u}$ is convex (because F has convex values), it follows that $\lambda h_1 + (1-\lambda)h_2 \in \Omega(u)$.

Step 2: Ω maps Q into Q.

To see this, take $h \in \Omega Q$. Then there exists $u \in Q$ with $h \in \Omega u$. And there exists $\upsilon : J_a \times J_b \to E$ Pettis integrable with $\upsilon(x,y) \in F(x,y,u(x,y))$

$$h(x,y) = \begin{cases} \Psi(x,y) & \text{if } (x,y) \in \tilde{J}, \\ \sum_{i=1}^m g_i(x,y)u(x-\xi_i, y-\mu_i) & \text{if } \ \upsilon \in S_{F,u}, \\ +\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}\upsilon(s,t)dsdt; & (x,y) \in J_a \times J_b. \end{cases}$$

We can consider that $h(x,y) \neq 0$ and by Proposition 2.6 there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(h(x,y)) = \|h(x,y)\|$ for $(x,y) \in J_a \times J_b$, we have

$$\begin{aligned} \|h(x,y)\| &= \varphi(h(x,y)) \\ &= \varphi\Big(\sum_{i=1}^m g_i(x,y)u(x-\xi_i, y-\mu_i) \\ &+ \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}\upsilon(s,t)dsdt\Big) \\ &= \varphi\left(\sum_{i=1}^m g_i(x,y)u(x-\xi_i, y-\mu_i)\right) \\ &+ \varphi\left(\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}\upsilon(s,t)dsdt\right) \\ &\leq mGR + \frac{p^*}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}dsdt \\ &\leq mGR + \frac{p^* a^{\alpha_1} b^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)} \leq R. \end{aligned}$$

on the other hand, for $(x,y) \in \tilde{J}$, we have

$$\|h(x,y)\| = \varphi(h(x,y)) \leq R.$$

Next, suppose that $(x_1,y_1),(x_2,y_2) \in J_a \times J_b$ with $x_1 < x_2$ and $y_1 < y_2$, and let $h \in \Omega u$, so $h(x_1,y_1) - h(x_2,y_2) \neq 0$. Then there exists $\varphi \in E^*$ such that

$$\|h(x_1,y_1) - h(x_2,y_2)\| = \varphi(h(x_1,y_1) - h(x_2,y_2)),$$

and $\|\varphi\| = 1$. Thus

$$
\begin{aligned}
&\|h(x_2,y_2)-h(x_1,y_1)\| \\
&\quad = \varphi(\sum_{i=1}^{m} g_i(x_2,y_2)u(x_2-\xi_i,y_2-\mu_i) \\
&\qquad +\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{x_2}\int_0^{y_2}(x_2-s)^{\alpha_1-1}(y_2-t)^{\alpha_2-1}\upsilon(s,t)dsdt \\
&\qquad -\sum_{i=1}^{m} g_i(x_1,y_1)u(x_1-\xi_i,y_1-\mu_i) \\
&\qquad +\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{x_1}\int_0^{y_1}(x_1-s)^{\alpha_1-1}(y_1-t)^{\alpha_2-1}\upsilon(s,t)dsdt) \\
&\quad = \varphi(\sum_{i=1}^{m} g_i(x_2,y_2)u(x_2-\xi_i,y_2-\mu_i)-\sum_{i=1}^{m} g_i(x_1,y_1)u(x_1-\xi_i,y_1-\mu_i)) \\
&\qquad +\varphi(\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}(x_2-s)^{\alpha_1-1}(y_2-t)^{\alpha_2-1}\upsilon(s,t)dsdt \\
&\qquad +\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{x_1}\int_0^{y_1}[(x_2-s)^{\alpha_1-1}(y_2-t)^{\alpha_2-1}-(x_1-s)^{\alpha_1-1}(y_1-t)^{\alpha_2-1}] \\
&\qquad \times\upsilon(s,t)dsdt+\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{x_1}\int_{y_1}^{y_2}(x_2-s)^{\alpha_1-1}(y_2-t)^{\alpha_2-1}\upsilon(s,t)dsdt \\
&\qquad +\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_{x_1}^{x_2}\int_0^{y_1}(x_2-s)^{\alpha_1-1}(y_2-t)^{\alpha_2-1}\upsilon(s,t)dsdt) \\
&\quad \leq \sum_{i=1}^{m}\|g_i(x_2,y_2)u(x_2-\xi_i,y_2-\mu_i)-g_i(x_1,y_1)u(x_1-\xi_i,y_1-\mu_i)\| \\
&\qquad +\frac{p^*}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{x_1}\int_0^{y_1}[(x_2-s)^{\alpha_1-1}(y_2-t)^{\alpha_2-1}-(x_1-s)^{\alpha_1-1}(y_1-t)^{\alpha_2-1}]dsdt \\
&\qquad +\frac{p^*}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}(x_2-s)^{\alpha_1-1}(y_2-t)^{\alpha_2-1}dsdt \\
&\qquad +\frac{p^*}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{x_1}\int_{y_1}^{y_2}(x_2-s)^{\alpha_1-1}(y_2-t)^{\alpha_2-1}dsdt \\
&\qquad +\frac{p^*}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_{x_1}^{x_2}\int_0^{y_1}(x_2-s)^{\alpha_1-1}(y_2-t)^{\alpha_2-1}dsdt \\
&\quad \leq R\sum_{i=1}^{m}\|g_i(x_2,y_2)-g_i(x_1,y_1)\|+\frac{p^*}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}[x_2^{\alpha_1}y_2^{\alpha_2}-x_1^{\alpha_1}y_1^{\alpha_2}].
\end{aligned}
$$

This implies that $h\in Q$, hence $\Omega Q\subset Q$

Step 3: Ω has weakly sequentially closed graph.

Let (u_n,w_n) be a sequence in $Q\times Q$ with $u_n(x,y)\to u(x,y)$ in (E,w) for each $(x,y)\in J_a\times J_b$, $w_n(x,y)\to w(x,y)$ in (E,w) for each $(x,y)\in J_a\times J_b$ and $w_n\in\Omega(u_n)$ for $n\in\{1,2,\ldots\}$. We show that $w\in\Omega(u)$.

Since $w_n\in\Omega(u_n)$, there exists $\upsilon_n\in S_{F,u_n}$ such that

$$
w_n(x,y)=\sum_{i=1}^{m} g_i(x,y)u_n(x-\xi_i,y-\mu_i)+\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{x}\int_0^{y}(x-s)^{\alpha_1-1}(y-t)^{\alpha 2-1}\upsilon_n(s,t)dsdt.
$$

We show that there exists $v \in S_{F,u}$ such that

$$w(x,y) = \sum_{i=1}^{m} g_i(x,y)u(x-\xi_i, y-\mu_i) + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha 2-1}v(s,t)dsdt.$$

Since $F(\cdot,\cdot,\cdot)$ has compact values, there exists a subsequence $v_{n_m} \in S_{F,u_n}$ such that v_{n_m} is Pettis integrable and

$$v_{n_m}(x,y) \in F(x,y,u_n(x,y)) \text{ a.e.}(x,y) \in J_a \times J_b$$

and

$$v_{n_m}(\cdot,\cdot) \to v(\cdot,\cdot) \text{ in } (E,w) \text{ as } m \to \infty.$$

As $F(x,y,\cdot)$ has weakly sequentially closed graph, $v(x,y) \in F(x,y,u(x,y))$. Then Lebesgue Dominated Convergence theorem for Pettis integral implies that

$$\varphi(w_n(x,y)) \to \varphi\left(\sum_{i=1}^{m} g_i(x,y)u(x-\xi_i, y-\mu_i) + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}v(s,t)dsdt\right)$$

i.e. $w_n(x,y) \to \Omega u(x,y)$ in (E,w). Since this holds, for each $(x,y) \in J_a \times J_b$, we have $w \in \Omega u$.

Step 4: the implication (6) holds.

Let V be a subset of Q such that $\overline{V} = \overline{conv}(\Omega(V)\cup\{0\})$. Obviously $V(x,y) \subset \overline{conv}(\Omega(V(x,y))\cup \{0\}), \forall (x,y) \in J$. Further, as V is bounded and equicontinuous, the function $(x,y) \to v(x,y) = \beta(V(x,y))$ is continuous on J.

If $(x,y) \in \tilde{J}$ then

$$\Omega V(x,y) = \{\Omega u(x,y) : u \in V\} = \{\Psi(x,y) : (x,y) \in \tilde{J}\}.$$

and since Ψ is continuous on $[-\xi,0]\times[-\mu,0]$, the set $\overline{\{\Psi(x,y),(x,y) \in [-\xi,0]\times[-\mu,0]\}} \subset E$ is compact. Now by (H3) and the properties of the measure β, for any $(x,y) \in J_a \times J_b$, we have

$$\begin{aligned}
v(x,y) &\leq \beta((\Omega V)(x,y) \cup \{0\}) \\
&\leq \beta((\Omega V)(x,y)) \\
&\leq \beta\{\Omega u(x,y) : u \in V\} \\
&\leq \beta\left\{\textstyle\sum_{i=1}^{m} g_i(x,y)u(x-\xi_i, y-\mu_i); u \in V\right\} \\
&+\beta\left\{\tfrac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\textstyle\int_0^x\int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}v(s,t)dsdt; v(x,y) \in F(x,y,u), u \in V\right\} \\
&\leq \textstyle\sum_{i=1}^{m} \beta(\{g_i(x,y)u(x-\xi_i, y-\mu_i); u \in V\})
\end{aligned}$$

$$+\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\beta\left\{\int_0^x\int_0^y(x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}v(s,t)dsdt; v(x,y)\in F(x,y,u), u\in V\right\}$$
$$\leq \sum_{i=1}^m |g_i(x,y)|\beta(V(x,y))$$
$$+\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y(x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}p(s,t)\beta(V(s,t))dsdt$$
$$\leq mG\|v\|_\infty + \frac{p^* a^{\alpha_1} b^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\|v\|_\infty$$

In particular,

$$\|v\|_\infty \leq \|v\|_\infty\left(mG + \frac{p^* a^{\alpha_1} b^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\right).$$

By (8) it follows that $\|v\|_\infty = 0$, that is $v(x,y) = \beta(V(x,y)) = 0$ for each $(x,y)\in J$ and then V is weakly relatively compact in $C(J,E)$. Applying now Theorem 2.7 we conclude that T has a fixed point which is a solution of problem (4)-(5). □

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