

\mathcal{L} -Random and Fuzzy Normed Spaces and Classical Theory

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ABSTRACT

In this paper we study \mathcal{L} -random and \mathcal{L} -fuzzy normed spaces and prove open mapping and closed graph theorems for these spaces.

RESUMEN

En este artículo estudiamos espacios normados \mathcal{L} -random and \mathcal{L} -fuzzy. Probamos el teorema de la aplicación abierta y el teorema del gráfico cerrado.

Key words and phrases: *\mathcal{L} -random normed space, \mathcal{L} -fuzzy normed space, completeness, quotient space, open mapping and closed graph.*

Math. Subj. Class.: *PLEASE INFORM.*

1 Introduction and Preliminaries

In this paper we study \mathcal{L} -random and \mathcal{L} -fuzzy normed spaces and study completeness for these spaces. Further we prove open mapping and closed graph theorems in this setting. The ideas here are motivated from the functional analysis literature. The plan in sections 1-3 is to present in detail the \mathcal{L} -random normed space setting. In section 4 we see from the definition how easily the theory extends to the \mathcal{L} -fuzzy normed space situation.

Let $\mathcal{L} = (L, \geq_L)$ be a complete lattice, i.e. a partially ordered set in which every nonempty subset admits a supremum and infimum, and $0_{\mathcal{L}} = \inf L$, $1_{\mathcal{L}} = \sup L$. The space of lattice random distribution functions, denoted by Δ_L^+ , is defined as the set of all mappings $F : \mathbb{R} \cup \{-\infty, +\infty\} \rightarrow L$ such that F is continuous and non-decreasing on \mathbb{R} , $F(0) = 0_{\mathcal{L}}$, $F(+\infty) = 1_{\mathcal{L}}$.

Now $D_L^+ \subseteq \Delta_L^+$ is defined as $D_L^+ = \{F \in \Delta_L^+ : l^-F(+\infty) = 1_{\mathcal{L}}\}$, where $l^-f(x)$ denotes the left limit of the function f at the point x . The space Δ_L^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \geq G$ if and only if $F(t) \geq_L G(t)$ for all t in \mathbb{R} . The maximal element for Δ_L^+ in this order is the distribution function given by

$$\varepsilon_0(t) = \begin{cases} 0_{\mathcal{L}}, & \text{if } t \leq 0, \\ 1_{\mathcal{L}}, & \text{if } t > 0. \end{cases}$$

Define the mapping \mathcal{T}_{\wedge} from L^2 to L by:

$$\mathcal{T}_{\wedge}(x, y) = \begin{cases} x, & \text{if } y \geq_L x, \\ y, & \text{if } x \geq_L y. \end{cases}$$

Recall (see [4], [5]) that if $\{x_n\}$ is a given sequence in L , $(\mathcal{T}_{\wedge})_{i=1}^n x_i$ is defined recurrently by $(\mathcal{T}_{\wedge})_{i=1}^1 x_i = x_1$ and $(\mathcal{T}_{\wedge})_{i=1}^n x_i = \mathcal{T}_{\wedge}((\mathcal{T}_{\wedge})_{i=1}^{n-1} x_i, x_n)$ for $n \geq 2$.

A negation on \mathcal{L} is any decreasing mapping $\mathcal{N} : L \rightarrow L$ satisfying $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $\mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L$, then \mathcal{N} is called an involutive negation. In the following \mathcal{L} is endowed with a (fixed) negation \mathcal{N} .

Definition 1.1. A lattice random normed space (briefly, \mathcal{L} -random normed space) is a triple $(X, \mathcal{P}, \mathcal{T})$, where X is a vector space, \mathcal{T} is a t -norm on the lattice \mathcal{L} and \mathcal{P} is a mapping from $X \times [0, \infty)$ into D_L^+ such that the following conditions hold:

- (LRN1) $\mathcal{P}(x, t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
- (LRN2) $\mathcal{P}(\alpha x, t) = \mathcal{P}\left(x, \frac{t}{|\alpha|}\right)$ for all x in X , $\alpha \neq 0$ and $t \geq 0$;
- (LRN3) $\mathcal{P}(x + y, t + s) \geq_L \mathcal{T}(\mathcal{P}(x, t), \mathcal{P}(y, s))$ for all $x, y \in X$ and $t, s \geq 0$.

We note from (LPN2) that $\mathcal{P}(-x, t) = \mathcal{P}(x, t)$ ($x \in X, t \geq 0$).

Example 1.2. Let $L = [0, 1] \times [0, 1]$ and operation \leq_L defined by:

$$L = \{(a_1, a_2) : (a_1, a_2) \in [0, 1] \times [0, 1] \text{ and } a_1 + a_2 \leq 1\},$$

$$(a_1, a_2) \leq_L (b_1, b_2) \iff a_1 \leq b_1, a_2 \geq b_2, \quad \forall a = (a_1, a_2), b = (b_1, b_2) \in L.$$

Then (L, \leq_L) is a complete lattice (see [2]). In this complete lattice, we denote its units by $0_L = (0, 1)$ and $1_L = (1, 0)$. Let $(X, \|\cdot\|)$ be a normed space. Let $\mathcal{F}(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$ for all $a = (a_1, a_2), b = (b_1, b_2) \in [0, 1] \times [0, 1]$ and μ be a mapping defined by

$$\mathcal{P}(x, t) = \left(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|} \right), \quad \forall t \in \mathbb{R}^+.$$

Then $(X, \mathcal{P}, \mathcal{F})$ is a \mathcal{L} -random normed space.

Definition 1.3. Let $(X, \mathcal{P}, \mathcal{F})$ be a \mathcal{L} -random normed space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $t > 0$ and $\varepsilon \in L \setminus \{0_{\mathcal{L}}\}$, there exists a positive integer N such that $\mathcal{P}(x_n - x, t) >_L \mathcal{N}(\varepsilon)$ whenever $n \geq N$.

(2) A sequence $\{x_n\}$ in X is called *Cauchy sequence* if, for every $t > 0$ and $\varepsilon \in L \setminus \{0_{\mathcal{L}}\}$, there exists a positive integer N such that $\mathcal{P}(x_n - x_m, t) >_L \mathcal{N}(\varepsilon)$ whenever $n \geq m \geq N$.

(3) A \mathcal{L} -random normed space $(X, \mathcal{P}, \mathcal{F})$ is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X .

Theorem 1.4. If $(X, \mathcal{P}, \mathcal{F})$ is a \mathcal{L} -random normed space and $\{x_n\}$ is a sequence such that $x_n \rightarrow x$, then $\lim_{n \rightarrow \infty} \mathcal{P}(x_n, t) = \mathcal{P}(x, t)$.

Proof. The proof is the same as in [9]. □

Let $(X, \mathcal{P}, \mathcal{F})$ be a \mathcal{L} -random normed space. For $t > 0$ we define the open ball $B(x, r, t)$ with center x and radius $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ as

$$B(x, r, t) = \{y \in X : \mathcal{P}(x - y, t) >_L \mathcal{N}(r)\}.$$

Henceforth we assume that \mathcal{F} is a continuous t -norm on the lattice \mathcal{L} such that for every $\mu \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$, there is a $\lambda \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that

$$\mathcal{F}^{n-1}(\mathcal{N}(\lambda), \dots, \mathcal{N}(\lambda)) >_L \mathcal{N}(\mu).$$

Lemma 1.5. Let $(X, \mathcal{P}, \mathcal{F})$ be a \mathcal{L} -random normed space. Let \mathcal{N} be a continuous negator on \mathcal{L} . Define $E_{\lambda, \mathcal{P}} : V \rightarrow \mathbf{R}^+ \cup \{0\}$ by

$$E_{\lambda, \mathcal{P}}(x) = \inf\{t > 0 : \mathcal{P}(x, t) >_L \mathcal{N}(\lambda)\}$$

for each $\lambda \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ and $x \in V$. Then we have the following properties.

(i) For any $\mu \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ there exists $\lambda \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that

$$E_{\mu, \mathcal{P}}(x + y) \leq E_{\lambda, \mathcal{P}}(x) + E_{\lambda, \mathcal{P}}(y)$$

for any $x, y \in V$.

(ii) The sequence $(x_n)_{n \in \mathbb{N}}$ is convergent w.r.t. a \mathcal{L} -random norm \mathcal{P} if and only if $E_{\lambda, \mathcal{P}}(x_n - x) \rightarrow 0$. Also the sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy w.r.t. a \mathcal{L} -random norm \mathcal{P} if and only if it is Cauchy w.r.t. $E_{\lambda, \mathcal{P}}$.

Proof. For (i), by the continuity of the t-norm \mathcal{T} and the negator \mathcal{N} , for every $\mu \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ we can find a $\lambda \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that

$$\mathcal{T}(\mathcal{N}(\lambda), \mathcal{N}(\lambda)) \geq_L \mathcal{N}(\mu).$$

By Definition 1.1 we have

$$\begin{aligned} \mathcal{P}(x + y, E_{\lambda, \mathcal{P}}(x) + E_{\lambda, \mathcal{P}}(y) + 2\delta) &\geq_L \mathcal{T}(\mathcal{P}(x, E_{\lambda, \mathcal{P}}(x) + \delta), \mathcal{P}(y, E_{\lambda, \mathcal{P}}(y) + \delta)) \\ &\geq_L \mathcal{T}(\mathcal{N}(\lambda), \mathcal{N}(\lambda)) \\ &\geq_L \mathcal{N}(\mu), \end{aligned}$$

for every $\delta > 0$, which implies that

$$E_{\mu, \mathcal{P}}(x + y) \leq E_{\lambda, \mathcal{P}}(x) + E_{\lambda, \mathcal{P}}(y) + 2\delta.$$

Since $\delta > 0$ was arbitrary, we have

$$E_{\mu, \mathcal{P}}(x + y) \leq E_{\lambda, \mathcal{P}}(x) + E_{\lambda, \mathcal{P}}(y).$$

For (ii), we have

$$\mathcal{P}(x_n - x, \eta) >_L \mathcal{N}(\lambda) \iff E_{\lambda, \mathcal{P}}(x_n - x) < \eta$$

for every $\eta > 0$. □

2 Quotient Spaces

Definition 2.1. Let $(V, \mathcal{P}, \mathcal{T})$ be a \mathcal{L} -random normed space, W a linear manifold in V and let $Q : V \rightarrow V/W$ be the natural map, $Qx = x + W$. For $t > 0$, we define:

$$\bar{\mathcal{P}}(x + W, t) = \sup\{\mathcal{P}(x + y, t) : y \in W\}.$$

Theorem 2.2. *Let W be a closed subspace of a \mathcal{L} -random normed space $(V, \mathcal{P}, \mathcal{T})$. If $x \in V$ and $\epsilon > 0$, then there is an x' in V such that $x' + W = x + W$, $E_{\lambda, \mathcal{P}}(x') < E_{\lambda, \bar{\mathcal{P}}}(x + W) + \epsilon$.*

Proof. By the properties of sup, there always exists $y \in W$ such that $E_{\lambda, \mathcal{P}}(x + y) < E_{\lambda, \bar{\mathcal{P}}}(x + W) + \epsilon$. Now it is enough to put $x' = x + y$. \square

Theorem 2.3. *Let W be a closed subspace of a \mathcal{L} -random normed space $(V, \mathcal{P}, \mathcal{T})$ and $\bar{\mathcal{P}}$ be given in the above definition. Then:*

- (1) $\bar{\mathcal{P}}$ is a \mathcal{L} -random normed space, on V/W .
- (2) $\bar{\mathcal{P}}(Qx, t) \geq_L \mathcal{P}(x, t)$.
- (3) If $(V, \mathcal{P}, \mathcal{T})$ is a complete \mathcal{L} -random normed space, then so is $(V/W, \bar{\mathcal{P}}, \mathcal{T})$.

Proof. It is clear that $\bar{\mathcal{P}}(x + W, t) >_L 0_{\mathcal{L}}$. Let $\bar{\mathcal{P}}(x + W, t) = 1_{\mathcal{L}}$. By definition there is a sequence $\{x_n\}$ in W such that $\mathcal{P}(x + x_n, t) \rightarrow 1_{\mathcal{L}}$. Thus, $x + x_n \rightarrow 0$ or equivalently $x_n \rightarrow (-x)$ and since W is closed, $x \in W$ and $x + W = W$, the zero element of V/W . Then we have

$$\begin{aligned} \bar{\mathcal{P}}((x + W) + (y + W), t) &= \bar{\mathcal{P}}((x + y) + W, t) \\ &\geq_L \mathcal{P}((x + m) + (y + n), t) \\ &\geq_L \mathcal{T}(\mathcal{P}(x + m, t_1), \mathcal{P}(y + n, t_2)) \end{aligned}$$

for $m, n \in W$, $x, y \in V$ and $t_1 + t_2 = t$. Now if we take the sup, then we have

$$\bar{\mathcal{P}}((x + W) + (y + W), t) \geq_L \mathcal{T}(\bar{\mathcal{P}}(x + W, t_1), \bar{\mathcal{P}}(y + W, t_2)).$$

Therefore $\bar{\mathcal{P}}$ is a \mathcal{L} -random norm on V/W .

- (2) By Definition 2.1, we have

$$\bar{\mathcal{P}}(Qx, t) = \bar{\mathcal{P}}(x + W, t) = \sup\{\mathcal{P}(x + y, t) : y \in W\} \geq_L \mathcal{P}(x, t).$$

Note that, by Lemma 1.5,

$$\begin{aligned} E_{\lambda, \bar{\mathcal{P}}}(Qx) &= \inf\{t > 0 : \bar{\mathcal{P}}(Qx, t) >_L \mathcal{N}(\lambda)\} \leq \inf\{t > 0 : \mathcal{P}(x, t) >_L \mathcal{N}(\lambda)\} \\ &= E_{\lambda, \mathcal{P}}(x). \end{aligned}$$

- (3) Let $\{x_n + W\}$ be a Cauchy sequence in V/W . Then there exists $n_0 \in \mathbf{N}$ such that for every $n \geq n_0$, $E_{\lambda, \bar{\mathcal{P}}}((x_n + W) - (x_{n+1} + W)) \leq 2^{-n}$. Let $y_1 = 0$. Choose $y_2 \in W$ such that

$$E_{\lambda, \mathcal{P}}(x_1 - (x_2 - y_2), t) \leq E_{\lambda, \bar{\mathcal{P}}}((x_1 - x_2) + W) + 1/2.$$

However $E_{\lambda, \bar{\mathcal{P}}}((x_1 - x_2) + W) \leq 1/2$ and so $E_{\lambda, \mathcal{P}}(x_1 - (x_2 - y_2)) \leq 1/2^2$.

Now suppose y_{n-1} has been chosen, so choose $y_n \in W$ such that

$$E_{\lambda, \mathcal{P}}((x_{n-1} + y_{n-1}) - (x_n + y_n)) \leq E_{\lambda, \mathcal{P}}((x_{n-1} - x_n) + W) + 2^{-n+1}.$$

Hence we have

$$E_{\lambda, \mathcal{P}}((x_{n-1} + y_{n-1}) - (x_n + y_n)) \leq 2^{-n+2}.$$

However for every positive integer $m > n$ and by Lemma 1.5 for $\lambda \in L$ there exists $\gamma \in L$, such that

$$\begin{aligned} E_{\lambda, \mathcal{P}}((x_m + y_m) - (x_n + y_n)) &\leq E_{\gamma, \mathcal{P}}((x_{n+1} + y_{n+1}) - (x_n + y_n)) + \\ &\quad \cdots + E_{\gamma, \mathcal{P}}((x_m + y_m) - (x_{m-1} + y_{m-1})) \\ &\leq \sum_{i=n}^m 2^{-i}. \end{aligned}$$

By Lemma 1.5, $\{x_n + y_n\}$ is a Cauchy sequence in V . Since V is complete, there is an x_0 in V such that $x_n + y_n \rightarrow x_0$ in V . On the other hand,

$$x_n + W = Q(x_n + y_n) \rightarrow Q(x_0) = x_0 + W.$$

Therefore, every Cauchy sequence $\{x_n + W\}$ is convergent in V/W and so V/W is complete. Thus $(V/W, \mathcal{P}, \mathcal{F})$ is a complete \mathcal{L} -random normed space. \square

Theorem 2.4. *Let W be a closed subspace of a \mathcal{L} -random normed space $(V, \mathcal{P}, \mathcal{F})$. If two of the spaces V , W and V/W are complete, then so is the third one.*

Proof. If V is a complete \mathcal{L} -random normed space, then so are V/W and W . Hence all that needs to be checked is that V is complete whenever both W and V/W are complete. Suppose that W and V/W are complete \mathcal{L} -random normed spaces and let $\{x_n\}$ be a Cauchy sequence in V . Since $E_{\lambda, \mathcal{P}}((x_n - x_m) + W) \leq E_{\lambda, \mathcal{P}}(x_n - x_m)$ for each $m, n \in \mathbf{N}$, the sequence $\{x_n + W\}$ is Cauchy in V/W and so converges to $y + W$ for some $y \in W$. Thus there is a $n_0 \in \mathbf{N}$ such that for every $n \geq n_0$, we have $E_{\lambda, \mathcal{P}}((x_n - y) + W) < 2^{-n}$. Now by the last theorem there exist a sequence $\{y_n\}$ in V such that $y_n + W = (x_n - y) + W$, $E_{\lambda, \mathcal{P}}(y_n) < E_{\lambda, \mathcal{P}}((x_n - y) + W) + 2^{-n}$. Thus we have $\lim_n E_{\lambda, \mathcal{P}}(y_n) \leq 0$ by Lemma 1.5, $\mathcal{P}(y_n, t) \rightarrow 1_{\mathcal{L}}$ for every $t > 0$, i.e. $\lim_n y_n = 0$. Therefore, $\{x_n - y_n - y\}$ is a Cauchy sequence in W and thus is convergent to a point $z \in W$. This implies that $\{x_n\}$ converges to $z + y$ and hence V is complete. \square

3 Open Mapping and Closed Graph Theorems

Definition 3.1. A linear operator $T : (V, \mathcal{P}, \mathcal{F}) \rightarrow (V', \mathcal{P}', \mathcal{F}')$ is said to be \mathcal{L} -random bounded if there exist constants $h \in \mathbf{R}^+$ such that for every $x \in V$ and for every $t > 0$,

$$\mathcal{P}'(Tx, t) \geq_L \mathcal{P}(x, t/h). \quad (3.1)$$

Note that, by (3.1) we have

$$\begin{aligned} E_{\lambda, \mathcal{P}'}(Tx) &= \inf\{t > 0 : \mathcal{P}'(Tx, t) >_L \mathcal{N}(\lambda)\} \leq \inf\{t > 0 : \mathcal{P}(x, t/h) >_L \mathcal{N}(\lambda)\} = \\ &= h \inf\{t > 0 : \mathcal{P}(x, t) >_L \mathcal{N}(\lambda)\} \\ &= hE_{\lambda, \mathcal{P}}(x). \end{aligned}$$

Theorem 3.2. *Every linear operator $T : (V, \mathcal{P}, \mathcal{T}) \rightarrow (V', \mathcal{P}', \mathcal{T}')$ is \mathcal{L} -random bounded if and only if it is continuous.*

Proof. By (3.1) every \mathcal{L} -random bounded linear operator is continuous. Now, we prove the converse. Let the linear operator T be continuous but not \mathcal{L} -random bounded. Then, for each n in \mathbf{N} there is a x_n in V such that $E_{\lambda, \mathcal{P}'}(Tx_n) \geq nE_{\lambda, \mathcal{P}}(p_n)$. If we let $y_n = \frac{x_n}{nE_{\lambda, \mathcal{P}}(x_n)}$ then it is easy to see $y_n \rightarrow 0$ but Ty_n do not tend to 0.

Theorem 3.3. (Open mapping theorem) *If T is a \mathcal{L} -random bounded linear operator from a complete \mathcal{L} -random normed space $(V, \mathcal{P}, \mathcal{T})$ onto a complete \mathcal{L} -random normed space $(V', \mathcal{P}', \mathcal{T}')$ then T is an open mapping.*

Proof. The theorem will be proved in several steps.

Step 1: Let E be a neighborhood of the 0 in V . We show $0 \in \overline{(T(E))^o}$. Let W be a balanced neighborhood of 0 such that $W + W \subset E$. Since $T(V) = V'$ and W is absorbing, it follows that $V' = \cup_n T(nW)$, so by Theorem 3.17 in [6], there exists a $n_0 \in \mathbf{N}$ such that $\overline{T(n_0W)}$ has nonempty interior. Therefore, $0 \in \overline{(T(W))^o} - \overline{(T(W))^o}$. On the other hand,

$$\overline{(T(W))^o} - \overline{(T(W))^o} \subset \overline{T(W)} - \overline{T(W)} = \overline{T(W) + T(W)} \subset \overline{T(E)}.$$

Thus the set $\overline{T(E)}$ includes the neighborhood $\overline{(T(W))^o} - \overline{(T(W))^o}$ of 0.

Step 2: We show $0 \in (T(E))^o$. Since $0 \in E$ and E is an open set, there exists $0_{\mathcal{L}} <_L \alpha <_L 1_{\mathcal{L}}$ and $t_0 \in (0, \infty)$ such that $B(0, \alpha, t_0) \subset E$. However $0_{\mathcal{L}} <_L \alpha <_L 1_{\mathcal{L}}$ so a sequence $\{\epsilon_n\}$ can be found such that $\mathcal{T}^{m-n}(\mathcal{N}(\epsilon_{n+1}), \mathcal{N}(\epsilon_n)) \rightarrow 1_{\mathcal{L}}$, $\mathcal{N}(\alpha) <_L \lim_n \mathcal{T}^{n-1}(\mathcal{N}(\epsilon_1), \mathcal{N}(\epsilon_n))$ in which $m > n$. On the other hand, $0 \in \overline{T(B(0, \epsilon_n, t'_n))}$, where $t'_n = \frac{1}{2^n} t_0$, so by step 1, there exist $0_{\mathcal{L}} <_L \sigma_n <_L 1_{\mathcal{L}}$ and $t_n > 0$ such that $B(0, \sigma_n, t_n) \subset \overline{T(B(0, \epsilon_n, t'_n))}$. Since the set $\{B(0, r, 1/n)\}$ is a countable local base at zero and $t'_n \rightarrow 0$ as $n \rightarrow \infty$, so t_n and σ_n can be chosen such that $t_n \rightarrow 0$ and $\sigma_n \rightarrow 0_{\mathcal{L}}$ as $n \rightarrow \infty$.

Now we show $B(0, \sigma_1, t_1) \subset (T(E))^o$. Suppose $y_0 \in B(0, \sigma_1, t_1)$. Then $y_0 \in \overline{T(B(0, \epsilon_1, t'_1))}$ and so for $0_{\mathcal{L}} <_L \sigma_2$ and $t_2 > 0$ the ball $B(y_0, \sigma_2, t_2)$ intersects $T(B(0, \epsilon_1, t'_1))$. Therefore there exists $x_1 \in B(0, \epsilon_1, t'_1)$ such that $Tx_1 \in B(y_0, \sigma_2, t_2)$, i.e. $\mathcal{P}'(y_0 - Tx_1, t_2) >_L \mathcal{N}(\sigma_2)$ or equivalently $y_0 - Tx_1 \in B(0, \sigma_2, t_2) \subset \overline{T(B(0, \epsilon_1, t'_1))}$. By the similar argument there exist x_2 in $B(0, \epsilon_2, t'_2)$ such that

$$\mathcal{P}'(y_0 - (Tx_1 + Tx_2), t_3) = \mathcal{P}'((y_0 - Tx_1) - Tx_2, t_3) >_L \mathcal{N}(\sigma_3).$$

If this process is continued, it leads to a sequence $\{x_n\}$ such that $x_n \in B(0, \epsilon_n, t'_n)$, $\mathcal{P}'\left(y_0 - \sum_{j=1}^{n-1} Tx_j, t_n\right) \geq_L \mathcal{N}(\sigma_n)$. Now if $n, m \in \mathbf{N}$ and $m > n$, then

$$\mathcal{P}\left(\sum_{j=1}^n x_j - \sum_{j=n+1}^m x_j, t\right) = \mu\left(\sum_{j=n+1}^m x_j, t\right) \geq_L \mathcal{F}^{m-n}(\mathcal{P}(x_{n+1}, t_{n+1}), \mathcal{P}(x_m, t_m))$$

where $t_{n+1} + t_{n+2} + \dots + t_m = t$. Put $t'_0 = \min\{t_{n+1}, t_{n+2}, \dots, t_m\}$. Since $t'_n \rightarrow 0$, there exists $n_0 \in \mathbf{N}$ such that $0 < t'_n \leq t'_0$ for $n > n_0$. Therefore, for $m > n$ we have

$$\begin{aligned} \mathcal{F}^{m-n}(\mathcal{P}(x_{n+1}, t'_0), \mathcal{P}(x_m, t'_0)) &\geq_L \mathcal{F}^{m-n}(\mathcal{P}(x_{n+1}, t'_{n+1}), \mathcal{P}(x_m, t'_m)) \\ &\geq_L \mathcal{F}^{m-n}(\mathcal{N}(\epsilon_{n+1}), \mathcal{N}(\epsilon_m)). \end{aligned}$$

Hence,

$$\lim_{n \rightarrow \infty} \mathcal{P}\left(\sum_{j=n+1}^m x_j, t\right) \geq_L \lim_{n \rightarrow \infty} \mathcal{F}^{m-n}(\mathcal{N}(\epsilon_{n+1}), \mathcal{N}(\epsilon_m)) = 1_{\mathcal{L}}.$$

That is, $\mathcal{P}\left(\sum_{j=n+1}^m x_j, t\right) \rightarrow 1_{\mathcal{L}}$, for all $t > 0$. Thus the sequence $\left\{\sum_{j=1}^n x_j\right\}$ is a Cauchy sequence and consequently the series $\left\{\sum_{j=1}^{\infty} x_j\right\}$ converges to some point $x_0 \in V$, because V is a complete space.

By fixing $t > 0$, there exists $n_0 \in \mathbf{N}$ such that $t > t_n$ for $n > n_0$, because $t_n \rightarrow 0$. Thus

$$\mathcal{P}'\left(y_0 - T\left(\sum_{j=1}^{n-1} x_j\right), t\right) \geq_L \mathcal{P}'\left(y_0 - T\left(\sum_{j=1}^{n-1} x_j\right), t_n\right) \geq_L \mathcal{N}(\sigma_n)$$

and thus $\mathcal{P}'\left(y_0 - T\left(\sum_{j=1}^{n-1} x_j\right), t\right) \rightarrow 1_{\mathcal{L}}$. Therefore,

$$y_0 = \lim_n T\left(\sum_{j=1}^{n-1} x_j\right) = T\left(\lim_n \sum_{j=1}^{n-1} x_j\right) = Tx_0.$$

But, by Proposition 1 of [7],

$$\begin{aligned} \mathcal{P}(x_0, t_0) &= \lim_n \mathcal{P}\left(\sum_{j=1}^n x_j, t_0\right) \geq_L \mathcal{F}^n(\lim_n(\mathcal{P}(x_1, t'_1), \mathcal{P}(x_n, t'_n))) \\ &\geq_L \lim_n \mathcal{F}^{n-1}(\mathcal{N}(\epsilon_1), \dots, \mathcal{N}(\epsilon_n)) \geq_L \mathcal{N}(\alpha) \end{aligned}$$

Hence $x_0 \in B(0, \alpha, t_0)$.

Step 3: Let G be an open subset of V and $x \in G$. Then we have

$$T(G) = Tx + T(-x + G) \supset Tx + (T(-x + G))^o.$$

Hence $T(G)$ is open, because it includes a neighborhood of each of its point. \square

Corollary 3.4. *Every one-to-one \mathcal{L} -random bounded linear operator from a complete \mathcal{L} -random normed space onto a complete \mathcal{L} -random normed space has a \mathcal{L} -random bounded inverse.*

Definition 3.5. Let \mathcal{F} and \mathcal{F}' be two continuous t -norms. Then \mathcal{F}' dominates \mathcal{F} , denoted by $\mathcal{F}' \gg_L \mathcal{F}$, if for all $x_1, x_2, y_1, y_2 \in \mathcal{L}$,

$$\mathcal{F}[\mathcal{F}'(x_1, x_2), \mathcal{F}'(y_1, y_2)] \leq_L \mathcal{F}'[\mathcal{F}(x_1, y_1), \mathcal{F}(x_2, y_2)].$$

Theorem 3.6. (Closed graph theorem) *Let T be a linear operator from the complete \mathcal{L} -random normed space $(V, \mathcal{P}, \mathcal{F})$ into the complete \mathcal{L} -random normed space $(V', \mathcal{P}', \mathcal{F}')$. Suppose for every sequence $\{x_n\}$ in V such that $x_n \rightarrow x$ and $Tx_n \rightarrow y$ for some elements $x \in V$ and $y \in V'$ it follows that $Tx = y$. Then T is \mathcal{L} -random bounded.*

Proof. For any $t > 0$, $x \in V$ and $y \in V'$, define

$$\Phi((x, y), t) = \mathcal{F}'(\mathcal{P}(x, t), \mathcal{P}'(y, t)),$$

where $\mathcal{F}' \gg_L \mathcal{F}$. First we show that $(V \times V', \Phi, \mathcal{F})$ is a complete \mathcal{L} -random normed space. The properties of (LRN1) and (LRN2) are immediate from the definition. For the triangle inequality (LRN3), suppose that $x, z \in V$, $y, u \in V'$ and $t, s > 0$, then

$$\begin{aligned} \mathcal{F}(\Phi((x, y), t), \Phi((z, u), s)) &= \mathcal{F}[\mathcal{F}'(\mathcal{P}(x, t), \mathcal{P}'(y, t)), \mathcal{F}'(\mathcal{P}(z, s), \mathcal{P}'(u, s))] \\ &\leq_L \mathcal{F}'[\mathcal{F}(\mathcal{P}(x, t), \mathcal{P}(z, s)), \mathcal{F}(\mathcal{P}'(y, t), \mathcal{P}'(u, s))] \\ &\leq_L \mathcal{F}'(\mathcal{P}(x+z, t+s), \mathcal{P}'(y+u, t+s)) \\ &= \Phi((x+z, y+u), t+s). \end{aligned}$$

Now if $\{(x_n, y_n)\}$ is a Cauchy sequence in $V \times V'$, then for every $\epsilon \in L \setminus \{0_\mathcal{L}\}$ and $t > 0$ there exists $n_0 \in \mathbf{N}$ such that $\Phi((x_n, y_n) - (x_m, y_m), t) >_L \mathcal{N}(\epsilon)$ for $m, n > n_0$. Thus for $m, n > n_0$,

$$\begin{aligned} \mathcal{F}'(\mathcal{P}(x_n - x_m, t), \mathcal{P}'(y_n - y_m, t)) &= \Phi((x_n - x_m, y_n - y_m), t) \\ &= \Phi((x_n, y_n) - (x_m, y_m), t) >_L \mathcal{N}(\epsilon). \end{aligned}$$

Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in V and V' , respectively, and there exist $x \in V$ and $y \in V'$ such that $x_n \rightarrow x$ and $y_n \rightarrow y$ and consequently $(x_n, y_n) \rightarrow (x, y)$. Hence $(V \times V', \Phi, \mathcal{F})$ is a complete \mathcal{L} -random normed space. The remainder of the proof is the same as the classical case. \square

4 \mathcal{L} -fuzzy normed space

We conclude the paper with the setting of \mathcal{L} -fuzzy normed spaces. Consider the \mathcal{L} -fuzzy normed space $(X, \mathcal{F}, \mathcal{F})$ in which \mathcal{F} is a \mathcal{L} -fuzzy set on $X \times]0, +\infty[$ satisfying the following

conditions for every x, y in X and t, s in $(0, +\infty)$:

- (a) $0_{\mathcal{L}} <_L \mathcal{F}(x, t)$;
- (b) $\mathcal{F}(x, t) = 1_{\mathcal{L}}$ if and only if $x = 0$;
- (c) $\mathcal{F}(\alpha x, t) = \mathcal{F}(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$;
- (d) $\mathcal{F}(\mathcal{F}(x, t), \mathcal{F}(y, s)) \leq_L \mathcal{F}(x + y, t + s)$;
- (e) $\mathcal{F}(x, \cdot) :]0, \infty[\rightarrow L$ is continuous;
- (f) $\lim_{t \rightarrow 0} \mathcal{F}(x, t) = 0_{\mathcal{L}}$ and $\lim_{t \rightarrow \infty} \mathcal{F}(x, t) = 1_{\mathcal{L}}$.

In this case \mathcal{F} is called a \mathcal{L} -fuzzy norm. For some details on the \mathcal{L} -fuzzy normed spaces, please see [1]

It is clear that all the results in section 2 and 3 can be written for \mathcal{L} -fuzzy normed spaces.

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