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Some New Characterizations for $\mathsf{PGL}(2,q)$

B. Khosravi [1,2], M. Khatami[2] and Z. Akhlaghi[2]
[1] *School of Mathematics,*
Institute for Research in Fundamental Sciences (IPM),
P.O. Box: 19395-5746, Tehran, Iran.
email: `khosravibbb@yahoo.com`
and
[2] *Dept. of Pure Math., Faculty of Math. and Computer Sci.,*
Amirkabir University of Technology (Tehran Polytechnic),
424, Hafez Ave., Tehran 15914, IRAN.

ABSTRACT

Many authors introduced some characterizations for finite groups. In this paper as the main result we prove that the finite group $\mathsf{PGL}(2,q)$ is uniquely determined by its noncommuting graph. Also we prove that $\mathsf{PGL}(2,q)$ is characterizable by its noncyclic graph. Throughout the proof of these results we prove that $\mathsf{PGL}(2,q)$ is uniquely determined by its order components and using this fact we give positive answer to a conjecture of Thompson and another conjecture of Shi and Bi for the group $\mathsf{PGL}(2,q)$.

RESUMEN

Muchos autores introdujeron algunas caracterizaciones de los grupos finitos. En este trabajo como principal resultado se demuestra que grupo finito $\mathsf{PGL}(2,q)$ es determinado nicamente por su gráfica no conmutativa. También se demuestra que $\mathsf{PGL}(2,q)$

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es caracterizable por su gráfico no cíclico. A lo largo de la prueba de estos resultados se demuestra que $PGL(2, Q)$ es determinado únicamente por los componentes de su orden y con ello damos respuesta positiva a una conjetura de Thompson y otra conjetura de Shi Bi y para el grupo $PGL(2, q)$.

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1 Introduction

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. If G is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$. We construct the *prime graph* of G which is denoted by $\Gamma(G)$ as follows: the vertex set is $\pi(G)$ and two distinct primes p and q are joined with an edge if and only if G contains an element of order pq. Let $t(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_1, \pi_2, ..., \pi_{t(G)}$ be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$, then we assume that $2 \in \pi_1$.

Now we can express $|G|$ as a product of coprime natural numbers m_i, such that $1 \leq i \leq t(G)$ and $\pi(m_i) = \pi_i$. These integers are called *order components* of G. The set of order components of G is denoted by $OC(G)$.

One of the other graphs which associated with a non-abelian group G is the noncommuting graph that is denoted by $\nabla(G)$ and is constructed as follows: the vertex set of $\nabla(G)$ is $G \setminus Z(G)$ with two vertices x and y are joined by an edge whenever the commutator of x and y is not identity. In [1] the authors put forward the following conjecture:

Conjecture A. Let S be a finite non-abelian simple group and G be a finite group such that $\nabla(G) \cong \nabla(S)$. Then $G \cong S$.

The validity of this conjecture has been proved for all simple groups with non-connected prime graphs. Also it is proved that some finite simple groups with connected prime graphs, say A_{10}, $U_4(7)$, $L_4(8)$, $L_4(4)$ and $L_4(9)$, can be uniquely determined by their noncommuting graghs (see [19, 20, 21, 22]).

In this paper as the main result we prove that the almost simple group $PGL(2, q)$, where $q = p^n$ for a prime number p and a natural number n, is characterizable by its noncommuting graph. As a consequence of our results we prove the validity of a conjecture of Thompson and another conjecture of Shi and Bi for the group $PGL(2, q)$.

Let G be a noncyclic group and $Cyc(G) = \{x \in G | \langle x, y\rangle \text{ is cyclic for all } y \in G\}$. In [2], the authors introduced the cyclic graph of G, which is denoted by $\Gamma_1(G)$ as follows: take $G \setminus Cyc(G)$ as the vertex set and join two vertices if they do not generate a cyclic subgroup. In this graph the degree of each vertex x is equal to $|G| \setminus |Cyc_G(x)|$, where $Cyc_G(x) = \{y \in G | \langle x, y\rangle \text{ is cyclic}\}$. It is proved that some finite simple groups, S_n, D_{2^k}, D_{2n} ,where n is odd, are characterizable by the noncyclic graph. We show that $\mathsf{PGL}(2, q)$ is uniquely determined by its noncyclic graph.

In this paper, all groups are finite and by simple groups we mean non-abelian simple groups. All further unexplained notations are standard and refer to [6], for example.

2. Preliminary results

In this section we bring some preliminary lemmas which are necessary in the proof of the main theorem.

Remark 2.1. Let N be a normal subgroup of G and p, q be incident vertices of $\Gamma(G/N)$. Then p, q are incident in $\Gamma(G)$. In fact if xN is of order pq, then there exists a power of x which is of order pq.

Definition 2.2. ([8]) A finite group G is called a 2-Frobenius group if it has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where K and G/H are Frobenius groups with kernels H and K/H, respectively.

Lemma 2.3. Let G be a Frobenius group of even order and let H, K be Frobenius complement and Frobenius kernel of G, respectively. Then $t(G) = 2$, and the prime graph components of G are $\pi(H)$, $\pi(K)$ and G has one of the following structures:
(a) $2 \in \pi(K)$ and all Sylow subgroups of H are cyclic;
(b) $2 \in \pi(H)$, K is an abelian group, H is a solvable group, the Sylow subgroups of odd order of H are cyclic groups and the 2-Sylow subgroups of H are cyclic or generalized quaternion groups;
(c) $2 \in \pi(H)$, K is an abelian group and there exists $H_0 \leq H$ such that $|H : H_0| \leq 2$, $H_0 = Z \times SL(2, 5)$, $\pi(Z) \cap \{2, 3, 5\} = \emptyset$ and the Sylow subgroups of Z are cyclic.

Also the next lemma follows from [8] and the properties of Frobenius groups [9]:
Lemma 2.4. Let G be a 2-Frobenius group, i.e., G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K and G/H are Frobenius groups with kernels H and K/H, respectively. Then

(a) $t(G) = 2$, $\pi_1 = \pi(G/K) \cup \pi(H)$ and $\pi_2 = \pi(K/H)$;

(b) G/K and K/H are cyclic, $|G/K| \mid (|K/H| - 1)$ and $G/K \leq Aut(K/H)$;

(c) H is nilpotent and G is a solvable group.

Lemma 2.5. ([4, Lemma 8]) Let G be a finite group with $t(G) \geq 2$ and let N be a normal subgroup of G. If N is a π_i-group for some prime graph component of G and $m_1, m_2, \ldots, m_r$ are some order components of G but not π_i-numbers, then $m_1 m_2 \cdots m_r$ is a divisor of $|N| - 1$.

Lemma 2.6. ([3, Lemma 1.4]) Suppose G and M are two finite groups satisfying $t(M) \geq 2$, $N(G) = N(M)$, where $N(G)=\{n \mid G$ has a conjugacy class of size n $\}$, and $Z(G) = 1$. Then

$|G| = |M|$.

Lemma 2.7. ([3, Lemma 1.5]) Let G_1 and G_2 be finite groups satisfying $|G_1| = |G_2|$ and $N(G_1) = N(G_2)$. Then $t(G_1) = t(G_2)$ and $OC(G_1) = OC(G_2)$.

Lemma 2.8. ([11]) Let G be a finite group and M be a finite group with $t(M) = 2$ satisfying $OC(G) = OC(M)$. Let $OC(M) = \{m_1, m_2\}$. Then one of the following holds:
(a) G is a Frobenius or 2-Frobenius group;
(b) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/K is a π_1-group, H is a nilpotent π_1-group, and K/H is a non-abelian simple group. Moreover $OC(K/H) = \{m'_1, m'_2, \ldots, m'_s, m_2\}$, where $m'_1 m'_2 \ldots m'_s | m_1$. Also $G/K \leq Out(K/H)$.

Lemma 2.9. ([1]) Let G be a finite non-abelian group. If H is a group such that $\nabla(G) \cong \nabla(H)$, then H is a finite non-abelian group such that $|Z(H)|$ divides

$$gcd(|G| - |Z(G)|, |G| - |C_G(x)|, |C_G(x)| - |Z(G)| : x \in G \setminus Z(G)).$$

Lemma 2.10. ([18]) Let G be a non-abelian group such that $\nabla(G) \cong \nabla(PSL(2, 2^n))$, where n is a natural number. Then $G \cong PSL(2, 2^n)$.

Lemma 2.11.([7, Remark 1]) The equation $p^m - q^n = 1$, where p and q are primes and $m, n > 1$ has only one solution, namely $3^2 - 2^3 = 1$.

Lemma 2.12. ([2]) Let G be a finite noncyclic group. If H is a group such that $\Gamma_1(G) \cong \Gamma_1(H)$, then H is a finite noncyclic group such that $|Cyc(H)|$ divides

$$gcd(|G| - |Cyc(G)|, |G| - |Cyc_G(x)|, |Cyc_G(x)| - |Cyc(G)| : x \in G \setminus Cyc(G)).$$

Lemma 2.13. ([2]) Let G and H be two finite noncyclic groups such that $\Gamma_1(G) \cong \Gamma_1(H)$. If $|G| = |H|$, then $\pi_e(G) = \pi_e(H)$.

3. Main Results

We note that if $q = 2^n$, then $PGL(2, q) = PSL(2, q)$ and we know that $PSL(2, q)$ is characterizable by its noncommuting graph (see [18]). Therefore throughout this section we suppose M is the almost simple group $PGL(2, q)$, where $q = p^n$ for an odd prime number p and a natural number n.

Theorem 3.1. Let G be a group such that $\nabla(G) \cong \nabla(M)$. Then $|G| = |M|$.

Proof. First note that G is a finite non-abelian group. Since $\nabla(G) \cong \nabla(M)$, we have $|G| - |Z(G)| = |M| - |Z(M)|$. Then it is enough to prove that $|Z(G)| = |Z(M)|$.

By Lemma 2.9, $|Z(G)|$ divides $|M| - |Z(M)|$. Since $|Z(M)| = 1$, we have $|Z(G)|$ divides $q(q^2 - 1) - 1$. Let P be a Sylow p-subgroup of M. We know that $Z(P) \neq 1$. So there exists $1 \neq x \in Z(P)$.

We claim that $C_M(x) = P$. It is obvious that $P \leq C_M(x)$, since $x \in Z(P)$. On the contrary we suppose that $y \in C_M(x) \setminus P$. So we can conclude that $o(xy) = o(x)o(y)$. Without lose of generality we suppose $|y| = r$, where $r \neq p$ is a prime number. Then M has an element of order rp. But p is an isolated vertex in $\Gamma(M)$, a contradiction. Therefore our claim is proved.

By Lemma 2.9 we have $|Z(G)|$ divides $|C_M(x)| - |Z(M)|$. Then $|Z(G)|$ divides $q - 1$. We know that $Z(G)$ divides $q(q^2 - 1) - 1$, which implies that $|Z(G)| = 1$ and so $|G| = |M|$. $\Box$

Theorem 3.2. Let G be a group such that $\nabla(G) \cong \nabla(M)$, where $M = PGL(2, q)$. Then $OC(G) = OC(M)$.

Proof. Since $\nabla(G) \cong \nabla(M)$, the set of vertex degrees of two graphs are the same. Therefore

$$\{|G| - |C_G(x)| \mid x \in G\} = \{|M| - |C_M(y)| \mid y \in M\}.$$

On the other hand Theorem 3.1 implies that $|G| = |M|$, and so $N(G) = N(M)$. Now using Lemma 2.7 we have $OC(G) = OC(M)$. $\Box$

Theorem 3.3. Let G be a finite group and $OC(G) = OC(M)$. If $q = p^n \neq 3$ then G is neither a Frobenius group nor a 2-Frobenius group. If $q = 3$ and G is a 2-Frobenius group, then $G \cong S_4$.

Proof. If G is a Frobenius group, then by Lemma 2.3, $OC(G) = \{|H|, |K|\}$ where K and H are Frobenius kernel and Frobenius complement of G, respectively. Therefore $OC(G) = \{q, q^2 - 1\}$ and since $|H| \mid (|K| - 1)$ it follows that $|H| < |K|$ and so $|H| = q$ and $|K| = q^2 - 1$. Also $q \mid (q^2 - 2)$ implies that $q = 2$, which is a contradiction, since q is odd. Therefore G is not a Frobenius group.

Let G be a 2-Frobenius group. Hence $G = ABC$, where A and AB are normal subgroups of G; AB and BC are Frobenius groups with kernels A, B and complements B, C, respectively. By Lemma 2.4, we have $|B| = q$ and $|A||C| = q^2 - 1$. Also $|B| \mid (|A| - 1)$ and so $|A| = qt + 1$, for some $t > 0$. On the other hand, $|A| \mid (q^2 - 1)$, which implies that $q^2 - 1 = k(qt + 1)$, for some $k > 0$. Therefore $q \mid (k + 1)$ and so $q - 1 \leq k$. If $t > 1$, then $q^2 - 1 = k(qt + 1) \geq (q - 1)(qt + 1) > (q - 1)(q + 1)$, which is a contradiction. Hence $t = 1$ and $|A| = q + 1$ and $|C| = q - 1$.

If there exists an odd prime r such that $r \mid (q + 1)$, then let R be a Sylow r-subgroup of A. Since A is a nilpotent group, it follows that R is a normal subgroup of G. Now Lemma 2.5, implies that $q \mid (|R| - 1)$ and $|R| \mid (q + 1)/2$, which is a contradiction. Therefore $q + 1 = 2^\alpha$, for some $\alpha > 0$. Similarly $Z(A) \neq 1$ is a characteristic subgroup of A and hence A is abelian. Let $X = \{x \in A | o(x) = 2\} \cup \{1\}$. Then X is a non-identity characteristic subgroup of A. Therefore A is an elementary abelian 2-subgroup of G and $|A| = 2^\alpha = q + 1$. By Lemma 2.11, if $q = p^n$ such that $n > 1$, then the equation $2^\alpha - q = 1$ does not have any solution.

Now let $n = 1$. Suppose $F = GF(2^{\alpha})$ and so A is the additive group of F. Also $|B| = q = p = 2^{\alpha} - 1$ and so B is the multiplicative group of F. Now C acts by conjugation on A and similarly C acts by conjugation on B and this action is faithful. Therefore C keeps the structure of the field F and so C is isomorphic to a subgroup of the automorphism group of F. Hence $|C| = 2^{\alpha} - 2 \leq |Aut(F)| = \alpha$. Therefore $\alpha \leq 2$. If $\alpha = 2$, then $G = S_4$, the symmetric group on 4 letters. $\square$

Lemma 3.4. Let G be a finite group and $M = PGL(2, q)$, where $q > 3$ or $q = 3$ and M is not a 2-Frobenius group. If $OC(G) = OC(M)$, then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H is a simple group. Moreover the odd order component of M is equal to an odd order component of K/H. In particular, $t(K/H) \geq 2$. Also $|G/H|$ divides $|Aut(K/H)|$, and in fact $G/H \leq Aut(K/H)$.

Proof. The first part of the lemma follows from Lemma 2.8 and Theorem 3.3, since the prime graph of G has two components. If K/H has an element of order pq, where p and q are primes, then by Remark 2.1, K has an element of order pq. Therefore G has an element of order pq. So by the definition of prime graph component, the odd order component of G is equal to an odd order component of K/H. Also $K/H \trianglelefteq G/H$ and $C_{G/H}(K/H) = 1$, which implies that

$$G/H = \frac{N_{G/H}(K/H)}{C_{G/H}(K/H)} \cong T \quad , \quad T \leq Aut(K/H). \ \square$$

Theorem 3.5. Let G be a finite group such that $OC(G) = OC(M)$, where $M = PGL(2, q)$. Then $G \cong PGL(2, q)$.

Proof. If $q = 3$ and G is a 2-Frobenius group, then Theorem 3.3 implies that $G = S_4 \cong PGL(2, 3)$, as desired. Otherwise Lemma 3.4 implies that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H is a simple subgroup and $t(K/H) \geq 2$.

Now using the classification of finite simple groups and the results in Tables 1-3 in [10], we consider the following cases.

Case 1. Let $K/H \cong A_m$, where $m = p', p' + 1$ or $p' + 2$ and $p' \geq 5$ is a prime number and m and $m - 2$ are not primes at the same time.

Then $q = p'$, and consequently $n = 1$ and $q = p = p'$. On the other hand, $|A_m| \mid |G| = p(p^2 - 1)$. If $m > p$, then $|A_m| > (p + 1)p(p - 1)$, which is a contradiction. Therefore $m = p$ and $|A_p| \mid |G| = p(p^2 - 1)$, and so $|A_p| = p!/2 \leq p(p^2 - 1)$. Hence $(p - 2)!/2 \leq p + 1$. But $p \geq 7$, since $p - 2$ is not a prime. So $(p - 2)(p - 3) < (p - 2)!/2 \leq p + 1$, which is a contradiction. This completes the proof.

Case 2. Let $K/H \cong A_{p'}$, where p' and $p' - 2$ are primes.

If $p = p'$, for $p' \geq 7$, then we can get a contradiction similarly to the previous case. So $p = 5$ and $K/H \cong A_5 \cong PSL(2,5)$. Since $K/H \leq G/H \leq Aut(K/H)$, we have $PSL(2,5) \leq G/H \leq PGL(2,5)$. Hence G/H is isomorphic to $PSL(2,5)$ or $PGL(2,5)$. If $G/H \cong PSL(2,5)$, then $|H| = 2$. But $H \trianglelefteq G$, which implies that $H \subseteq Z(G)$ and we get a contradiction. So $G/H \cong PGL(2,5)$, which implies that $H = 1$ and $G \cong PGL(2,5)$.

Let $p = p' - 2$. Since $p' \mid |A_{p'}|$, we have $p' \mid |G| = p(p^2 - 1)$. But we know that $p = p' - 2$ is the greatest prime divisor of $|G|$, which is a contradiction.

Case 3. Let K/H be a sporadic simple group.

Using the tables in [10] we see that the odd order components of sporadic simple groups are prime.

Let S be a sporadic simple group and $K/H \cong S$. Since q is equal to the greatest odd order component of K/H, we have $q = m_i$, such that $m_i = max\{m_2, m_3, ..., m_{t(S)}\}$. So q is a prime number.

If $S = J_4$, then $q = p = 43$. Since $11^2 \mid |K/H|$, we have $11^2 \mid (p^2 - 1) = 43^2 - 1$, which is a contradiction.

If $S = Co_2$, then $q = p = 23$. Since $7 \mid |K/H|$, we have $7 \mid (23^2 - 1)$, which is a contradiction.

The proof of other cases are similar and we omit them for convenience.

If K/H is isomorphic to ${}^2A_3(2)$, ${}^2F_4(2)'$, $A_2(4)$, ${}^2A_5(2)$, $E_7(2)$, $E_7(3)$ or ${}^2E_6(2)$, then similarly we get a contradiction.

In the sequel of the proof we consider simple groups of Lie type. Since the proofs of these cases are similar we state only a few of them.

In all of the following cases p' is an odd prime number and q' is a prime power.

Case 4. Let $K/H \cong A_{p'-1}(q')$, where $(p', q') \neq (3,2), (3,4)$. By hypothesis we have $q = (q'^{p'} - 1)/((q' - 1)(p', q' - 1))$. Hence $q < q'^{p'} - 1 < q'^{p'}$. Then $q^2 - 1 < q'^{2p'}$. On the other hand, we know $q'^{p'(p'-1)/2} \mid (q^2 - 1)$ and therefore $q'^{p'(p'-1)/2} < q'^{2p'}$. So $p'(p' - 1)/2 < 2p'$ and hence $p' < 5$. So $p' = 3$ and $q = (q'^2 + q' + 1)/(3, q' - 1)$, which implies that $q < 2q'^2$. Therefore $q^2 - 1 < 4q'^4 - 1$. On the other hand, $q'^3(q'^2 - 1)(q' - 1) \mid (q^2 - 1)$ and consequently $q'^3(q'^2 - 1)(q' - 1) < 4q'^4 - 1$. So $q' = 2, 3$ or 4. Since $(p', q') \neq (3,2), (3,4)$, we have $q' = 3$ and $q = 13$. Then $3^3(3^2 - 1)(3 - 1) \mid (13^2 - 1)$, which is a contradiction.

Case 5. Let $K/H \cong {}^2A_{p'}(q')$, where $(q' + 1) \mid (p' + 1)$ and $(p', q') \neq (3,3), (5,2)$. In this case we have $q = (q'^{p'} + 1)/(q' + 1)$. Therefore $q < q'^{p'} + 1 < 2q'^{p'} \leq q'^{p'+1}$ and hence $q^2 - 1 < q'^{2(p'+1)}$. On the other hand, we have $q'^{p'(p'+1)/2} \mid (q^2 - 1)$. So we conclude that $q'^{p'(p'+1)/2} < q'^{2(p'+1)}$. Hence $p'(p' + 1)/2 < 2(p' + 1)$, which implies that $p' = 3$. Then $(q' + 1) \mid 4$ and hence $q' = 3$. So $(p', q') = (3,3)$, which is impossible.

Case 6. Let $K/H \cong B_n(q')$, where $n = 2^m \geq 4$ and q' is odd. Therefore $q = (q'^n + 1)/2$. So $q < 2q'^n < q'^{n+1}$. Therefore $q^2 - 1 < q'^{2(n+1)}$. On the other hand, we have $q'^{n^2} \mid (q^2 - 1)$ and consequently $q'^{n^2} < q'^{2(n+1)}$. So $n^2 < 2(n + 1)$, which implies that $n = 2$, and this is a contradiction.

Case 7. Let $K/H \cong C_n(q')$, where $n = 2^m \geq 2$. Then $q = (q'^n + 1)/(2, q' - 1)$. Therefore $q \leq q'^n + 1 < 2q'^n \leq q'^{n+1}$, which implies that $q^2 - 1 < q'^{2(n+1)}$. On the other hand, we have $q'^{n^2} \mid (q^2 - 1)$, which implies that $q'^{n^2} < q'^{2(n+1)}$. So we have $n^2 < 2(n+1)$ and hence $n = 2$. Therefore $q < 2q'^2$ and so $q'^4(q'^2 - 1) < q^2 - 1 < 4q'^4$, which is impossible.

Case 8. Let $K/H \cong {}^2D_{p'}(3)$, where $p' = 2^n + 1 \geq 5$. So we have $q = (3^{p'} + 1)/4$ or $q = (3^{p'-1} + 1)/2$.

If $q = (3^{p'} + 1)/4$, then $q < 3^{p'+1}$. On the other hand, we have $3^{p'(p'-1)} \mid (q^2 - 1)$, which implies that $3^{p'(p'-1)} \leq q^2 - 1 < 3^{2(p'+1)}$. Therefore $p'(p' - 1) < 2(p' + 1)$, and hence $p' \leq 3$, which is impossible.

If $q = (3^{p'-1} + 1)/2$, then $q < 3^{p'}$. On the other hand, $3^{p'(p'-1)} \mid (q^2 - 1)$, which implies that $3^{p'(p'-1)} < 3^{2p'}$, and so $p'(p' - 1) < 2p'$, which is impossible.

Case 9. Let $K/H \cong {}^2B_2(q')$, where $q' = 2^{2n+1} > 2$. In this case we have $q = q' \pm \sqrt{2q'} + 1$ or $q = q' - 1$.

If $q = q' \pm \sqrt{2q'} + 1$, then $q^2 - 1 = q'^2 + 4q' \pm 2\sqrt{2q'}(q' + 1)$. On the other hand, we have $q'^2 \mid (q^2 - 1)$ and so $q' \mid (q'^2 + 4q' \pm 2\sqrt{2q'}(q' + 1))$, which implies that $q' \leq 2\sqrt{2q'}$. Hence $q'^2 \leq 8q'$. Therefore $q' = 8$ and so $q = 5$ or 13, which is a contradiction by $q'^2 \mid (q^2 - 1)$.

If $q = q' - 1$, then $q'^2 | (q'^2 - 2q')$, which is a contradiction.

Case 10. Let $K/H \cong {}^2F_4(q')$, where $q' = 2^{2n+1} > 2$. In this case we have $q = q'^2 \pm \sqrt{2q'^3} + q' \pm \sqrt{2q'} + 1$. Therefore $q < 4q'^2 < q'^3$ and so $q^2 - 1 < q'^6$. On the other hand, $q'^{12} \mid (q^2 - 1)$, which is a contradiction.

Case 11. Let $K/H \cong A_1(q')$, where $4|q'$. By hypothesis we have $q = q' - 1$ or $q = q' + 1$.

If $q = q' - 1$, then $q^2 - 1 = q'^2 - 2q'$. But we know $q'(q' + 1) \mid (q^2 - 1)$, which is a contradiction.

If $q = q' + 1$, then $q^2 - 1 = q'^2 + 2q'$. Since $q'(q' - 1) \mid (q^2 - 1)$, we conclude that $(q' - 1) \mid 3$. So $q' = 4$ and hence $K/H \cong A_1(4) \cong A_5$. By the proof of Case 2 we have $K/H \cong \mathrm{PGL}(2, 5)$.

Case 12. If $K/H \cong A_1(q')$, where $4|(q' - 1)$, then $q = (q' + 1)/2$ or $q = q'$.

If $q = (q' + 1)/2$, then $q^2 - 1 = (q'^2 - 3 + 2q')/4$. On the other hand, $q'(q' - 1) \mid (q^2 - 1)$

and hence $q'(q'-1) \le (q'^2-3+2q')/4$. So $q'^2-2q'+1 \le 0$, which is a contradiction.

If $q = q'$, then $K/H \cong A_1(q) = \mathsf{PSL}(2,q)$. Since $K/H \le G/H$ and $|G| = 2|\mathsf{PSL}(2,q)|$, we conclude that $|H| = 1$ or 2. Let $|H| = 2$. Since $H \trianglelefteq G$ we have $H \subseteq Z(G)$, which is a contradiction. So $H = 1$.

By Lemma 2.8, $G/K \le \mathrm{Out}(K/H)$ and $|G/K| = 2$. If G/K contains a field automorphism, then $2p \in \pi_e(G)$, which is a contradiction. If G/K contains a diagonal-field automorphism, then G is the non-split extension of $\mathsf{PSL}(2,q)$ by $\mathbb{Z}_2$ and we know that the prime graph of G is the prime graph of $\mathsf{PSL}(2,q)$ (see [12]), which is a contradiction. So a diagonal automorphism generates G/K and consequently $G \cong \mathsf{PGL}(2,q)$.

If $K/H \cong A_1(q')$, where $4|(q'+1)$, then similarly we conclude that $G \cong \mathsf{PGL}(2,q)$. □

Theorem 3.6. Let G be a group such that $\nabla(G) \cong \nabla(M)$, where $M = \mathsf{PGL}(2,q)$ and q is a prime power. Then $G \cong M$.

Proof. If $q = 2^n$, where n is an integer, then $\mathsf{PGL}(2,q) \cong \mathsf{PSL}(2,q)$ and so Lemma 2.10 implies that $G \cong M$. If q is odd, then obviously the theorem follows from Theorems 3.2 and 3.5. □

Remark 3.7. It is a well known conjecture of J. G. Thompson that if G is a finite group with $Z(G) = 1$ and M is a non-abelian simple group satisfying $N(G) = N(M)$, then $G \cong M$.

We can give a positive answer to this conjecture for the group $\mathsf{PGL}(2,q)$ by our characterization of this group.

Corollary 3.8. Let G be a finite group with $Z(G) = 1$ and $M = \mathsf{PGL}(2,q)$, where q is a prime power. If $N(G) = N(M)$, then $G \cong M$.

Proof. By Lemmas 2.6 and 2.7, if G and M are two finite groups satisfying the conditions of Corollary 3.8, then $OC(G) = OC(M)$. So using Theorem 3.5 we get the result. □

Remark 3.9. W. Shi and J. Bi in [16] put forward the following conjecture:
Conjecture. Let G be a group and M be a finite simple group. Then $G \cong M$ if and only if
(i) $|G| = |M|$, and,
(ii) $\pi_e(G) = \pi_e(M)$, where $\pi_e(G)$ denotes the set of orders of elements in G.

This conjecture is valid for sporadic simple groups [13], alternating groups [17], and some simple groups of Lie type [14, 15, 16]. As a consequence of Theorem 3.5, we prove the validity of this conjecture for the almost simple group $\mathsf{PGL}(2,q)$, where q is a prime power.

Corollary 3.10. Let G be a finite group and $M = \mathsf{PGL}(2,q)$, where q is a prime power. If

$|G| = |M|$ and $\pi_e(G) = \pi_e(M)$, then $G \cong M$.

Proof. By assumption we have $OC(G) = OC(M)$. Thus the corollary follows from Theorem 3.5. □

Proposition 3.11. Let G be a group such that $\Gamma_1(G) \cong \Gamma_1(M)$, where $M = PGL(2,q)$ and q is a prime power. Then $G \cong M$.

proof. First we show that $|G| = |M|$. By Lemma 2.12 we have $|Cyc(G)|$ divides $|M| - |Cyc(M)|$. Since $Cyc(M) \leq Z(M) = 1$, it follows that $|Cyc(G)|$ divides $|M|-1$. On the other hand, by Lemma 2.12, $|Cyc(G)|$ divides $|Cyc_M(x)| - |Cyc(M)|$, where $x \in M \setminus Cyc(M)$. Let x be a p-element of M. We claim that $\langle x\rangle = Cyc_M(x)$. We know that $\langle x\rangle \subseteq Cyc_M(x)$ and so it is enough to prove that $Cyc_M(x) \subseteq \langle x\rangle$. On the contrary let $y \in Cyc_M(x) \setminus \langle x\rangle$ and hence $\langle y, x\rangle$ is cyclic. If y is a p-element, then we know that $\langle y, x\rangle$ has only one subgroup of order p and so $\langle x\rangle = \langle y\rangle$, which is a contradiction. Therefore y is not a p-element. So we have an element of order $po(y)$, which is a contradiction by the structure of $\Gamma(M)$. So $p = |\langle x\rangle| = |Cyc_M(x)|$. Therefore $|Cyc(G)|$ divides $p-1$ and $p-1$ divides $|M|$. We know that $|Cyc(G)|$ divides $|M|-1$ and so $|Cyc(G)| = 1$ and $|G| = |M|$. Now using Lemma 2.13 we conclude that $\pi_e(G) = \pi_e(M)$ and by Corollary 3.10 the proof is complete. □

Remark 3.12. We note that in the main theorem of [5] it is proved that $PGL(2,q)$ is uniquely determined by $\pi_e(G)$.

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References

[1] A. Abdollahi, S. Akbari and H. R. Maimani, *Non-commuting graph of a group*, J. Algebra, 298 (2) (2006), 468-492.

[2] A. Abdollahi and Mohammadi Hassanabadi, *Noncyclic graph of a group*, Comm. Algebra, 35 (2007), 1-25.

[3] G. Y. Chen, *On Thompson's conjecture*, J. Algebra, 185 (1) (1996), 184-193.

[4] G. Y. Chen, *Further reflections on Thompson's conjecture*, J. Algebra, 218 (1) (1999), 276-285.

[5] G. Y. Chen, V. D. Mazurov, W. J. Shi, A. V. Vasil'ev and A. Kh. Zhurtov, *Recognition of the finite almost simple groups* $PGL(2,q)$ *by their spectrum*, J. Group Theory, 10 (2007), 71-85.

[6] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups, Oxford University Press*, Oxford (1985).

[7] P. Crescenzo, *A diophantine equation which arises in the theory of finite groups*, Advances in Math., 17 (1) (1975), 25-29.

[8] K. W. Gruenberg and K. W. Roggenkamp, *Decomposition of the augmentation ideal and of the relation modules of a finite group*, Proc. London Math. Soc. (3), 31 (2) (1975), 149-166.

[9] B. Huppert, *Endliche Gruppen I*, Springer Verlag, Berlin, 1967.

[10] A. Iranmanesh, S. H. Alavi and B. Khosravi, *A Characterization of* $\mathsf{PSL}(3, q)$ *where* q *is an odd prime power*, J. Pure Appl. Algebra, 170 (2-3) (2002), 243-254.

[11] A. Khosravi and B. Khosravi, *A new characterization of almost sporadic groups*, J. Algebra Appl., 1 (3) (2002), 267-279.

[12] M. S. Lucido and E. Jabara, *Finite groups with hall covering*, J. Aust. Math. Soc., 78 (1) (2005), 1-16.

[13] W. Shi, *A new characterization of the sporadic simple groups*, Group Theory, Proceeding of the 1987 Singapore Group Theory Conference, Walter de Gruyter, Berlin, New York, 1989, 531-540.

[14] W. Shi, *A new characterization of some simple groups of Lie type*, Contemp. Math., 82 (1989), 171-180.

[15] W. Shi, *Pure quantitative characterization of finite simple groups (I)*, Progr. Natur. Sci., 4 (3) (1994), 316-326.

[16] W. Shi and J. Bi, *A characteristic property for each finite projective special linear group*, Lecture Notes in Math., 1456 (1990), 171-180.

[17] W. Shi and J. Bi, *A new characterization of the alternating groups*, Southeast Asian Bull. Math., 16 (1) (1992), 81-90.

[18] L. Wang and W. J. Shi, *A new characterization of* $L_2(q)$ *by its noncommuting graph*, Front. Math. China, 2 (1) (2007), 143-148.

[19] L. Wang and W. Shi, *A new characterization of* A_{10} *by its noncommuting graph*, Comm. Algebra, 36 (2) (2008), 523-528.

[20] L. C. Zhang, G. Y. Chen, S. M. Chen and X. F. Liu, *Notes on finite simple groups whose orders have three or four prime divisors*, J. Algebra Appl., 8 (3) (2009), 389–399.

[21] L. C. Zhang and W. J. Shi, *Noncommuting graph characterization of some simple groups with connected prime graphs*, Int. Electron. J. Algebra, 5 (2009), 169–181.

[22] L. C. Zhang, W. J. Shi and X. L. Liu, *A characterization of* $L_4(4)$ *by its noncommuting graph*, Chinese Annals of Mathematics, 30A (4) (2009), 517–524. (in chinese)