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Fractional Order Differential Inclusions via the Topological Transversality Method

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ABSTRACT

The aim of this paper is to present new results on the existence of solutions for a class of boundary value problem for differential inclusions involving the Caputo fractional derivative. Our approach is based on the topological transversality method.

RESUMEN

El objetivo de este trabajo es presentar nuevos resultados sobre la existencia de soluciones para una clase de problemas de contorno para inclusiones diferenciales derivados de la participación de Caputo fraccionada. Nuestro enfoque se basa en el método de la transversalidad topológica.

Keywords and phrases: Fractional differential inclusions; fixed point, Caputo fractional derivative, existence, topological transversality theorem.

Mathematics Subject Classification: 26A33, 26A42, 34A60, 34B15.

1 Introduction

This paper deals with the existence of solutions for boundary value problems (BVP for short) for fractional order differential inclusion of the form

$$^{c}D^{\alpha}y(t) \in F(t, y(t)), \quad t \in J := [0, T], \quad 1 < \alpha \leq 2, \tag{1.1}$$

$$y(0) = y_0, \quad y(T) = y_T \tag{1.2}$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative, $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a Carathéodory multifunction, $y_0, y_T \in \mathbb{R}^n$. Here $\mathcal{P}(\mathbb{R}^n)$ denotes the family of all nonempty subsets of $\mathcal{P}(\mathbb{R}^n)$.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [11, 16, 20, 24, 27, 28]). There has been a significant development in fractional differential equations and inclusions in recent years; see the monographs of Kilbas *et al* [21], Lakshmikantham *et al.* [22], Miller and Ross [25], Podlubny [28], Samko *et al* [29] and the survey by Agarwal *et al.* [1], Benchohra *et al.* [5, 6, 7], Chang and Nieto [10], Diethelm *et al* [11, 12], Ouahab [26], Yu and Gao [30] and Zhang [31] and the references therein. Very recently, in [4, 8] the authors studied the existence and uniqueness of solutions of some classes of functional differential equations with infinite delay and fractional order, and in [3] a class of perturbed functional differential equations involving the Caputo fractional derivative has been considered. These papers have relied on different methods such as Banach fixed point theorem, Schaefer's theorem, Leray-Schauder nonlinear alternative.

In this paper we use a powerful method due to Granas [17] to prove the existence of solution to BVP (1.1)-(1.2). Granas' method is commonly known as topological transversality and relies on the idea of an essential map. The method has been highly useful for proving existence of solutions for initial and boundary value problem for integer order differential equations, see for example [9, 14, 18, 19].

This paper is organized as follows: in Section 2 we introduce some backgrounds on fractional calculus and the topological transversality theorem. In Section 3 we present our main results and an illustrative example will be presented in Section 4. This paper initiates the application of the topological transversality method to boundary value problems for fractional order differential inclusions.

2 Preliminaries

We now introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper.
We denote by $\|y\|$ the norm of any element $y \in \mathbb{R}^n$.

$C(J, \mathbb{R}^n)$ is the Banach space of all continuous functions from J into $\mathbb{R}^n$ with the usual norm

$$|y|_\infty = \sup\{|y(t)| : 0 \le t \le T\}.$$

$AC^1(J, \mathbb{R}^n)$ is the space of differentiable functions $y : J \to \mathbb{R}^n$, whose first derivative, y' is absolutely continuous.
$L^1(J, \mathbb{R}^n)$ denote the Banach space of functions $y : J \longrightarrow \mathbb{R}^n$ that are Lebesgue integrable with the norm

$$\|y\|_{L^1} = \int_0^T \|y(t)\| dt.$$

2.1 Some Properties of Fractional Calculus

Definition 1. ([21, 28]). Given an interval $[a, b]$ of $\mathbb{R}$. The fractional (arbitrary) order integral of the function $h \in L^1([a, b], \mathbb{R})$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds,$$

where Γ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = [h * \varphi_\alpha](t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \le 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 2. ([21]). For a given function h on the interval $[a, b]$, the Caputo fractional-order derivative of h, is defined by

$$({}^cD_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s) ds,$$

where $n = [\alpha] + 1$.

Lemma 3. (Lemma 2.22 [21]). Let $\alpha > 0$, then the differential equation

$${}^cD^\alpha h(t) = 0$$

has solutions

$$h(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}, c_i \in \mathbb{R}, \quad i = 0, 1, 2, \ldots, n-1, \quad n = [\alpha] + 1.$$

Lemma 4. (Lemma 2.22 [21]). Let $\alpha > 0$, then

$$I^\alpha \; {}^cD^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1},$$

for arbitrary $c_i \in \mathbb{R}$, $\quad i = 0, 1, 2, \ldots, n-1, \quad n = [\alpha] + 1$.

2.2 Set-valued maps.

Let X and Y be Banach spaces. A set-valued map $G : X \to \mathcal{P}(Y)$ is said to be compact if $G(X) = \overline{\bigcup\{G(y); y \in X\}}$ is compact. G has convex (closed, compact) values if $G(y)$ is convex(closed, compact) for every $y \in X$. G is bounded on bounded subsets of X if $G(B)$ is bounded in Y for every bounded subset B of X. A set-valued map G is upper semicontinuous (usc for short) at $z_0 \in X$ if for every open set O containing Gz_0, there exists a neighborhood V of z_0 such that $G(V) \subset O$. G is usc on X if it is usc at every point of X if G is nonempty and compact-valued then G is usc if and only if G has a closed graph. The set of all bounded closed convex and nonempty subsets of X is denoted by $bcc(X)$. A closed valued set-valued map $G : J \to \mathcal{P}(X)$ is measurable if for each $y \in X$, the function $t \mapsto \mathrm{dist}(y, G(t))$ is measurable on J. If $X \subset Y$, G has a fixed point if there exists $y \in X$ such that $y \in Gy$. Also, $\|G(y)\|_{\mathcal{P}} = \sup\{|x|; x \in G(y)\}$.

Definition 5. A multivalued map $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is said to be L^1-Carathéodory if

(i) $t \longmapsto F(t, y)$ is measurable for each $x \in \mathbb{R}^n$;

(ii) $y \longmapsto F(t, y)$ is upper semicontinuous for almost each $t \in J$;

(iii) for each $q > 0$, there exists $\varphi_q \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t, y)\|_{\mathcal{P}} = \sup\{\|v\| : v \in F(t, y)\} \leq \varphi_q(t) \ \text{ for all } \|y\| \leq q \text{ and for } a.e.\ t \in [0, 1].$$

For each $y \in C(J, \mathbb{R}^n)$, define the set of selections of F by

$$S^1_{F,y} = \{v \in L^1(J, \mathbb{R}^n) : v(t) \in F(t, y(t)) \ a.e.\ t \in J\},$$

denotes the set of selections of F.

Remark 6. Note that for an L^1-Carathéodory multifunction $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ the set $S^1_{F,y}$ is not empty (see [23]).

For more details on set-valued maps we refer to [2, 13].

2.3 Topological transversality theory.

Let X be a Banach space, C a convex subset of X and U an open subset of C. $K_{\partial U}(\bar{U}, \mathcal{P}(C))$ denotes the set of all set-valued maps $G : \overline{U} \to \mathcal{P}(C)$ which are compact, usc with closed convex values and have no fixed points on ∂U (i.e., $u \in Gu$ for all $u \in \partial U$). A compact homotopy is a set-valued map $H : [0, 1] \times \bar{U} \to \mathcal{P}(C)$ which is compact, usc with closed convex values.

If $u \in H(\lambda, u)$ for every $\lambda \in [0, 1]$, $u \in \partial U$, H is said to be fixed point free on ∂U.

Two set-valued maps $F, G \in K_{\partial U}(\bar{U}, \mathcal{P}(C))$ are called homotopic in $K_{\partial U}(\bar{U}, \mathcal{P}(C))$ if there exists a compact homotopy $H : [0, 1] \times \bar{U} \to \mathcal{P}(C)$ which is fixed point free on ∂U and such that $H(0, .) = F$ and $H(1, .) = G$. The function $G \in K_{\partial U}(\bar{U}, \mathcal{P}(C))$ is called essential if every $F \in K_{\partial U}(\bar{U}, \mathcal{P}(C))$ such that $G|_{\partial U} = F|_{\partial U}$, has a fixed point. Otherwise G is called inessential.

Theorem 7. [17] Let $G : \bar{U} \to \mathcal{P}(C)$ be the constant set-valued map $G(u) \equiv u_0$. Then, if $u_0 \in U$, G is essential.

Theorem 8. (Topological transversality theorem) [17]. Let F, G be two homotopic maps in $K_{\partial U}(\bar{U}, \mathcal{P}(C)))$. Then F is essential if and only if G is essential.

For further details of the Topological Transversality Theory we refer the reader to [18].

3 Main results

In this section, we are concerned with the existence of solutions for the problem (1.1)-(1.2). Consider the following spaces

$$AC^1_B(J, \mathbb{R}^n) = \{y \in AC^1(J, \mathbb{R}^n);\ y(0) = y_0,\ \ y(T) = y_T\},$$

$$AC^{1,\alpha}(J, \mathbb{R}^n) = \{y \in AC^1_B(J, \mathbb{R}^n);\ \int_0^T |{}^cD^\alpha y(t)|dt < \infty\}.$$

$AC^{1,\alpha}(J, \mathbb{R}^n)$ is a Banach space with norm

$$\|y\|_{AC^{1,\alpha}} = \max\{\|y\|_\infty, \|y'\|_\infty, \|{}^cD^\alpha y\|_{L^1}\}.$$

For the existence of solutions for the problem (1.1)-(1.2), we have the following result which is useful in what follows.

Definition 9. A function $y \in AC^{1,\alpha}(J, \mathbb{R}^n)$ is said a solution to BVP (1.1)-(1.2) if there exists a function $v \in L^1(J, \mathbb{R})$ with $v(t) \in F(t, y(t))$, for a.e. $t \in J$, such that
${}^cD^\alpha y(t) = v(t)$, a.e $t \in J$, $1 < \alpha \leq 2$, and the function y satisfies condition (1.2).

Let $h : J \to \mathbb{R}^n$ be continuous, and consider the linear fractional order differential equation

$$ {}^cD^\alpha y(t) = h(t), \quad t \in J, \quad 1 < \alpha \leq 2. \tag{3.1}$$

We shall refer to (3.1)-(1.2) as (LP). For the existence of solutions for the problem (1.1)-(1.2), we have the following result which is useful in what follows.

Lemma 10. Let $1 < \alpha \leq 2$ and let $h : J \to \mathbb{R}^n$ be continuous. A function y is a solution of the fractional integral equation

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^T G(t,s)h(s)ds + y_0 + \frac{(y_T - y_0)t}{T}, \tag{3.2}$$

if and only if y is a solution of (LP), where $G(t,s)$ is the Green's function defined by

$$G(t,s) = \begin{cases} (t-s)^{\alpha-1} - \frac{t(T-s)^{\alpha-1}}{T}, & 0 \leq s \leq t \leq T, \\ \frac{-t(T-s)^{\alpha-1}}{T}, & 0 \leq t \leq s \leq T. \end{cases} \tag{3.3}$$

Proof. Assume y satisfies (3.1), then Lemma 4 implies that

$$y(t) = c_0 + c_1 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds.$$

From (1.2), a simple calculation gives

$$c_0 = y_0,$$

and

$$c_1 = -\frac{1}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s) ds + \frac{y_T - y_0}{T}.$$

Hence we get equation (3.2). Inversely, it is clear that if y satisfies equation (3.2), then equations (3.1)-(1.2) hold.

Our main result is

Theorem 11. Assume the following hypotheses hold:

(A_1) The function $F : J \times \mathbb{R}^n \to bcc(\mathbb{R}^n)$ is a L^1-Carathéodory multi-valued map;

(A_2) There exist a function $p \in L^1(J, \mathbb{R}_+)$, and a continuous nondecreasing function $\psi : [0, \infty) \longrightarrow (0, \infty)$, such that

$$\|F(t, y)\|_{\mathcal{P}} \le p(t)\psi(\|y\|) \text{ for each } (t, y) \in J \times \mathbb{R}^n;$$

(A_3) $\displaystyle\limsup_{r\to+\infty} \frac{r}{\psi(r)} = +\infty.$

Then, the fractional BVP (1.1)-(1.2) has a least one solution on J.

Proof. This proof will be given in several steps.

Step 1: Consider the set-valued operator $\mathcal{F} : C(J, \mathbb{R}^n) \to \mathcal{P}(L^1(J, \mathbb{R}^n))$ defined by

$$(\mathcal{F}y)(t) = F(t, y(t)).$$

$\mathcal{F}$ is well defined, upper semicontinuous, with convex values and sends bounded subsets of $C(J, \mathbb{R}^n)$ into bounded subsets of $L^1(J, \mathbb{R}^n)$. In fact, we have

$$\mathcal{F}y := \{u : J \to \mathbb{R}^n, \text{ measurable } u(t) \in F(t, y(t)), \ a.e. \ t \in J\}.$$

Let $z \in C(J, \mathbb{R}^n)$. and $u \in \mathcal{F}z$. Then

$$\|u(t)\| \le p(t)\psi(\|z(t)\|) \le p(t)\psi(\|z\|_0).$$

Hence $\|u\|_{L^1} \le k_0 := \|p\|_{L^1}\psi(\|z\|_0)$. This shows that $\mathcal{F}$ is well defined.
It is clear that $\mathcal{F}$ is convex valued.

Now, let B be a bounded subset of $C(J, \mathbb{R}^n)$. Then, there exists $k > 0$ such that $\|u\|_0 \leq k$ for $u \in B$. So, for $w \in \mathcal{F}u$ we have $\|w\|_{L^1} \leq k_1$, where $k_1 = \|p\|_{L^1}\psi(k)$.
Also, we can argue as in [15] to show that $\mathcal{F}$ is usc.

Step 2: *A priori bounds on solutions.*

We shall show that if y be a possible solution of (1.1)-(1.2), then there exists a positive constant R^*, independent of y, such that

$$\|y\|_{AC^{1,\alpha}} \leq R^*.$$

Let y be a possible solution of (1.1)-(1.2), by Lemma 10, there exits $v \in S^1_{F,y}$ such that, for each $t \in J$

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^T G(t,s)v(s)ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{1}{T} y_T, \tag{3.4}$$

where G is given by (3.3). Let

$$G_0 := \sup\{\|G(t,s)\|;\ (t,s) \in J \times J\},$$

$$p_0 = \sup\{p(t) : t \in J\}.$$

Hence for $t \in J$

$$\begin{aligned} \|y(t)\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^T G(t,s)\|v(s)\|ds - \left(\frac{t}{T} - 1\right) \|y_0\| + \frac{1}{T}\|y_T\| \\ &\leq \frac{G_0}{\Gamma(\alpha)} \int_0^T p(s)\psi(\|y(s)\|)ds + \left|\left(\frac{t}{T} - 1\right)\right| \|y_0\| + \frac{1}{T}\|y_T\| \\ &\leq \frac{G_0}{\Gamma(\alpha)} \int_0^T p(s)\psi(\|y(s)\|)ds + \|y_0\| + \frac{1}{T}\|y_T\|. \end{aligned}$$

Since ψ is nondecreasing we have

$$\|y\|_\infty \leq \frac{G_0\psi(\|y\|_\infty)p_0T}{\Gamma(\alpha)} + \|y_0\| + \frac{1}{T}\|y_T\|.$$

Thus

$$\frac{\|y\|_\infty}{\psi(\|y\|_\infty)} \leq \frac{G_0p_0T}{\Gamma(\alpha)} + \frac{\|y_0\|}{\psi(\|y\|_\infty)} + \frac{\|y_T\|}{T\psi(\|y\|_\infty)} = \tilde{R}$$

So

$$\frac{\|y\|_\infty}{\psi(\|y\|_\infty)} \leq \tilde{R}. \tag{3.5}$$

Now, the condition ψ in (A_3) shows that there exists $R_1^* > 0$ such that for all $R > R_1^*$

$$\frac{R}{\psi(R)} > \tilde{R}. \tag{3.6}$$

Comparing these last two inequalities (3.5) and (3.6) we see that $R_0 \leq R_1^*$. Consequently, we obtain

$$\|y(t)\| \leq R_1^* \ \text{ for all } t \in J.$$

From (3.4) we have for each $t \in J$

$$y'(t) = \frac{1}{\Gamma(\alpha)} \int_0^T \frac{\partial G(t,s)}{\partial t} f(s, y(s)) ds - \frac{y_0}{T}, \tag{3.7}$$

Using a similar argument as before we can show that there exists $R_2^* > 0$ such that

$$\|y'(t)\| \leq R_2^* \quad \text{for all } t \in J. \tag{3.8}$$

Now from (1.1) and (A_1) we have

$$\int_0^T \|{}^c D^\alpha y(t)\| dt \leq \psi(R_1^*) \int_0^T p(s) ds := R_3^*. \tag{3.9}$$

Hence

$$\|y\|_{AC^{1,\alpha}} \leq \max\{R_1^*, R_2^*, R_3^*\} := R^*.$$

Step 3: *Existence of solutions.*

For $0 \leq \lambda \leq 1$ consider the one-parameter family of problems

$${}^c D^\alpha y(t) \in \lambda F(t, y(t)), \text{ a.e. } t \in J, \quad 1 < \alpha \leq 2, \quad (1_\lambda)$$

$$y(0) = \lambda y_0, \quad y(T) = \lambda y_T \qquad (2_\lambda)$$

which reduces to (1.1)-(1.2) for $\lambda = 1$. For $0 \leq \lambda \leq 1$, we define the operator $\mathcal{F}_\lambda : C(J, \mathbb{R}^n) \to \mathcal{P}(L^1(J, \mathbb{R}^n))$ by $(\mathcal{F}_\lambda y)(t) = \lambda F(t, y(t))$.

Step 1 shows that $\mathcal{F}_\lambda$ is usc, has convex values and sends bounded subsets of $C(J, \mathbb{R}^n)$ into bounded subsets of $L^1(J, \mathbb{R}^n)$ and if y is a solution of $(1_\lambda)-(2_\lambda)$ for some $\lambda \in [0, 1]$, then $\|y\|_{AC^{1,\alpha}} \leq R^*$, where R^* does not depend on λ.

For $\lambda \in [0, 1]$, we define the operators

$$\mathcal{J} : AC^{1,\alpha}(J, \mathbb{R}^n) \to C(J, \mathbb{R}^n) \text{ by } (\mathcal{J}y)(t) = y(t),$$

$$L : AC^{1,\alpha}(J, \mathbb{R}^n) \to L^1(J, \mathbb{R}^n) \text{ by } (Ly)(t) = {}^c D^\alpha y(t).$$

It is clear that $\mathcal{J}$ is continuous and completely continuous and L is linear, continuous and has a bounded inverse denoted by L^{-1}. Let

$$V := \{y \in AC^{1,\alpha}(J, \mathbb{R}^n); \|y\|_{AC^{1,\alpha}} < R^* + 1\}.$$

Define a map $H : [0, 1] \times \overline{V} \to AC^{1,\alpha}(J, \mathbb{R}^n)$ by

$$H(\lambda, y) = (L^{-1} \circ \mathcal{F}_\lambda \circ \mathcal{J})(y).$$

We can show that the fixed points of $H(\lambda,\cdot)$ are solutions of $(1_\lambda)-(2_\lambda)$. Moreover, H is a compact homotopy between $H(0,\cdot)\equiv 0$ and $H(1,\cdot)$. In fact, H is compact since $\mathcal{J}$ is completely continuous, $\mathcal{F}_\lambda$ is continuous and L^{-1} is continuous. Since solutions of (1_λ) satisfy

$$\|y\|_{AC^{1,\alpha}} \leq R^*$$

we see that $H(\lambda,.)$ has no fixed points on ∂V. Now $H(0,\cdot)$ is essential by Theorem 7. Hence by Theorem 8, $H(1,\cdot)$ is essential. This implies that $L^{-1}\circ\mathcal{F}_1\circ\mathcal{J}$ has a fixed point which is a solution to problem (1.1)-(1.2).

4 An Example

As an application of our results we consider the following boundary value problem

$$^cD^\alpha y(t) \in F(t,y), \quad t\in J:=[0,1], \quad 1<\alpha\leq 2, \tag{4.1}$$

$$y(0)=1, \quad y(1)=2, \tag{4.2}$$

where $^cD^\alpha$ is the Caputo fractional derivative. Set

$$F(t,y)=\{v\in\mathbb{R} : f_1(t,y)\leq v\leq f_2(t,y)\},$$

where $f_1,f_2 : J\times\mathbb{R}\to\mathbb{R}$ are measurable in t. We assume that for each $t\in J$, $f_1(t,\cdot)$ is lower semi-continuous (i.e, the set $\{y\in\mathbb{R} : f_1(t,y)>\mu\}$ is open for each $\mu\in\mathbb{R}$), and assume that for each $t\in J$, $f_2(t,\cdot)$ is upper semi-continuous (i.e the set $\{y\in\mathbb{R} : f_2(t,y)<\mu\}$ is open for each $\mu\in\mathbb{R}$). Assume that there exists $p\in L^1(J,\mathbb{R}^+)$ and $\delta\in(0,1)$ such that

$$\max(|f_1(t,y)|,\ |f_2(t,y)|)\leq p(t)|y|^\delta, \quad t\in J, \text{ and all } y\in\mathbb{R}.$$

It is clear that F is compact and convex valued, and it is upper semi-continuous (see [13]). Since assumptions $(A_1)-(A_3)$ hold, Theorem 11 implies that the BVP (4.1)-(4.2) has at least one solution.

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