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An Elementary Study of a Class of Dynamic Systems with Two Time Delays

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ABSTRACT

An elementary analysis is developed to determine the stability region of a certain class of ordinary differential equations with two delays. Our analysis is based on determining stability switches first where an eigenvalue is pure complex, and then checking the conditions for stability loss or stability gain. In the case of both stability losses and stability gains Hopf bifurcation occurs giving the possibility of the birth of limit cycles.

RESUMEN

Se realiza un análisis básico para determinar la estabilidad de la región de una cierta clase de ecuaciones diferenciales ordinaras con dos retrasos. Nuestro análisis se basa en la determinación de switches de estabilidad, en primer lugar cuando un autovalor es complejo puro, y luego revisando las condiciones para la pérdida o ganancia de estabilidad. En el caso de ambas pérdidas de estabilidad y ganancias de estabilidad, se obtiene la bifurcación de Hopf dando la posibilidad del nacimiento de ciclos límites.

Keywords and Phrases: dynamic systems, time delays, stabiliy analysis.

2010 AMS Mathematics Subject Classification: 34K20, 37C75

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1 Introduction

Dynamic models with time delays have many applications in many fields of quantitative sciences (see for example, Cushing (1977) and Invernizzi and Medio (1991)). The case of a single delay is well established in the literature (Hayes (1950) and Burger (1956)), however the presence of multiple delays makes analysis much more complicated. Sufficient and necessary conditions were derived for several classes of models giving a complete description of the stability region (Hale (1979), Hale and Huang (1993) and Piotrowska (2007)).

In this paper a special class of dynamic systems is considered which are governed by delay differential equations with two delays. It is well known (Hayes (1950) and Cooke and Grossman (1982)) that stability can be lost or gained on a curve of stability switches, where an eigenvalue is pure complex. We will therefore determine these curves and then by bifurcation analysis characterize those segments where stability is gained or lost. In this way the stability region can be completely described.

This paper is the continuation of our previous work (Matsumoto and Szidarovszky (2011)) where an elementary analysis was presented with a single delay.

The paper is organized in the following way. Section 2 determines the curves where stability switches are possible and characterizes those segments where stability is lost or gained in the nonsymmetric cases. Section 3 discusses the symmetric case and Section 4 concludes the paper.

2 Stability Switches and Stability Region

We will examine the asymptotical stability of the delay differential equation

$$\dot{x}(t) + Ax(t-\tau_1) + Bx(t-\tau_2) = 0 \tag{2.1}$$

where A and B are positive constants. The characteristic equation can be obtained by looking for the solution in the exponential form $\alpha e^{\lambda t}$. By substitution,

$$\alpha\lambda e^{\lambda t} + A\alpha e^{\lambda(t-\tau_1)} + B\alpha e^{\lambda(t-\tau_2)} = 0$$

or

$$\lambda + Ae^{-\lambda\tau_1} + Be^{-\lambda\tau_2} = 0. \tag{2.2}$$

Introduce the new variables

$$\omega = \frac{A}{A+B},\ 1-\omega = \frac{B}{A+B},\ \bar{\lambda} = \frac{\lambda}{A+B}$$

$$\gamma_1 = \tau_1(A+B) \text{ and } \gamma_2 = \tau_2(A+B)$$

to reduce equation (2.2) to the following:

$$\bar{\lambda} + \omega e^{-\bar{\lambda}\gamma_1} + (1-\omega)e^{-\bar{\lambda}\gamma_2} = 0. \tag{2.3}$$

Because of symmetry we can assume that $\omega \geq 1/2$.In order to find the stability region in the (γ_1, γ_2) plane we will first characterize the cases when an eigenvalue is pure complex, that is, when $\bar{\lambda} = i\upsilon$. We can assume that $\upsilon > 0$, since if $\bar{\lambda}$ is an eigenvalue, its complex conjugate is also an eigenvalue. Substituting $\bar{\lambda} = i\upsilon$ into equation (2.3) we have

$$\iota\upsilon + \omega e^{-i\upsilon\gamma_1} + (1-\omega)e^{-i\upsilon\gamma_2} = 0.$$

In the special case of $\gamma_1 = 0$, the equation becomes

$$\iota\upsilon + \omega + (1-\omega)e^{-i\upsilon\gamma_2} = 0.$$

The real and imaginary parts imply that

$$\omega + (1-\omega)\cos(\upsilon\gamma_2) = 0$$

$$\upsilon - (1-\omega)\sin(\upsilon\gamma_2) = 0.$$

We can assume first $\omega > 1/2$, so from the first equation

$$\cos(\upsilon\gamma_2) = -\frac{\omega}{1-\omega} < -1$$

so no stability switch is possible. If $\omega = 1/2$, then

$$\cos(\upsilon\gamma_2) = -1$$

implying that $\sin(\upsilon\gamma_2) = 0$ and so $\upsilon = 0$ showing that there is no pure complex root. Hence for $\gamma_1 = 0$ the system is asymptotically stable with all $\gamma_2 \geq 0$.

Assume now that $\gamma_1 > 0$, $\gamma_2 \geq 0$. The real and imaginary parts give two equations:

$$\omega\cos(\upsilon\gamma_1) + (1-\omega)\cos(\upsilon\gamma_2) = 0 \tag{2.4}$$

and

$$\upsilon - \omega\sin(\upsilon\gamma_1) - (1-\omega)\sin(\upsilon\gamma_2) = 0. \tag{2.5}$$

We consider the case of $\omega > 1/2$ first and the symmetric case of $\omega = 1/2$ will be discussed later. Introduce the variables

$$x = \sin(\upsilon\gamma_1) \text{ and } y = \sin(\upsilon\gamma_2),$$

then (2.4) implies that

$$\omega^2(1-x^2) = (1-\omega)^2(1-y^2)$$

or

$$-\omega^2x^2 + (1-\omega)^2y^2 = 1 - 2\omega. \tag{2.6}$$

From (2.5),

$$\upsilon - \omega x - (1-\omega)y = 0$$

implying that

$$y = \frac{\upsilon - \omega x}{1 - \omega} \tag{2.7}$$

Combining (2.6) and (2.7) yields

$$-\omega^2 x^2 + (1-\omega)^2 \left(\frac{\upsilon - \omega x}{1-\omega}\right)^2 = 1 - 2\omega$$

from which we can conclude that

$$x = \frac{\upsilon^2 + 2\omega - 1}{2\upsilon\omega} \tag{2.8}$$

and then from (2.7),

$$y = \frac{\upsilon^2 - 2\omega + 1}{2\upsilon(1-\omega)}. \tag{2.9}$$

Equations (2.8) and (2.9) provide a parameterized curve in the (γ_1, γ_2) plane:

$$\sin(\upsilon\gamma_1) = \frac{\upsilon^2 + 2\omega - 1}{2\upsilon\omega} \text{ and } \sin(\upsilon\gamma_2) = \frac{\upsilon^2 - 2\omega + 1}{2\upsilon(1-\omega)}. \tag{2.10}$$

In order to guarantee feasibility we have to satisfy

$$-1 \le \frac{\upsilon^2 + 2\omega - 1}{2\upsilon\omega} \le 1 \tag{2.11}$$

and

$$-1 \le \frac{\upsilon^2 - 2\omega + 1}{2\upsilon(1-\omega)} \le 1. \tag{2.12}$$

Simple calculation shows that with $\omega > 1/2$ these relations hold if and only if

$$2\omega - 1 \le \upsilon \le 1.$$

From (2.10) we have four cases for γ_1 and γ_2, since

$$\gamma_1 = \frac{1}{\upsilon}\left\{\sin^{-1}\left(\frac{\upsilon^2 + 2\omega - 1}{2\upsilon\omega}\right) + 2k\pi\right\}$$

or

$$\gamma_1 = \frac{1}{\upsilon}\left\{\pi - \sin^{-1}\left(\frac{\upsilon^2 + 2\omega - 1}{2\upsilon\omega}\right) + 2k\pi\right\} \quad (k = 0, 1, 2, ...)$$

and similarly

$$\gamma_2 = \frac{1}{\upsilon}\left\{\sin^{-1}\left(\frac{\upsilon^2 - 2\omega + 1}{2\upsilon(1-\omega)}\right) + 2n\pi\right\}$$

or

$$\gamma_2 = \frac{1}{\upsilon}\left\{\pi - \sin^{-1}\left(\frac{\upsilon^2 - 2\omega + 1}{2\upsilon(1-\omega)}\right) + 2n\pi\right\} \quad (n = 0, 1, 2, ...).$$

However from (2.4) we can see that $\cos(\upsilon\gamma_1)$ and $\cos(\upsilon\gamma_2)$ must have different signs, so we have only two possibilities:

$$L_1(k,n):\ \begin{cases} \gamma_1 = \dfrac{1}{\upsilon}\left\{\sin^{-1}\left(\dfrac{\upsilon^2+2\omega-1}{2\upsilon\omega}\right)+2k\pi\right\} \\[2ex] \gamma_2 = \dfrac{1}{\upsilon}\left\{\pi-\sin^{-1}\left(\dfrac{\upsilon^2-2\omega+1}{2\upsilon(1-\omega)}\right)+2n\pi\right\} \end{cases} \tag{2.13}$$

and

$$L_2(k,n):\ \begin{cases} \gamma_1 = \dfrac{1}{\upsilon}\left\{\pi-\sin^{-1}\left(\dfrac{\upsilon^2+2\omega-1}{2\upsilon\omega}\right)+2k\pi\right\} \\[2ex] \gamma_2 = \dfrac{1}{\upsilon}\left\{\sin^{-1}\left(\dfrac{\upsilon^2-2\omega+1}{2\upsilon(1-\omega)}\right)+2n\pi\right\} \end{cases} \tag{2.14}$$

For each $\upsilon \in [2\omega-1, 1]$ these equations determine the values of γ_1 and γ_2. At the initial point $\upsilon = 2\omega - 1$, we have

$$\frac{\upsilon^2+2\omega-1}{2\upsilon\omega} = 1 \text{ and } \frac{\upsilon^2-2\omega+1}{2\upsilon(1-\omega)} = -1$$

and if $\upsilon = 1$, then

$$\frac{\upsilon^2+2\omega-1}{2\upsilon\omega} = 1 \text{ and } \frac{\upsilon^2-2\omega+1}{2\upsilon(1-\omega)} = 1.$$

Therefore the starting point and end point of $L_1(k,n)$ are given as

$$\gamma_1^s = \frac{1}{2\omega-1}\left(\frac{\pi}{2}+2k\pi\right),\ \gamma_2^s = \frac{1}{2\omega-1}\left(\frac{3\pi}{2}+2n\pi\right)$$

and

$$\gamma_1^e = \frac{\pi}{2}+2k\pi \text{ and } \gamma_2^e = \frac{\pi}{2}+2n\pi.$$

Similarly, the starting and end points of $L_2(k,n)$ are as follows:

$$\gamma_1^S = \frac{1}{2\omega-1}\left(\frac{\pi}{2}+2k\pi\right),\ \gamma_2^S = \frac{1}{2\omega-1}\left(-\frac{\pi}{2}+2n\pi\right)$$

and

$$\gamma_1^E = \frac{\pi}{2}+2k\pi \text{ and } \gamma_2^E = \frac{\pi}{2}+2n\pi.$$

Figure 1 illustrates the loci $L_1(k,n)$ and $L_2(k,n)$ of the corresponding points (γ_1,γ_2), when υ increases from $2\omega-1$ to unity. The parameter value $\omega = 0.8$ is selected. The red curves show $L_1(0,n)$ and the blue curves show $L_2(0,n)$ with $n = 0,1,2,....$ Notice that γ_2^S is infeasible at $n = 0$ and only the segment of $L_2(0,0)$ between $\upsilon = \sqrt{2\omega-1}$ and $\upsilon = 1$ is feasible. With fixed value of k, $L_1(k,n)$ and $L_2(k,n)$ have the same end point, however the starting point of $L_1(k,n)$ is the same as that of $L_2(k,n+1)$. Therefore the segments $L_1(k,n)$ and $L_2(k,n)$ with fixed k form a continuous curve with $n = 0,1,2,....$

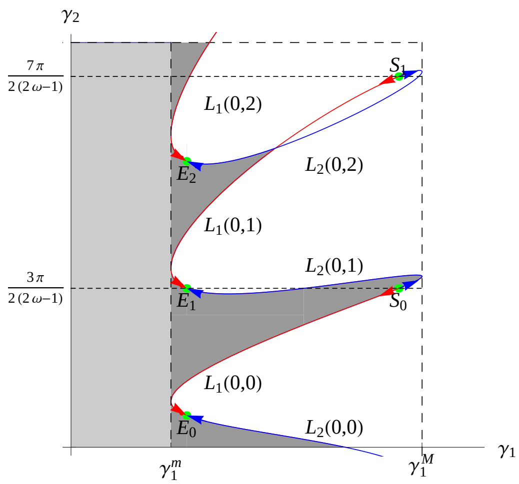

Figure 1. Partition curve in the (γ_1, γ_2) plane with fixing $k = 0$.

Consider first the segment $L_1(k,n)$. Since $\left(\upsilon^2 - 2\omega + 1\right)/(2\upsilon(1-\omega))$ is strictly increasing in υ, γ_2 is strictly decreasing in υ. By differentiation and substitution of equation (2.4), we have

$$\begin{aligned} \left.\frac{\partial\gamma_1}{\partial\upsilon}\right|_{L_1} &= -\tfrac{1}{\upsilon^2}\left(\sin^{-1}\left(\tfrac{\upsilon^2+2\omega-1}{2\upsilon\omega}\right) + 2k\pi\right) + \frac{1}{\upsilon\sqrt{1-\left(\frac{\upsilon^2+2\omega-1}{2\upsilon\omega}\right)^2}}\frac{2\upsilon(2\upsilon\omega)-(\upsilon^2+2\omega-1)2\omega}{2^2\upsilon^2\omega^2} \\ &= -\frac{1}{\upsilon^2}\upsilon\gamma_1 + \frac{1}{\upsilon\cos(\upsilon\gamma_1)}\frac{\upsilon^2-2\omega+1}{2\upsilon^2\omega} \\ &= -\tfrac{1}{\upsilon^2}\left(\upsilon\gamma_1 + \tan(\upsilon\gamma_2)\right). \end{aligned} \tag{2.15}$$

Consider next segment $L_2(k,n)$, similarly to (2.15) we can shown that

$$\left.\frac{\partial\gamma_1}{\partial\upsilon}\right|_{L_2} = -\frac{1}{\upsilon^2}(\upsilon\gamma_1 + \tan(\upsilon\gamma_2))$$

which is the same as in $L_1(k,n)$, since from (2.14), $\cos(\upsilon\gamma_1) < 0$. Similarly

$$\left.\frac{\partial\gamma_2}{\partial\upsilon}\right|_{L_2} = -\frac{1}{\upsilon^2}\left(\upsilon\gamma_2 + \tan(\upsilon\gamma_1)\right) \tag{2.16}$$

where we used again equation (2.4).

In order to visualize the curves $L_1(k,n)$ and $L_2(k,n)$, we change the coordinates (γ_1,γ_2) to $(\upsilon\gamma_1, \upsilon\gamma_2)$ to get the transformed segments:

$$\ell_1(k,n): \begin{cases} \upsilon\gamma_1 = \sin^{-1}\left(\dfrac{\upsilon^2+2\omega-1}{2\upsilon\omega}\right) + 2k\pi \\ \\ \upsilon\gamma_2 = \pi - \sin^{-1}\left(\dfrac{\upsilon^2-2\omega+1}{2\upsilon(1-\omega)}\right) + 2n\pi \end{cases} \tag{2.17}$$

and

$$\ell_2(k,n):\begin{cases}\upsilon\gamma_1=\pi-\sin^{-1}\left(\dfrac{\upsilon^2+2\omega-1}{2\upsilon\omega}\right)+2k\pi\\[2ex]\upsilon\gamma_2=\sin^{-1}\left(\dfrac{\upsilon^2-2\omega+1}{2\upsilon(1-\omega)}\right)+2n\pi\end{cases}\tag{2.18}$$

They also form a continuous curve with each fixed value of k, and they are periodic in both directions $\upsilon\gamma_1$ and $\upsilon\gamma_2$. Figure 2 shows them with $k=0$ where the curves $\ell_1(0,n)$ are shown in red color while the curves $\ell_2(0,n)$ with blue color.

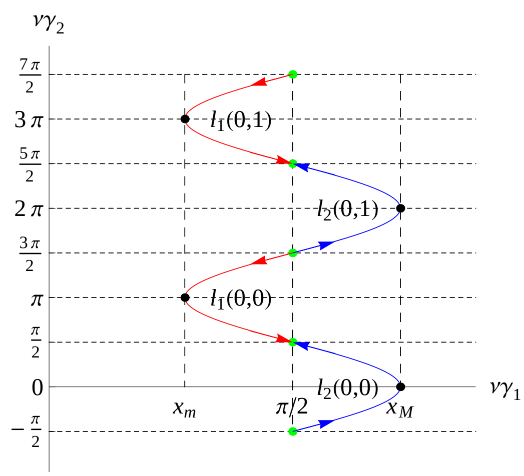

Figure 2. Partition curve in the $(\upsilon\gamma_1,\upsilon\gamma_2)$ plane with fixing $k=0$

We will next examine the directions of the stability switches on the different segments of the curves $L_1(k,n)$ and $L_2(k,n)$. We fix the value of γ_2 and select γ_1 as the bifurcation parameter, so the eigenvalues are functions of $\gamma_1:\bar{\lambda}=\lambda(\gamma_1)$. By differentiating the characteristic equation (2.3) implicitly with respect to γ_1 we have

$$\frac{d\bar{\lambda}}{d\gamma_1}+\omega e^{-\bar{\lambda}\gamma_1}(-\frac{d\bar{\lambda}}{d\gamma_1}\gamma_1-\bar{\lambda})+(1-\omega)e^{-\bar{\lambda}\gamma_2}\left(-\frac{d\bar{\lambda}}{d\gamma_1}\gamma_2\right)=0$$

implying that

$$\frac{d\bar{\lambda}}{d\gamma_1}=\frac{\bar{\lambda}\omega e^{-\bar{\lambda}\gamma_1}}{1-\omega\gamma_1e^{-\bar{\lambda}\gamma_1}-(1-\omega)\gamma_2e^{-\bar{\lambda}\gamma_2}}\tag{2.19}$$

From (2.3) we have

$$(1-\omega)e^{-\bar{\lambda}\gamma_2}=-\bar{\lambda}-\omega e^{-\bar{\lambda}\gamma_1},$$

so

$$\frac{d\bar{\lambda}}{d\gamma_1}=\frac{\bar{\lambda}\omega e^{-\bar{\lambda}\gamma_1}}{1+\bar{\lambda}\gamma_2+\omega(\gamma_2-\gamma_1)e^{-\bar{\lambda}\gamma_1}}$$

If $\bar{\lambda} = \iota\upsilon$, then
$$\frac{d\bar{\lambda}}{d\gamma_1} = \frac{i\upsilon\omega(\cos(\upsilon\gamma_1) - i\sin(\upsilon\gamma_1))}{1 + i\upsilon\gamma_2 + \omega(\gamma_2 - \gamma_1)(\cos(\upsilon\gamma_1) - i\sin(\upsilon\gamma_1))}$$
and the real part of this expression has the same sign as
$$\begin{aligned}\upsilon\omega\sin(\upsilon\gamma_1)[1 + \omega(\gamma_2 - \gamma_1)\cos(\upsilon\gamma_1)] + \upsilon\omega\cos(\upsilon\gamma_1)[\upsilon\gamma_2 - \omega(\gamma_2 - \gamma_1)\sin(\upsilon\gamma_1)] \\ = \upsilon\omega\,[\sin(\upsilon\gamma_1) + \upsilon\gamma_2\cos(\upsilon\gamma_1)]\end{aligned}$$

Hence
$$\operatorname{Re}\left(\frac{d\bar{\lambda}}{d\gamma_1}\right) \gtreqless 0 \text{ if and only if } \sin(\upsilon\gamma_1) + \upsilon\gamma_2\cos(\upsilon\gamma_1) \gtreqless 0$$
Consider first the case of crossing any segment $L_1(k,n)$ from the left. Here $\upsilon\gamma_1 \in (0, \pi/2]$, so both $\sin(\upsilon\gamma_1)$ and $\cos(\upsilon\gamma_2)$ are positive. Hence stability is lost everywhere on any segment of $L_1(k,n)$. Consider the case when crossing the segments of $L_2(k,n)$ from the left. Here $\upsilon\gamma_1 \in [\pi/2, \pi]$, so $\cos(\upsilon\gamma_1) < 0$. Combining (2.16) and the conditions for the sign of $\operatorname{Re}[d\bar{\lambda}/d\gamma_1]$, we have that
$$\operatorname{Re}\left(\frac{d\bar{\lambda}}{d\gamma_1}\right) \gtreqless 0 \text{ if and only if } \frac{\partial\gamma_2}{\partial\upsilon} \gtreqless 0.$$
That is, stability is lost when γ_2 increases in υ and stability is gained when γ_2 decreases in υ. We can also show that at any intercept with $L_1(k,n)$ or $L_2(k,n)$ the complex root is single. Otherwise $\lambda = i\upsilon$ would satisfy both equations
$$\lambda + \omega e^{-\lambda\gamma_1} + (1-\omega)e^{-\lambda\gamma_2} = 0$$
and
$$1 - \omega\gamma_1 e^{-\lambda\gamma_1} - (1-\omega)\gamma_2 e^{-\lambda\gamma_2} = 0,$$
from which we have
$$e^{-\lambda\gamma_1} = \frac{1 + \lambda\gamma_2}{(\gamma_1 - \gamma_2)\omega} \text{ and } e^{-\lambda\gamma_2} = \frac{-1 - \lambda\gamma_1}{(\gamma_1 - \gamma_2)(1-\omega)}.$$
By substituting $\lambda = i\upsilon$ and comparing the real and imaginary parts yield
$$\sin(\upsilon\gamma_1) + \upsilon\gamma_2\cos(\upsilon\gamma_1) = \sin(\upsilon\gamma_2) + \upsilon\gamma_1\cos(\upsilon\gamma_2) = 0.$$
Therefore this intercept is at an extremum in υ of a segment $L_1(k,n)$ and also at an extremum of a segment $L_2(\bar{k},\bar{n})$ which is impossible.

For each $\gamma_2 > 0$, define
$$m(\gamma_2) = \min_{\gamma_1}\{(\gamma_1,\gamma_2) \in L_1(k,n) \cup L_2(k,n),\ \ k,n \geq 0\} \tag{2.20}$$
At $\gamma_1 = 0$ the system is asymptotically stable with all $\gamma_2 > 0$. With fixed value of γ_2 by increasing the value of γ_1 the first intercept with $m(\gamma_2)$ should be a stability loss, since there is no stability switch before. Then by increasing the value of γ_1 further, the next intercept is either a stability

gain or a stability loss. In the first case the equilibrium becomes asymptotically stable. In the second case the equilibrium remains unstable, which will not change even if the next intercept is an stability gain, since the real part of only one eigenvalue becomes negative.

Consider next a point (γ_1^*, γ_2^*) with $\gamma_1^*,\ \gamma_2^* > 0$ which is not located on any curve $L_1(k,n)$ or $L_2(k,n)$, and consider the horizontal line $\gamma_2 = \gamma_2^*$ and its segment with $\gamma_1 \in (0, \gamma_1^*)$. If it has no stability switch, then the equilibrium is asymptotically stable. This is the case even if the number of stability losses equals the number of stability gains, otherwise the equilibrium is unstable. The stability region is shown as the shaded region in Figure 1. Notice that this is the same result which was obtained earlier by Hale and Huang (1993) by using different approach.

3 The Symmetric Case

Assume next that $\omega = 1/2$. Then equations (2.4) and (2.5) imply that

$$\cos(\upsilon\gamma_1) + \cos(\upsilon\gamma_2) = 0$$

$$\upsilon - \tfrac{1}{2}\left(\sin(\upsilon\gamma_1) + \sin(\upsilon\gamma_2)\right) = 0$$

and the curves $L_1(k,n)$ and $L_2(k,n)$ are simplified as follows:

$$L_1(k,n) : \begin{cases} \gamma_1 = \dfrac{1}{\upsilon}\left(\sin^{-1}(\upsilon) + 2k\pi\right) \\ \\ \gamma_2 = \dfrac{1}{\upsilon}\left(\pi - \sin^{-1}(\upsilon) + 2n\pi\right) \end{cases} \tag{3.1}$$

and

$$L_2(k,n) : \begin{cases} \gamma_1 = \dfrac{1}{\upsilon}\left(\pi - \sin^{-1}(\upsilon) + 2k\pi\right) \\ \\ \gamma_2 = \dfrac{1}{\upsilon}\left(\sin^{-1}(\upsilon) + 2n\pi\right) \end{cases} \tag{3.2}$$

which are shown in Figure 3. The same argument as shown above for the nonsymmetric case can be applied here as well to show that stability region is left of $L_1(0,0)$ and below $L_2(0,0)$, where the shape of the stability region differs from that of the nonsymmetric case. It is illustrated in Figure 3 by the shaded domain.

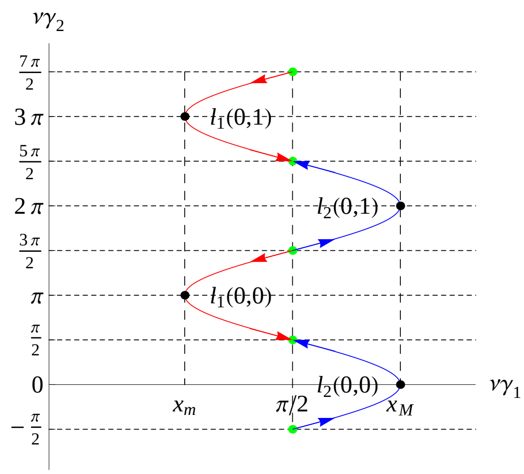

Figure 3. Partition curve in the (γ_1, γ_2) plane with $\omega = \frac{1}{2}$

Notice that at each segment of $\ell_2(k,n)$ there are at most two intercepts with the $\upsilon\gamma_2 = -\tan(\upsilon\gamma_1)$ curve, so the same holds for $L_2(k,n)$. At every other point $\text{Re}[d\bar{\lambda}/d\gamma_1] \neq 0$, so at these points Hopf bifurcation occurs giving the possibility of the birth of limit cycles.

4 Conclusions

Ordinary differential equation were examined with two delays. After finding the possible stability switches, their curves were determined. Hopf bifurcation was used to find segments with stability losses and stability gains. The boundary of the stability region are the $\gamma_2 = 0$, $\gamma_1 = 0$ and a continuous curve consisting of certain portions of the segments $L_1(0,n)$ and $L_2(0,n)$. All other points on the curves $L_1(k,n)$ and $L_2(k,n)$ for $k \geq 1$ do not lead to actual stability switches, since the system is already unstable.

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