*CUBO A Mathematical Journal*
*Vol.12, N$^{o}$03, (139–151). October 2010*

# Some Generalizations of Mulit-Valued Version of Schauder's Fixed Point Theorem with Applications

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**ABSTRACT**

In this article, a generalization of a Kakutani-Fan fixed point theorem for multi-valued mappings in Banach spaces is proved under weaker upper semi-continuity condition and it is further applied to derive a generalized version of Krasnoselskii's fixed point theorem and some nonlinear alternatives of Leray-Schauder type for multi-valued closed mappings in Banach spaces.

**RESUMEN**

En este artículo probamos una generalización para el teorema del punto fijo de Kakutani-Fan para aplicaciones multi-valuadas en espacios de Banach, bajo condición de semi-continuidad superior debil. Este resultado es aplicado para obtener una versión generalizada del teorema del punto fijo Krasnoselskii y algunas alternativas de tipo Leray-Schauder para aplicaciones multi-valuadas cerradas en espacios de Banach.

**Key words and phrases:** *Multi-valued mappings, fixed point theorem, nonlinear alternative.*

**Math. Subj. Class.:** *47H10.*

# 1 Introduction

Throughout this paper, unless otherwise mentioned, let $E$ be a Banach space and let $\mathscr{P}(E)$ denote the class of all subsets of $E$. Denote

$$\mathscr{P}_p(E) = \{A \subset E \mid A \text{ is non-empty and has a property } p\}.$$

Here, $p$ may be the property $p$ = closed (in short cl), or $p$ = compact (in short cp), or $p$ = convex (in short cv), or $p$ = bounded (in short bd) etc. Thus, $\mathscr{P}_{bd}(E)$,$\mathscr{P}_{cl}(E)$,$\mathscr{P}_{cv}(E)$, $\mathscr{P}_{cp}(E)$, $\mathscr{P}_{cl,bd}(E)$, $\mathscr{P}_{cp,cv}(E)$ denote the classes of all bounded, closed, convex, compact, closed-bounded and compact-convex subsets of $E$ respectively. Similarly, $\mathscr{P}_{cl,cv,bd}(E)$ and $\mathscr{P}_{rcp}(E)$ denote respectively the classes of closed, convex and bounded and relatively compact subsets of $E$.

A correspondence $Q : E \to \mathscr{P}_p(E)$ is called a multi-valued operator or multi-valued mapping on $E$ into $E$. A point $u \in E$ is called a fixed point of $Q$ if $u \in Qu$. For the sake of convenience, we denote $Q(A) = \bigcup_{x\in A} Tx$ for all subsets $A$ of $E$.

Let $E_1$ and $E_2$ be two Banach spaces and let $Q : E_1 \to \mathscr{P}_p(E_2)$ be a multi-valued operator. Then for any non-empty subset $A$ of $E_2$, define

$$Q^+(A) = \{x \in E_1 \mid Tx \subset A\},$$

$$Q^-(A) = \{x \in E_1 \mid Tx \cap A \neq \emptyset\},$$

and

$$Q^{-1}(A) = \{x \in E_1 \mid \cup_x Tx = A\}.$$

**Definition 1.1.** *A multi-valued operator $Q : E_1 \to \mathscr{P}_p(E_2)$ is called upper semi-continuous (resp. lower semi-continuous and continuous) if $Q^+(U)$ (resp. $Q^-(U)$ and $Q^{-1}(U)$) is open set in $E_1$ for every open subset $U$ of $E_2$.*

In what follows,we confine ourselves only to the fixed point theory related to upper semi-continuous multi-valued mappings in Banach spaces. The first fixed point theorem in this direction is due to Kakutani-Fan [11] which is as follows.

**Theorem 1.1.** *Let $K$ be a compact subset of a Banach space $E$ and let $Q : E \to \mathscr{P}_{cp,cv}(E)$ be an upper semi-continuous multi-valued operator. Then $Q$ has a fixed point.*

Note that following are the main three ingredients for the above Theorem 1.1.

(i) The domain space E,

(ii) The domain set $K$, and

(iii) The nature of the multi-valued operator $Q$.

Theorem 1.1 has been extended in the literature by generalizing or modifying the above three hypotheses with the same conclusion. In the following discussion, we do not change the hypothesis on the domain space, and thus keep us in the practical applicability of the so obtained fixed point theorem to other areas of mathematics. However, the generalizations of the above Theorem 1.1 with change of domain space may be found in the works of Browder-Fan [11] and Himmelberg [9] etc.

A first generalization of Theorem 1.1 is due to Bohnenblust-Karlin as given in Petruşel [12].

**Theorem 1.2** (Bohnenblust-Karlin)**.** *Let $X$ be a closed convex and bounded subset of a Banach algebra $E$ and let $Q : X \to \mathscr{P}_{cp,cv}(X)$ be a upper semi-continuous multi-valued operator with a relatively compact range. Then $Q$ has a fixed point.*

A multi-valued map $Q : X \to \mathscr{P}_{cp}(X)$ is called **compact** if $\overline{Q(X)}$ is a compact subset of $X$. $Q$ is called **totally bounded** if for any bounded subset $A$ of $X$, $Q(A) = \bigcup_{x \in A} Qx$ is a totally bounded subset of $X$. It is clear that every compact multi-valued operator is totally bounded, but the converse may not be true. However, these two notions are equivalent on a bounded subset of $X$. Finally, $Q$ is called **completely continuous** if it is upper semi-continuous and totally bounded on $X$.

The upper semi-continuity is further weakened to closed graph operators as follows. If $Q : E_1 \to E_2$ is a multi-valued operator, then the graph $Gr(Q)$ of the operator $Q$ is defined by

$$Gr(Q) = \{(x, y) \in E_1 \times E_2 \mid y \in Tx\}.$$

The graph $Gr(Q)$ of the operator $Q$ is said to be closed if $\{(x_n, y_n)\}$ be a sequence in $Gr(Q)$ such that $(x_n, y_n) \to (x, y)$, then we have that $(x, y) \in Gr(Q)$.

**Definition 1.2.** *A multi-valued operator $Q : E_1 \to \mathscr{P}_{cl}(E_2)$ is called closed if it has a closed graph in $E_1 \times E_2$.*

The following result concerning the upper semi-continuity of multi-valued mappings in Banach spaces is very much useful in the study of multi-valued analysis. The details appears in Deimling [5].

**Lemma 1.1.** *A multi-valued operator $Q : E_1 \to \mathscr{P}_{cl}(E_2)$ is upper semi-continuous if and only if it is closed and has compact range.*

**Theorem 1.3** (O'Regan [13]). *Let $X$ be a closed convex and bounded subset of a Banach algebra $E$ and let $Q : X \to \mathscr{P}_{cp,cv}(X)$ be a compact and closed multi-valued operator. Then $Q$ has a fixed point.*

The compactness of $Q$ in Theorem 1.3 is further weakened to condensing operators with the help of measure of noncompactness in the Banach space $E$. The Kuratowskii measure $\alpha$ and the ball or Hausdorff measure $\beta$ of noncompactness of a bounded set in the Banach space $E$ are the functions $\alpha, \beta : \mathscr{P}_{bd}(E) \to \mathbb{R}^+$ defined by

$$\alpha(A) = \inf\Big\{r > 0 : A \subset \cup_{i=1}^{n} A_i,\ \mathrm{diam}(S_i) \le r\ \forall\, i\Big\}, \tag{1.1}$$

and

$$\beta(A) = \inf\Big\{r > 0 \mid A \subset \bigcup_{i=1}^{n} \mathscr{B}_r(x_i) \text{ for some } x_i \in X\Big\} \tag{1.2}$$

for all $A \in \mathscr{P}_{bd}(E)$, where diam $(A_i) = \sup\{\|x - x\| : x, y \in A_i\}$ and $\mathscr{B}_r(x_i)$ are the open balls centered at $x_i$ of radius $r$.

**Definition 1.3.** *A multi-valued operator $Q : E \to \mathscr{P}_{cl,bd}(E)$ is called $\beta$-condensing if for all bounded sets $A$ in E, $Q(A)$ is bounded and $\beta(Q(A)) < \beta(A)$ for $\beta(A) > 0$ .*

**Theorem 1.4.** *Let $X$ be a closed convex and bounded subset of a Banach space $E$ and let $Q : X \to \mathscr{P}_{cl,cv}(X)$ be a upper semi-continuous and $\beta$-condensing multi-valued operator. Then $Q$ has a fixed point.*

In this article, we generalize Theorem 1.1 by weakening the upper semi-continuity as well as compactness of the multi-valued operator $Q$ in a Banach space $E$ and discuss some of its applications.

# 2 Fixed Point Theory

A function $d_H : \mathscr{P}_p(E) \times \mathscr{P}_p(E) \to \mathbb{R}^+$ defined by

$$d_H(A,B) = \max\Big\{\sup_{a\in A} D(a,B)\,,\ \sup_{b\in B} D(b,A)\Big\} \tag{2.1}$$

satisfies all the conditions of a metric on $\mathscr{P}_p(E)$ and is called a Hausdorff-Pompeiu metric on $E$, where $D(a,B) = \inf\{\|a - b\| : b \in B\}$. It is known that the hyperspace $\big(\mathscr{P}_{cl}(E), d_H\big)$ is a complete metric space.

The axiomatic way of defining the measures of noncompactness has been adopted in several papers in the literature. See Akhmerov *et al.* [2], Banas and Goebel [3], and the references given therein. In this paper, we define the measure of noncompactness in a Banach space on the lines of Dhage [6] which is slightly different manner from that given in the above monographs.

**Definition 2.1.** *A sequence* $\{A_n\}$ *of non-empty sets in* $\mathscr{P}_p(E)$ *is said to converge to a set* $A$*, called the limiting set, if* $d_H(A_n, A) \to 0$ *as* $n \to \infty$.

**Definition 2.2.** *A mapping* $\mu : \mathscr{P}_p(E) \to \mathbb{R}^+$ *is continuous if for any sequence* $\{A_n\}$ *in* $\mathscr{P}_p(E)$*, we have that*

$$d_H(A_n, A) \to 0 \quad implies \quad |\mu(A_n) - \mu(A)| \to 0 \quad as \quad n \to \infty.$$

**Definition 2.3.** A mapping $\mu : \mathscr{P}_p(E) \to \mathbb{R}^+$ is called nondecreasing if $A, B \in \mathscr{P}_p(E)$ are any two sets with $A \subseteq B$, then $\mu(A) \le \mu(B)$, where $\subseteq$ is a order relation by inclusion in $\mathscr{P}_p(E)$.

Now we are equipped with the necessary details to define the measures of noncompactness of a bounded subset of the Banach space $E$.

**Definition 2.4.** *A function* $\mu : \mathscr{P}_{cl,bd}(E) \to \mathbb{R}^+$ *is called a measure of noncompactness if it satisfies*

($\mu_1$) $\emptyset \neq \mu^{-1}(0) \subset \mathscr{P}_{rcp}(E)$,

($\mu_2$) $\mu(\overline{A}) = \mu(A)$*, where* $\overline{A}$ *denotes the closure of* $A$*,*

($\mu_3$) $\mu(Conv\, A) = \mu(A)$*, where* $Conv\, A$ *denotes the convex hull of* $A$*,*

($\mu_4$) $\mu$ *is nondecreasing, and*

($\mu_5$) *if* $\{A_n\}$ *is a decreasing sequence of sets in* $\mathscr{P}_{cl,bd}(E)$ *satisfying* $\lim_{n\to\infty} \mu(A_n) = 0$*, then the limiting set* $A_\infty = \lim_{n\to\infty} \overline{A}_n$ *is non-empty.*

Note that the functions $\alpha$ and $\beta$ defined by (1.1) and (1.2) satisfy the conditions ($\mu_1$) through ($\mu_5$). Hence $\alpha$ and $\beta$ are the measures of noncompactness on $E$. Moreover, they are locally Lipschitz and hence are locally continuous on $\mathscr{P}_{cl,bd}(E)$. Some nice properties of $\alpha$ and $\beta$ have been discussed in Akhmerov *et al.* [2] and Banas and Goebel [3].

We remark that if ($\mu_4$) holds, then $A_\infty \in \mathscr{P}_{rcp}(E)$. To see this, let $\lim_{n\to\infty} \mu(A_n) = 0$. As $A_\infty \subseteq A_n$ for each $n = 0, 1, 2, \ldots$; by the monotonicity of $\mu$, we obtain

$$\mu(A_\infty) \le \lim_{n\to\infty} \overline{A}_n = \lim_{n\to\infty} \mu(A_n) = 0.$$

Hence, by assumption ($\mu_1$), we get $A_\infty$ is nonempty and $A_\infty \in \mathscr{P}_{rcp}(E)$.

A measure $\mu$ is called *complete* or *full* if the kernel of $\mu$ consists of all possible relatively compact subsets of $E$. Next, a measure $\mu$ is called *sublinear* if it satisfies

($\mu_6$) $\mu(\lambda A) = |\lambda|\mu(A)$ for $\lambda \in \mathbb{R}$, and

($\mu_7$) $\mu(A+B) \le \mu(A)+\mu(B)$ for $A,B \in \mathscr{P}_{cl,bd}(E)$.

There do exist the sublinear measures of noncompactness in Banach spaces $E$. Indeed, the measures $\alpha$ and $\beta$ of noncompactness defined by (1.1) and (1.2) are sublinear on $E$.

Now we prove a fixed point theorem for the mappings in Banach spaces involving the measures of noncompactness. Before going to the main results, we give a useful definition.

**Definition 2.5.** *A multi-valued mapping $Q : E \to \mathscr{P}_{cl,bd}(E)$ is called $\mathscr{D}$-**set-Lipschitz** if there exists a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mu(Q(A)) \le \psi(\mu(A))$ for all $A \in \mathscr{P}_{cl,bd}(E)$ with $Q(A) \in \mathscr{P}_{cl,bd}(E)$, where $\psi(0) = 0$. Sometimes we call the function $\psi$ to be a $\mathscr{D}$-**function** of $Q$ on $E$. In the special case, when $\psi(r) = kr, k > 0$, $Q$ is called a $k$-**set-Lipschitz** mapping and if $k < 1$, then $Q$ is called a $k$-**set-contraction** on $E$. Further, if $\psi(r) < r$ for $r > 0$, then $Q$ is called a **nonlinear $\mathscr{D}$-set-contraction** on $E$.*

We need the following lemma in the sequel.

**Lemma 2.1** (Dhage [8]). *If $\psi$ is a $\mathscr{D}$-function with $\psi(r) < r$ for $r > 0$, then $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t \in [0,\infty)$.*

**Theorem 2.1.** *Let $X$ be a non-empty, closed, convex and bounded subset of a Banach space $E$ and let $Q : X \to \mathscr{P}_{cl,cv}(X)$ be a closed and nonlinear $\mathscr{D}$-set-contraction. Then $Q$ has a fixed point.*

*Proof.* Define a sequence $\{X_n\}$ of sets in $\mathscr{P}_{cl,bd}(E)$ by

$$X_0 = X,\ X_{n+1} = \overline{\mathrm{Conv}\, Q(X_n)}, n = 0,1,...$$

Clearly,

$$X_0 \supset X_1 \supset \cdots \supset X_n \supset X_{n+1} \cdots .$$

and so, $\{X_n\}$ is a decreasing sequence of subsets of $E$. Since

$$\mu(X_{n+1}) = \mu\left(\overline{\mathrm{Conv}\, Q(X_n)}\right) = \mu(Q(X_n)) \le \psi(\mu(X_n))$$

for all $n = 0,1,2,\ldots$, we have

$$\mu(X_{n+1}) \le \psi^n(\mu(X_0)).$$

Therefore

$$\limsup_{n\to\infty} \mu(X_{n+1}) \le \limsup_{n\to\infty} \psi^n(\mu(X_0)) = 0.$$

From the monotonicity of $\mu$ it follows that $\lim_{n\to\infty} X_n = X_\infty$ is a compact subset of $E$. As $X_{n+1} \subset X_n$ and $Q : X_n \to X_n$ for all $n = 0, 2, \ldots$, we have

$$X_\infty = \lim_{n\to\infty} X_n = \bigcap_{n=1}^{\infty} X_n \neq \emptyset$$

is a convex subset of $E$ and $Q : X_\infty \to \mathscr{P}_{cp,cv}(X_\infty)$ which is upper-semi-continuous in view of Lemma 1.1. Now the desired conclusion follows by an application of Theorem 1.1 to the operator $Q$ on $X_\infty$. This completes the proof. □

**Remark 2.1.** The fixed point set Fix($Q$) of the multi-valued operator $Q$ in above Theorem 2.1 is compact. In fact if $\mu(\mathrm{Fix}(Q)) > 0$, then from nonlinear $\mathscr{D}$-set-contraction it follows that $\mu(\mathrm{Fix}(Q)) = \mu(Q(\mathrm{Fix}(Q))) \le \psi(\mu(\mathrm{Fix}(Q)))$ which is a contradiction since $\psi(r) < r$ for $r > 0$.

As a consequence of Theorem 2.1 we obtain a fixed point theorem of Darbo [3] type for linear set-contractions,

**Corollary 2.1.** *Let $X$ be a closed, convex and bounded subset of a Banach space $E$ and let $Q : X \to \mathscr{P}_{cl,cv}(X)$ be a closed and $k$-set-contraction. Then $Q$ has a fixed point.*

Before stating the generalization of Theorem 2.1of Sadovskii [14] type, we give a useful definition.

**Definition 2.6.** *A multi-valued mapping $Q : E \to \mathscr{P}(E)$ is called $\mu$-condensing if for any bounded subset $A$ of $E$, $Q(A)$ is bounded and $\mu(Q(A)) < \mu(A)$ for $\mu(A) > 0$.*

**Theorem 2.2.** *Let $X$ be a nonempty, closed, convex and bounded subset of a Banach space $E$ and let $Q : X \to \mathscr{P}_{cl,cv}(X)$ be a closed and $\mu$-condensing mapping. Then $Q$ has a fixed point.*

Thus, we have a one way implication that *Sadovskii's type theorem* $\Rightarrow$ *Theorem 2.1* $\Rightarrow$ *Darbo's type theorem*. However, it is rather difficult to find the operators satisfying the conditions on Banach spaces given in Sadovskii's type fixed point theorem.

# 3 Applications

## 3.1 Hybrid fixed point theory

First, we derive a Krasnoselskii type fixed point theorem for the sum of two multi-valued mappings in Banach spaces. Before stating this result, we need the following definition.

**Definition 3.1.** *A multi-valued mapping $Q : E \to \mathscr{P}_{cl,cv}(E)$ is said to be nonlinear $\mathscr{D}$-contraction if there is a $\mathscr{D}$-function $\psi$ such that*

$$d_H(Qx, Qy) \le \psi(d(x, y))$$

*for all $x, y \in E$, where $\psi(r) < r$.*

**Theorem 3.1.** *Let $X$ be a closed, convex and bounded subset of a Banach space $E$ and let $\mu$ be a sublinear measure of noncompactness in it. Let $S, T : X \to \mathscr{P}_{cl,cv}(E)$ be two operators such that*

*(a) $S$ is closed and nonlinear $\mathscr{D}$-set-contraction,*

*(b) $T$ is compact and closed, and*

*(c) $Sx + Tx \subset X$ for all $x \in X$.*

*Then the operator inclusion $x \in Sx + Tx$ has a solution and the set of all solutions is compact in $E$.*

*Proof.* Define a mapping $Q : X \to \mathscr{P}_{cl,cv}(X)$ by

$$Qx = Sx + Tx. \tag{3.1}$$

We show that $Q$ satisfies all the conditions of Theorem 2.1. Obviously, by hypothesis (c), $Q$ defines a mapping $Q : X \to \mathscr{P}_{cl,cv}(X)$. Since $S$ and $T$ are closed, the sum $Q = S + T$ is also closed on $X$. As hypothesis (a) holds, there is a $\mathscr{D}$-function $\psi$ such that $\psi(r) < r$ for $r > 0$. Further, let $A$ be a non-empty subset of $X$.Then $A$ bounded and

$$Q(A) \subseteq X \quad \text{and} \quad Q(A) \subseteq S(A) + T(A),$$

and hence $Q(A)$ is bounded. By sublinearity of $\mu$, we obtain

$$\mu(Q(A)) \leq \mu(S(A)) + \mu(T(A)) \leq \psi(\mu(A))$$

where, $\psi(r) < r$ for $r > 0$. This shows that $Q$ is a nonlinear $\mathscr{D}$-set-contraction on $X$ into itself. Now an application of Theorem 2.1 yields that $Q$ has a fixed point. Consequently, the operator equation $x \in Sx + Tx$ has a solution. This completes the proof. □

The following lemma is obvious and the proof may be found in the monographs of Deimling [5] and Hu and Papageorgiou [10].

**Lemma 3.1.** *If $Q : E \to \mathscr{P}_{cp,cv}(E)$ is nonlinear contraction. Then for any bounded subset $A$ of $E$ with $Q(A)$ bounded, we have $\beta(Q(A)) \leq \psi(\beta(A))$, where $\beta$ is a ball measure of noncompactness in $E$ defined by (1.2).*

**Theorem 3.2.** *Let $X$ be a closed, convex and bounded subset of a Banach space $E$ and let $S, T : X \to \mathscr{P}_{cp,cv}(E)$ be two multi-valued operators such that* *(a)* *$S$ is a nonlinear $\mathscr{D}$-contraction,*

*(b)* *$T$ is compact and closed, and*

*(c)* *$Sx + Tx \subset X$ for all $x \in X$.*

*Then the operator inclusion $x \in Sx + Tx$ has a solution and the set of all solutions is compact in $E$.*

*Proof.* Since $S$ is nonlinear $\mathscr{D}$-contraction, it is closed on $X$ and there is a $\mathscr{D}$-function $\psi$ of $S$ on $X$ with the properties that $\psi(r) < r$ for $r > 0$. Again from Lemma 3.1, it follows that it is a also nonlinear $\mathscr{D}$-set-contraction with respect to the Hausdorff measure of noncompactness $\beta$ and with a $\mathscr{D}$-function $\psi$ on $X$. Now the desired conclusion follows by a direct application of Theorem 2.1. □

## 3.2 Nonlinear alternative

The following nonlinear alternative for multi-valued mappings in Banach spaces is well-known in the literature.

**Theorem 3.3** (O'Regan [13]). *Let $U$ be a open bounded subset of a Banach space $E$ with $0 \in U$ and let $Q : \overline{U} \to \mathscr{P}_{cl,cv}(E)$ be a compact and closed multi-valued operator. Then either*

*(i)* *the operator inclusion $x \in Qx$ has a solution in $\overline{U}$, or*

*(ii)* *there is an element $u \in \partial U$ such that $\lambda u \in Qu$ for some $\lambda > 1$, where $\partial U$ is the boundary of $U$ in $E$.*

A generalization of above Theorem 3.4 is

**Theorem 3.4.** *Let $U$ be a open bounded subset of a Banach space $E$ with $0 \in U$ and let $Q : \overline{U} \to \mathscr{P}_{cl,cv}(E)$ be a $\mu$-condensing and closed multi-valued operator. Then either*

*(i)* *the operator inclusion $x \in Qx$ has a solution in $\overline{U}$ and the set of all solutions is compact in $E$, or*

*(ii)* *there is an element $u \in \partial U$ such that $\lambda u \in Qu$ for some $\lambda > 1$, where $\partial U$ is the boundary of $U$ in $E$.*

*Proof.* The proof is similar to that given for Theorem 3.3 in O'Regan [13]( see also Agarwal *et al.* [1]) and now the conclusion follows by an application of Theorem 3.2. □

As a consequence of Theorem 3.4, we obtain

**Corollary 3.1.** *Let $\mathscr{B}_r(0)$ be a open ball in a Banach space $E$ centered at origin $0 \in E$ of radius $r$ and let $Q : \overline{\mathscr{B}_r}(0) \to \mathscr{P}_{cl,cv}(E)$ be a $\mu$-condensing and closed multi-valued operator. Then either*

- *(i) the operator inclusion $x \in Qx$ has a solution in $\overline{\mathscr{B}_r}(0)$ and the set of all solutions is compact in $E$, or*
- *(ii) there is an element $u \in E$ such that $\|u\| = r$ satisfying $\lambda u \in Qu$ for some $\lambda > 1$.*

**Corollary 3.2.** *Let $E$ be a Banach space and let $Q : E \to \mathscr{P}_{cl,cv}(E)$ be a $\mu$-condensing and closed multi-valued operator. Then, either*

- *(i) the operator inclusion $x \in Qx$ has a solution and the set of all solutions is compact in $E$, or*
- *(ii) the set $\mathscr{E} = \{u \in E \mid \lambda u \in Qu\}$ is in unbounded for some $\lambda > 1$.*

The above Corollary 3.1 includes the following fixed point result due to Martelli [10] which has been used by several authors in the literature for proving the existence theorems for differential and integral inclusions.

**Corollary 3.3.** *Let $E$ be a Banach space and let $Q : E \to \mathscr{P}_{cl,cv}(E)$ be a upper semi-continuous and $\alpha$-condensing( or $\beta$-condensing) multi-valued operator. Then, either*

- *(i) the operator inclusion $x \in Qx$ has a solution in $X$, or*
- *(ii) the set $\mathscr{E} = \{u \in E \mid \lambda u \in Qu\}$ is in unbounded for some $\lambda > 1$.*

Similarly, we can apply Theorem 3.4 to obtain the following nonlinear alternatives for sum of the two multi-valued operators in Banach spaces.

**Theorem 3.5.** *Let $U$ be a open bounded subset of a Banach space $E$ with $0 \in U$ and let $S, T : \overline{U} \to \mathscr{P}_{cl,cv}(E)$ be two multi-valued operators such that*

- *(a) $S$ is closed and nonlinear $\mathscr{D}$-set-contraction, and*
- *(b) $T$ is compact and closed.*

*Then, either*

- *(i) the operator inclusion $x \in Sx + Tx$ has a solution in $\overline{U}$ and the set of all solutions is compact in $E$, or* *(ii) there is an element $u \in \partial U$ such that $\lambda u \in Su + Tu$ for some $\lambda > 1$, where $\partial U$ is the boundary of $U$ in $E$.*

**Theorem 3.6.** *Let $U$ be a open bounded subset of a Banach space $E$ with $0 \in U$ and let $S, T : \overline{U} \to \mathscr{P}_{cp,cv}(E)$ be two multi-valued operators such that*

*(a) $S$ is nonlinear $\mathscr{D}$-contraction, and*

*(b) $T$ is compact and closed.*

*Then, either*

*(i) the operator inclusion $x \in Sx + Tx$ has a solution in $\overline{U}$ and the set of all solutions is compact in $E$, or*

*(ii) there is an element $u \in \partial U$ such that $\lambda u \in Su + Tu$ for some $\lambda > 1$, where $\partial U$ is the boundary of $U$ in $E$.*

**Corollary 3.4.** *Let $\mathscr{B}_r(0)$ be a open ball in a Banach space $E$ centered at origin $0 \in E$ of radius $r$ and let $S, T : \overline{\mathscr{B}_r}(0) \to \mathscr{P}_{cp,cv}(E)$ be two multi-valued operators such that*

*(a) $S$ is nonlinear $\mathscr{D}$-contraction, and*

*(b) $T$ is compact and closed.*

*Then, either*

*(i) the operator inclusion $x \in Sx + Tx$ has a solution in $\overline{\mathscr{B}_r}(0)$ and the set of all solutions is compact in $E$, or*

*(ii) there is an element $u \in E$ such that $\|u\| = r$ satisfying $\lambda u \in Su + Tu$ for some $\lambda > 1$.*

**Corollary 3.5.** *Let $E$ be a Banach space $E$ and let $S, T : E \to \mathscr{P}_{cp,cv}(E)$ be two multi-valued operators such that*

*(a) $S$ is nonlinear $\mathscr{D}$-contraction, and*

*(b) $T$ is compact and closed.*

*Then, either*

*(i) the operator inclusion $x \in Sx + Tx$ has a solution and the set of all solutions is compact in $E$, or*

*(ii) the set $\mathscr{E} = \{u \in E \mid \lambda u \in Su + Tu\}$ is in unbounded for some $\lambda > 1$.*

**Remark 3.1.** Note that our Theorem 3.6 and Corollary 3.4 improve the hybrid fixed point theorems for multi-valued mappings proved in Dhage [6, 7] under weaker upper semi-continuity conditions.

# 4 The Conclusion

Finally, while concluding, we remark that the multi-valued fixed point theorems of this paper have some nice applications to differential and integral inclusions for proving the existence as well as some characterizations of solutions such as global and local asymptotic attractivity of solutions on bounded and unbounded intervals of real line. The investigations of these and other similar problems form the scope for further research work in the theory of differential and integral inclusions under weaker upper semi-continuity conditions. Some of the results in this direction will be reported elsewhere.

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