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$q-$ Fractional Inequalities

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ABSTRACT

Here we give $q-$fractional Poincaré type, Sobolev type and Hilbert-Pachpatte type integral inequalities, involving $q-$fractional derivatives of functions. We give also their generalized versions.

RESUMEN

Estudiamos el tipo $q-$fraccional Poincaré, el tipo Sobolev y el tipo integral de inecuaciones de Hilbert-Pachpatte, involucrando a $q-$fraccional derivados de funciones. Damos también las versiones generalizadas.

Keywords: $q-$fractional derivative, $q-$fractional integral, $q-$fractional Poincaré inequality, $q-$fractional Sobolev inequality, $q-$fractio- nal Hilbert-Pachpatte inequality.

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1 Introduction

Here we follow [4] in all of this section, see also [3].

Let $q \in (0,1)$, we define

$$[\alpha]_q := \frac{1-q^\alpha}{1-q}, \qquad (\alpha \in \mathbb{R}). \tag{1}$$

The $q-$analog of the Pochhammer symbol ($q-$shifted factorial) is defined by:

$$(a;q)_0 = 1, \quad (a;q)_k = \prod_{i=0}^{k-1} \left(1 - aq^i\right) \qquad (k \in \mathbb{N} \cup \{\infty\}).$$

The expansion to reals is

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty} \qquad (\alpha \in \mathbb{R}); \tag{2}$$

also define the $q-$analog

$$(a-b)^{(\alpha)} = a^\alpha \frac{\left(\frac{b}{a};q\right)_\infty}{\left(q^\alpha \frac{b}{a};q\right)_\infty}, \quad a,b \in \mathbb{R},\, a \neq 0.$$

Notice that

$$(a-b)^{(\alpha)} = a^\alpha \left(\frac{b}{a};q\right)_\alpha.$$

The $q-$gamma function is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty} (1-q)^{1-x}, \qquad (x \in \mathbb{R} - \{0,-1,-2,...\}). \tag{3}$$

Clearly

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x). \tag{4}$$

The $q-$derivative of a function $f(x)$ is defined by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{x - qx}, \qquad (x \neq 0), \tag{5}$$

$$(D_q f)(0) = \lim_{x \to 0} (D_q f)(x), \tag{6}$$

and the $q-$derivatives of higher order:

$$D_q^0 f = f, \quad D_q^n f = D_q\left(D_q^{n-1} f\right), \qquad n = 1,2,3,... \tag{7}$$

The $q-$integral is defined by

$$(I_{q,0} f)(x) = \int_0^x f(t)\, d_q t = x(1-q) \sum_{k=0}^\infty f\left(xq^k\right) q^k, \qquad (0 < q < 1), \tag{8}$$

and

$$(I_{q,a} f)(x) = \int_a^x f(t)\, d_q t = \int_0^x f(t)\, d_q t - \int_0^a f(t)\, d_q t. \tag{9}$$

By [2], we see that: if $f(x) \geq 0$, then it is not necessarily true that

$$\int_a^b f(x)\, d_q x \geq 0.$$

In the case of $a = xq^n$, then (9) becomes

$$\int_{xq^n}^x f(t)\, d_q t = x(1-q) \sum_{k=0}^{n-1} f\left(xq^k\right) q^k, \tag{10}$$

see also [2].

Double $q-$integration is defined the usual iterative way.

Also we define

$$I_{q,a}^0 f = f, \quad I_{q,a}^n f = I_{q,a}\left(I_{q,a}^{n-1} f\right), \qquad n = 1, 2, 3, ... \tag{11}$$

The following are valid:

$$\left(D_q I_{q,a} f\right)(x) = f(x), \tag{12}$$

$$\left(I_{q,a} D_q f\right)(x) = f(x) - f(a). \tag{13}$$

Denote

$$\begin{aligned} [n]_q! &= [1]_q [2]_q ... [n]_q, \quad n \in \mathbb{N}; \\ [0]_q! &= 1, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}. \end{aligned}$$

In the next we work on $(0, b)$, $b > 0$, and let $a \in (0, b)$. Also the required $q-$derivatives and $q-$integrals do exist.

Definition 1. The fractional $q-$integral is

$$\left(I_{q,a}^\alpha f\right)(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_a^x \left(q\frac{t}{x}; q\right)_{\alpha-1} f(t)\, d_q t \tag{14}$$

$$= \frac{1}{\Gamma_q(\alpha)} \int_a^x (x - qt)^{(\alpha-1)} f(t)\, d_q t, \qquad \left(a < x,\, \alpha \in \mathbb{R}^+\right).$$

The usual fractional integral (see also [1]) is the limit case of (14) as $q \uparrow 1$, since

$$\lim_{q \uparrow 1} x^{\alpha-1} \left(q\frac{t}{x}; q\right)_{\alpha-1} = (x-t)^{\alpha-1}. \tag{15}$$

Clearly

$$\left(I_{q,a}^\alpha f\right)(a) = 0. \tag{16}$$

We mention

Theorem 2. Let $\alpha, \beta \in \mathbb{R}^+$. The $q-$fractional integration has the semigroup property

$$\left(I_{q,a}^{\beta} I_{q,a}^{\alpha} f\right)(x) = \left(I_{q,a}^{\alpha+\beta} f\right)(x), \qquad (a < x). \tag{17}$$

Corollary 3. For $\alpha \geq n$ $(n \in \mathbb{N})$ it holds

$$\left(D_q^n I_{q,a}^{\alpha} f\right)(x) = \left(I_{q,a}^{\alpha-n} f\right)(x), \qquad (a < x). \tag{18}$$

We mention the fractional $q-$derivative of Caputo type:

Definition 4. The fractional $q-$derivative of Caputo type is

$$\left({}_*D_{q,a}^{\alpha} f\right)(x) = \begin{cases} \left(I_{q,a}^{-\alpha} f\right)(x), & \alpha \leq 0; \\ \left(I_{q,a}^{\lceil\alpha\rceil-\alpha} D_q^{\lceil\alpha\rceil} f(x)\right), & \alpha > 0, \end{cases} \tag{19}$$

where $\lceil . \rceil$ denotes the ceiling of the number.

Next we mention the highlight of this introductory section. Again all here come from [4]. So the following is the fractional $q-$Taylor formula of Caputo type.

Theorem 5. Let $\alpha \in \mathbb{R}^+ - \mathbb{N}$, $a < x$. Then

$$\left(I_{q,a}^{\alpha}\ {}_*D_{q,a}^{\alpha} f\right)(x) = f(x) - \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{\left(D_q^k f\right)(a)}{[k]_q!} x^k \left(\frac{a}{x}; q\right)_k. \tag{20}$$

Also we give

Theorem 6. Let $\alpha \in \mathbb{R}^+ - \mathbb{N}$, $\beta \in \mathbb{R}^+$, $\alpha > \beta > 0$, $a < x$. Then

$$\left(I_{q,a}^{\beta}\ {}_*D_{q,a}^{\alpha} f\right)(x) = \left({}_*D_{q,a}^{\alpha-\beta} f\right)(x) - \tag{21}$$

$$\sum_{k=\lceil\alpha-\beta\rceil}^{\lceil\alpha\rceil-1} \frac{\left(D_q^k f\right)(a)}{\Gamma_q(k-\alpha+\beta+1)} x^{k-\alpha+\beta} \left(\frac{a}{x}; q\right)_{k-\alpha+\beta}.$$

2 Main Results

We need the following $q-$Hölder's inequality.

Proposition 7. Let $x > 0$, $0 < q < 1$; $p_1, q_1 > 1$ such that $\frac{1}{p_1} + \frac{1}{q_1} = 1$; $n \in \mathbb{N}$. Then

$$\int_{xq^n}^{x} |f(t)| |g(t)| d_q t \leq \left(\int_{xq^n}^{x} |f(t)|^{p_1} d_q t\right)^{\frac{1}{p_1}} \left(\int_{xq^n}^{x} |g(t)|^{q_1} d_q t\right)^{\frac{1}{q_1}}. \tag{22}$$

Proof. By the discrete Hölder's inequality we have

$$\int_{xq^n}^{x} |f(t)| |g(t)| d_q t = x(1-q) \sum_{k=0}^{n-1} \left|f\left(xq^k\right)\right| \left|g\left(xq^k\right)\right| q^k =$$

$$x(1-q)\sum_{k=0}^{n-1}\left(\left|f\left(xq^k\right)\right|\left(q^k\right)^{\frac{1}{p_1}}\right)\left(\left|g\left(xq^k\right)\right|\left(q^k\right)^{\frac{1}{q_1}}\right)\leq$$

$$\left(x(1-q)\sum_{k=0}^{n-1}\left|f\left(xq^k\right)\right|^{p_1}q^k\right)^{\frac{1}{p_1}}\left(x(1-q)\sum_{k=0}^{n-1}\left|g\left(xq^k\right)\right|^{q_1}q^k\right)^{\frac{1}{q_1}}=$$

$$\left(\int_{xq^n}^{x}|f(t)|^{p_1}d_qt\right)^{\frac{1}{p_1}}\left(\int_{xq^n}^{x}|g(t)|^{q_1}d_qt\right)^{\frac{1}{q_1}}.$$

□

We present a $q-$fractional Poincaré type inequality.

Theorem 8. Let $x>0$, $0<w\leq x$, $0<q<1$; $\alpha>0$, $p_1,q_1>1$ such that $\frac{1}{p_1}+\frac{1}{q_1}=1$; $n\in\mathbb{N}$. Set

$$\Delta(w):=f(w)-\sum_{k=0}^{\lceil\alpha\rceil-1}\frac{\left(D_q^kf\right)(wq^n)}{[k]_q!}w^k\left(q^n;q\right)_k.$$

Then

$$\int_0^x\frac{|\Delta(w)|^{q_1}}{w^{q_1(\alpha-1)}}d_qw\leq\frac{1}{\left(\Gamma_q(\alpha)\right)^{q_1}}\cdot\left(\int_0^x\left(\int_{wq^n}^{w}\left(\left(q\frac{t}{w};q\right)_{\alpha-1}\right)^{p_1}d_qt\right)^{q_1}d_qw\right)^{\frac{1}{p_1}}$$

$$\cdot\left(\int_0^x\left(\int_{wq^n}^{w}\left|{}_*D_{q,wq^n}^{\alpha}f(t)\right|^{q_1}d_qt\right)^{q_1}d_qw\right)^{\frac{1}{q_1}}. \tag{23}$$

Proof. By $q-$fractional Taylor's formula (20) we get

$$\Delta(w)=\left(I_{q,wq^n}^{\alpha}\ {}_*D_{q,wq^n}^{\alpha}f\right)(w)=\frac{w^{\alpha-1}}{\Gamma_q(\alpha)}\int_{wq^n}^{w}\left(q\frac{t}{w};q\right)_{\alpha-1}\left({}_*D_{q,wq^n}^{\alpha}f\right)(t)\,d_qt. \tag{24}$$

Here by (14) and (19), we see that

$$\left({}_*D_{q,wq^n}^{\alpha}f\right)(t)=\frac{t^{\lceil\alpha\rceil-\alpha-1}}{\Gamma_q\left(\lceil\alpha\rceil-\alpha\right)}\int_{wq^n}^{w}\left(q\frac{s}{t};q\right)_{\lceil\alpha\rceil-\alpha-1}D_q^{\lceil\alpha\rceil}f(s)\,d_q(s), \tag{25}$$

all $wq^n\leq t\leq w$.

Here we observe trivially that

$$\left|\int_{xq^n}^{x}f(t)\,d_qt\right|\leq\int_{xq^n}^{x}|f(t)|\,d_qt. \tag{26}$$

Furthermore we see that

$$\left(q\frac{t}{w};q\right)_{\alpha-1}=\frac{\left(q\frac{t}{w};q\right)_\infty}{\left(q^\alpha\frac{t}{w};q\right)_\infty}=\frac{\prod_{i=0}^{\infty}\left(1-q\frac{t}{w}q^i\right)}{\prod_{i=0}^{\infty}\left(1-q^\alpha\frac{t}{w}q^i\right)}=\frac{\prod_{i=0}^{\infty}\left(1-\frac{t}{w}q^{i+1}\right)}{\prod_{i=0}^{\infty}\left(1-\frac{t}{w}q^{i+\alpha}\right)}>0. \tag{27}$$

Hence by (22) we obtain

$$|\Delta(w)| \leq \frac{w^{\alpha-1}}{\Gamma_q(\alpha)} \int_{wq^n}^{w} \left(q\frac{t}{w};q\right)_{\alpha-1} \left|\left({}_*D^{\alpha}_{q,wq^n} f\right)(t)\right| d_qt \leq$$

$$\frac{w^{\alpha-1}}{\Gamma_q(\alpha)} \left(\int_{wq^n}^{w} \left(\left(q\frac{t}{w};q\right)_{\alpha-1}\right)^{p_1} d_qt\right)^{\frac{1}{p_1}} \cdot \left(\int_{wq^n}^{w} \left|\left({}_*D^{\alpha}_{q,wq^n} f\right)(t)\right|^{q_1} d_qt\right)^{\frac{1}{q_1}}. \tag{28}$$

Consequently we derive

$$\frac{|\Delta(w)|}{w^{\alpha-1}} \leq \frac{1}{\Gamma_q(\alpha)} \left(\int_{wq^n}^{w} \left(\left(q\frac{t}{w};q\right)_{\alpha-1}\right)^{p_1} d_qt\right)^{\frac{1}{p_1}} \cdot \tag{29}$$

$$\left(\int_{wq^n}^{w} \left|\left({}_*D^{\alpha}_{q,wq^n} f\right)(t)\right|^{q_1} d_qt\right)^{\frac{1}{q_1}},$$

and

$$\frac{|\Delta(w)|^{q_1}}{w^{q_1(\alpha-1)}} \leq \frac{1}{(\Gamma_q(\alpha))^{q_1}} \left(\int_{wq^n}^{w} \left(\left(q\frac{t}{w};q\right)_{\alpha-1}\right)^{p_1} d_qt\right)^{\frac{q_1}{p_1}} \cdot \tag{30}$$

$$\left(\int_{wq^n}^{w} \left|\left({}_*D^{\alpha}_{q,wq^n} f\right)(t)\right|^{q_1} d_qt\right).$$

Applying $q-$Hölder's inequality (which is also valid on $[0,x]$) on (30), we see that

$$\int_0^x \frac{|\Delta(w)|^{q_1}}{w^{q_1(\alpha-1)}} d_qw \leq \frac{1}{(\Gamma_q(\alpha))^{q_1}} \cdot$$

$$\int_0^x \left[\left(\int_{wq^n}^{w} \left(\left(q\frac{t}{w};q\right)_{\alpha-1}\right)^{p_1} d_qt\right)^{\frac{q_1}{p_1}} \cdot \left(\int_{wq^n}^{w} \left|\left({}_*D^{\alpha}_{q,wq^n} f\right)(t)\right|^{q_1} d_qt\right)\right] d_qw$$

$$\leq \frac{1}{(\Gamma_q(\alpha))^{q_1}} \cdot \left(\int_0^x \left(\int_{wq^n}^{w} \left(\left(q\frac{t}{w};q\right)_{\alpha-1}\right)^{p_1} d_qt\right)^{q_1} d_qw\right)^{\frac{1}{p_1}} \cdot$$

$$\left(\int_0^x \left(\int_{wq^n}^{w} \left|\left({}_*D^{\alpha}_{q,wq^n} f\right)(t)\right|^{q_1} d_qt\right)^{q_1} d_qw\right)^{\frac{1}{q_1}}, \tag{31}$$

proving the claim. □

Next we give a $q-$fractional Sobolev type inequality.

Theorem 9. Here all terms and assumptions as in Theorem 8. Additionaly let $r_1, r_2 > 1 : \frac{1}{r_1} + \frac{1}{r_2} = 1$. Then

$$\left(\int_0^x \left(\frac{|\Delta(w)|}{w^{\alpha-1}}\right)^{r_1} d_qw\right)^{\frac{1}{r_1}} \leq \frac{1}{\Gamma_q(\alpha)} \cdot$$

$$\left(\int_0^x \left(\int_{wq^n}^w \left(\left(q\frac{t}{w};q\right)_{\alpha-1}\right)^{p_1} d_qt\right)^{\frac{r_1^2}{p_1}} d_qw\right)^{\frac{1}{r_1^2}} \cdot$$

$$\left(\int_0^x \left(\int_{wq^n}^w \left|{}_*D_{q,wq^n}^{\alpha} f\left(t\right)\right|^{q_1} d_qt\right)^{\frac{r_1 r_2}{q_1}} d_qw\right)^{\frac{1}{r_1 r_2}} . \tag{32}$$

Proof. As in the proof of Theorem 8 we get (29), so that

$$\left(\frac{\left|\Delta\left(w\right)\right|}{w^{\alpha-1}}\right)^{r_1} \leq \frac{1}{\left(\Gamma_q\left(\alpha\right)\right)^{r_1}} \left(\int_{wq^n}^w \left(\left(q\frac{t}{w};q\right)_{\alpha-1}\right)^{p_1} d_qt\right)^{\frac{r_1}{p_1}} \cdot \tag{33}$$

$$\left(\int_{wq^n}^w \left|\left({}_*D_{q,wq^n}^{\alpha} f\right)\left(t\right)\right|^{q_1} d_qt\right)^{\frac{r_1}{q_1}} .$$

Hence

$$\int_0^x \left(\frac{\left|\Delta\left(w\right)\right|}{w^{\alpha-1}}\right)^{r_1} d_qw \leq \frac{1}{\left(\Gamma_q\left(\alpha\right)\right)^{r_1}} \cdot \tag{34}$$

$$\int_0^x \left[\left(\int_{wq^n}^w \left(\left(q\frac{t}{w};q\right)_{\alpha-1}\right)^{p_1} d_qt\right)^{\frac{r_1}{p_1}} \cdot \left(\int_{wq^n}^w \left|\left({}_*D_{q,wq^n}^{\alpha} f\right)\left(t\right)\right|^{q_1} d_qt\right)^{\frac{r_1}{q_1}}\right] d_qw$$

(by $q-$Hölder's inequality on $[0,x]$)

$$\leq \frac{1}{\left(\Gamma_q\left(\alpha\right)\right)^{r_1}} \left(\int_0^x \left(\int_{wq^n}^w \left(\left(q\frac{t}{w};q\right)_{\alpha-1}\right)^{p_1} d_qt\right)^{\frac{r_1^2}{p_1}} d_qw\right)^{\frac{1}{r_1}} \cdot \tag{35}$$

$$\left(\int_0^x \left(\int_{wq^n}^w \left|\left({}_*D_{q,wq^n}^{\alpha} f\right)\left(t\right)\right|^{q_1} d_qt\right)^{\frac{r_1 r_2}{q_1}} d_qw\right)^{\frac{1}{r_2}},$$

proving the claim. $\square$

It follows a $q-$fractional Hilbert-Pachpatte type inequality.

Theorem 10. Let for $i = 1, 2$ that $x_i > 0$, $0 < w_i \leq x_i$, $0 < q < 1$; $\alpha > 0$, $p_1, q_1 > 1$ such that $\frac{1}{p_1} + \frac{1}{q_1} = 1$; $n \in \mathbb{N}$. Call

$$\Delta_i\left(w_i\right) = f_i\left(w_i\right) - \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{\left(D_q^k f_i\right)\left(w_i q^n\right)}{[k]_q!} w_i^k \left(q^n;q\right)_k ,$$

$$F\left(w_1\right) = \int_{w_1 q^n}^{w_1} \left(q\frac{t_1}{w_1};q\right)_{\alpha-1} d_qt_1, \tag{36}$$

and

$$G(w_2)=\int_{w_2q^n}^{w_2}\left(q\frac{t_2}{w_2};q\right)_{\alpha-1}^{q_1}d_qt_2.$$

Then

$$\int_0^{x_1}\int_0^{x_2}\frac{|\Delta_1(w_1)|\,|\Delta_2(w_2)|}{(w_1w_2)^{\alpha-1}\left(\frac{F(w_1)}{p_1}+\frac{G(w_2)}{q_1}\right)}d_qw_1d_qw_2\leq \tag{37}$$

$$\frac{x_1^{\frac{1}{p_1}}x_2^{\frac{1}{q_1}}}{(\Gamma_q(\alpha))^2}\left(\int_0^{x_1}\left(\int_{w_1q^n}^{w_1}\left|{}_*D_{q,w_1q^n}^{\alpha}f_1\right|^{q_1}(t_1)\,d_qt_1\right)d_qw_1\right)^{\frac{1}{q_1}}\cdot$$

$$\left(\int_0^{x_2}\left(\int_{w_2q^n}^{w_2}\left|{}_*D_{q,w_2q^n}^{\alpha}f_2\right|^{p_1}(t_2)\,d_qt_2\right)d_qw_2\right)^{\frac{1}{p_1}}.$$

Proof. We notice by (20) that

$$\Delta_i(w_i)=\frac{w_i^{\alpha-1}}{\Gamma_q(\alpha)}\int_{w_iq^n}^{w_i}\left(q\frac{t_i}{w_i};q\right)_{\alpha-1}\left({}_*D_{q,w_iq^n}^{\alpha}f_i\right)(t_i)\,d_qt_i, \tag{38}$$

for $i=1,2$.

Therefore we derive

$$|\Delta_1(w_1)|\leq\frac{w_1^{\alpha-1}}{\Gamma_q(\alpha)}\int_{w_1q^n}^{w_1}\left(q\frac{t_1}{w_1};q\right)_{\alpha-1}\left|\left({}_*D_{q,w_1q^n}^{\alpha}f_1\right)(t_1)\right|d_qt_1\leq$$

$$\frac{w_1^{\alpha-1}}{\Gamma_q(\alpha)}\left(\int_{w_1q^n}^{w_1}\left(q\frac{t_1}{w_1};q\right)_{\alpha-1}^{p_1}d_qt_1\right)^{\frac{1}{p_1}}\cdot\left(\int_{w_1q^n}^{w_1}\left|{}_*D_{q,w_1q^n}^{\alpha}f_1\right|^{q_1}(t_1)\,d_qt_1\right)^{\frac{1}{q_1}}. \tag{39}$$

Similarly we obtain

$$|\Delta_2(w_2)|\leq\frac{w_2^{\alpha-1}}{\Gamma_q(\alpha)}\int_{w_2q^n}^{w_2}\left(q\frac{t_2}{w_2};q\right)_{\alpha-1}\left|\left({}_*D_{q,w_2q^n}^{\alpha}f_2\right)(t_2)\right|d_qt_2\leq$$

$$\frac{w_2^{\alpha-1}}{\Gamma_q(\alpha)}\left(\int_{w_2q^n}^{w_2}\left(\left(q\frac{t_2}{w_2};q\right)_{\alpha-1}\right)^{q_1}d_qt_2\right)^{\frac{1}{q_1}}\cdot\left(\int_{w_2q^n}^{w_2}\left|{}_*D_{q,w_2q^n}^{\alpha}f_2\right|^{p_1}(t_2)\,d_qt_2\right)^{\frac{1}{p_1}}. \tag{40}$$

Consequently we get

$$|\Delta_1(w_1)|\,|\Delta_2(w_2)|\leq\frac{(w_1w_2)^{\alpha-1}}{(\Gamma_q(\alpha))^2}(F(w_1))^{\frac{1}{p_1}}(G(w_2))^{\frac{1}{q_1}}\cdot$$

$$\left(\int_{w_1q^n}^{w_1}\left|{}_*D_{q,w_1q^n}^{\alpha}f_1\right|^{q_1}(t_1)\,d_qt_1\right)^{\frac{1}{q_1}}\cdot\left(\int_{w_2q^n}^{w_2}\left|{}_*D_{q,w_2q^n}^{\alpha}f_2\right|^{p_1}(t_2)\,d_qt_2\right)^{\frac{1}{p_1}} \tag{41}$$

(by Young's inequality)

$$\leq\frac{(w_1w_2)^{\alpha-1}}{(\Gamma_q(\alpha))^2}\left(\frac{F(w_1)}{p_1}+\frac{G(w_2)}{q_1}\right)\cdot$$

$$\left(\int_{w_1q^n}^{w_1}\left|{}_*D_{q,w_1q^n}^{\alpha}f_1\right|^{q_1}(t_1)\,d_qt_1\right)^{\frac{1}{q_1}}\cdot\left(\int_{w_2q^n}^{w_2}\left|{}_*D_{q,w_2q^n}^{\alpha}f_2\right|^{p_1}(t_2)\,d_qt_2\right)^{\frac{1}{p_1}}. \tag{42}$$

Therefore

$$\int_0^{x_1}\int_0^{x_2}\frac{|\Delta_1(w_1)|\,|\Delta_2(w_2)|}{(w_1w_2)^{\alpha-1}\left(\frac{F(w_1)}{p_1}+\frac{G(w_2)}{q_1}\right)}d_qw_1d_qw_2\leq$$

$$\frac{1}{(\Gamma_q(\alpha))^2}\left(\int_0^{x_1}\left(\int_{w_1q^n}^{w_1}\left|{}_*D_{q,w_1q^n}^{\alpha}f_1\right|^{q_1}(t_1)\,d_qt_1\right)^{\frac{1}{q_1}}d_qw_1\right)\cdot$$

$$\left(\int_0^{x_2}\left(\int_{w_2q^n}^{w_2}\left|{}_*D_{q,w_2q^n}^{\alpha}f_2\right|^{p_1}(t_2)\,d_qt_2\right)^{\frac{1}{p_1}}d_qw_2\right)\leq \tag{43}$$

$$\frac{x_1^{\frac{1}{p_1}}x_2^{\frac{1}{q_1}}}{(\Gamma_q(\alpha))^2}\left(\int_0^{x_1}\left(\int_{w_1q^n}^{w_1}\left|{}_*D_{q,w_1q^n}^{\alpha}f_1\right|^{q_1}(t_1)\,d_qt_1\right)d_qw_1\right)^{\frac{1}{q_1}}\cdot$$

$$\left(\int_0^{x_2}\left(\int_{w_2q^n}^{w_2}\left|{}_*D_{q,w_2q^n}^{\alpha}f_2\right|^{p_1}(t_2)\,d_qt_2\right)d_qw_2\right)^{\frac{1}{p_1}}, \tag{44}$$

proving the claim. □

We continue with a generalized $q-$fractional Poincaré type inequality.

Theorem 11. Let $x>0$, $0<w\leq x$, $0<q<1$; $\alpha>\beta>0$, $p_1,q_1>1:\frac{1}{p_1}+\frac{1}{q_1}=1$; $n\in\mathbb{N}$. Set

$$K(w)=\left({}_*D_{q,wq^n}^{\alpha-\beta}f\right)(w)-\sum_{k=\lceil\alpha-\beta\rceil}^{\lceil\alpha\rceil-1}\frac{\left(D_q^kf\right)(wq^n)}{\Gamma_q(k-\alpha+\beta+1)}w^{k-\alpha+\beta}(q^n;q)_{k-\alpha+\beta}.$$

Then

$$\int_0^x\left(\frac{|K(w)|}{w^{\beta-1}}\right)^{q_1}d_qw\leq\frac{1}{(\Gamma_q(\beta))^{q_1}}\cdot \tag{45}$$

$$\left(\int_0^x\left(\int_{wq^n}^w\left(\left(q\frac{t}{w};q\right)_{\beta-1}\right)^{p_1}d_qt\right)^{q_1}d_qw\right)^{\frac{1}{p_1}}\cdot$$

$$\left(\int_0^x\left(\int_{wq^n}^w\left|{}_*D_{q,wq^n}^{\alpha}f(t)\right|^{q_1}d_qt\right)^{q_1}d_qw\right)^{\frac{1}{q_1}}.$$

Proof. By (21) we get

$$K(w)=I_{q,wq^n}^{\beta}\left({}_*D_{q,wq^n}^{\alpha}f\right)(w)=\frac{w^{\beta-1}}{\Gamma_q(\beta)}\int_{wq^n}^w\left(q\frac{t}{w};q\right)_{\beta-1}\left({}_*D_{q,wq^n}^{\alpha}f\right)(t)\,d_qt. \tag{46}$$

Rest of proof goes as in the proof of Theorem 8. □

Next comes a generalized $q-$fractional Sobolev's type inequality.

Theorem 12. Here all terms and assumptions as in Theorem 11. Additionaly let $r_1, r_2 > 1$: $\frac{1}{r_1} + \frac{1}{r_2} = 1$. Then

$$\left(\int_0^x \left(\frac{|K(w)|}{w^{\beta-1}}\right)^{r_1} d_q w\right)^{\frac{1}{r_1}} \le \frac{1}{\Gamma_q(\beta)}. \tag{47}$$

$$\left(\int_0^x \left(\int_{wq^n}^w \left(\left(q\frac{t}{w};q\right)_{\beta-1}\right)^{p_1} d_q t\right)^{\frac{r_1^2}{p_1}} d_q w\right)^{\frac{1}{r_1^2}}.$$

$$\left(\int_0^x \left(\int_{wq^n}^w \left|{}_*D^{\alpha}_{q,wq^n} f(t)\right|^{q_1} d_q t\right)^{\frac{r_1 r_2}{q_1}} d_q w\right)^{\frac{1}{r_1 r_2}}.$$

Proof. As in the Theorem 9, using (46). □

We finish with a generalized $q-$fractional Hilbert-Pachpatte type inequality.

Theorem 13. Let for $i = 1, 2$ that $x_i > 0$, $0 < w_i \le x_i$, $0 < q < 1$; $\alpha > \beta > 0$, $p_1, q_1 > 1$: $\frac{1}{p_1} + \frac{1}{q_1} = 1$; $n \in \mathbb{N}$. Call

$$K_i(w_i) = \left({}_*D^{\alpha-\beta}_{q,w_i q^n} f_i\right)(w_i) - \sum_{k=\lceil\alpha-\beta\rceil}^{\lceil\alpha\rceil-1} \frac{\left(D_q^k f_i\right)(w_i q^n)}{\Gamma_q(k-\alpha+\beta+1)} w_i^{k-\alpha+\beta} (q^n;q)_{k-\alpha+\beta},$$

$$F^*(w_1) = \int_{w_1 q^n}^{w_1} \left(q\frac{t_1}{w_1};q\right)_{\beta-1}^{p_1} d_q t_1, \tag{48}$$

$$G^*(w_2) = \int_{w_2 q^n}^{w_2} \left(q\frac{t_2}{w_2};q\right)_{\beta-1}^{q_1} d_q t_2.$$

Then

$$\int_0^{x_1}\int_0^{x_2} \frac{|K_1(w_1)|\,|K_2(w_2)|}{(w_1 w_2)^{\beta-1}\left(\frac{F^*(w_1)}{p_1} + \frac{G^*(w_2)}{q_1}\right)} d_q w_1 d_q w_2 \le \frac{x_1^{\frac{1}{p_1}} x_2^{\frac{1}{q_1}}}{(\Gamma_q(\beta))^2}. \tag{49}$$

$$\left(\int_0^{x_1}\left(\int_{w_1 q^n}^{w_1} \left|{}_*D^{\alpha}_{q,w_1 q^n} f_1\right|^{q_1}(t_1)\, d_q t_1\right) d_q w_1\right)^{\frac{1}{q_1}}.$$

$$\left(\int_0^{x_2}\left(\int_{w_2 q^n}^{w_2} \left|{}_*D^{\alpha}_{q,w_2 q^n} f_2\right|^{p_1}(t_2)\, d_q t_2\right) d_q w_2\right)^{\frac{1}{p_1}}.$$

Proof. Similar to the proof of Theorem 10, using (21). □

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