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Existence of Periodic Solutions for a Class of Second-Order Neutral Differential Equations with Multiple Deviating Arguments[1]

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ABSTRACT

Using Kranoselskii fixed point theorem and Mawhin's continuation theorem we establish the existence of periodic solutions for a second order neutral differential equation with multiple deviating arguments.

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RESUMEN

Usando el teorema del punto fijo de Kranoselskii y el teorema de continuación de Mawhin establecemos la existencia de soluciones periódicas de una ecuación diferencial neutral de segundo orden con argumento de desviación multiple.

Key words and phrases: *Periodic solution, Multiple deviating arguments, Neutral differential equation, Kranoselskii fixed point theorem, Mawhin's continuation theorem.*

Math. Subj. Class.: *34K15; 34C25.*

1 Introduction

In this paper, we discuss the second-order neutral differential equation with multiple deviating arguments of the form

$$x''(t)+cx''(t-\tau)+a(t)x(t)+g(t,x(t-\tau_1(t)),x(t-\tau_2(t))\cdots,x(t-\tau_n(t)))=p(t), \qquad \textbf{(1.1)}$$

where $|c|<1$, τ is a constant, $\tau_i(t)(i=1,2,\cdots,n)$, $a(t)$ and $p(t)$ are real continuous functions defined on $\mathbf{R}$ with positive period T and $g(t,x_1,x_2,\cdots,x_n)\in C(\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\cdots\times\mathbf{R},\mathbf{R})$ and is $T-$periodic in t.

Periodic solutions for differential equations were studied in [2-4, 6-10, 12, 15] and we note that most of the results in the literatue concern delay problems. There are only a few papers[1, 5, 11, 13, 14] which discuss neutral problems.

For the sake of completeness, we first state Kranoselskii fixed point theorem and Mawhin's continuation theorem [3].

Theorem A (Kranoselskii). Suppose that Ω is a Banach space and X is a bounded, convex and closed subset of Ω. Let $U,S:X\to\Omega$ satisfy the following conditions:

(1) $Ux+Sy\in X$ for any $x,y\in X$;

(2) U is a contraction mapping;

(3) S is completely continuous.

Then $U+S$ has a fixed point in X.

Let X and Y be two Banach space and $L:DomL\subset X\longrightarrow Y$ is a linear mapping and $N:X\longrightarrow Y$ is a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $dimKerL=codimImL<+\infty$, and ImL is closed in Y. If L is a Fredholm mapping of index zero, there exist continuous projectors $P:X\longrightarrow X$ and $Q:Y\longrightarrow Y$ such that

$ImP = KerL$ and $ImL = KerQ = Im(I-Q)$. It follows that $L|_{DomL\cap KerP} : (I-P)X \longrightarrow ImL$ has an inverse which will be denoted by K_P. If Ω is an open and bounded subset of X, the mapping N will be called $L-$compact on Ω if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N(\overline{\Omega})$ is compact. Since ImQ is isomorphic to $KerL$, there exists an isomorphism $J : ImQ \longrightarrow KerL$.

Theorem B (Mawhin's continuation theorem[3]). Let L be a Fredholm mapping of index zero, and let N be $L-$compact on $\overline{\Omega}$. Suppose

(1) for each $\lambda \in (0,1)$ and $x \in \partial\Omega, Lx \neq \lambda Nx$ and

(2) for each $x \in \partial\Omega \cap Ker(L), QNx \neq 0$ and $deg(QN, \Omega\cap Ker(L), 0) \neq 0$.

Then the equation $Lx = Nx$ has at least one solution in $\overline{\Omega}\cap D(L)$.

2 Main Results

Now we make the following assumption on $a(t)$:

(H_1) $\quad (\frac{\pi}{T})^2 > M = \max_{t\in[0,T]} a(t) \geq a(t) \geq m = \min_{t\in[0,T]} a(t) > 0.$

Our main results are the following theorems.

Theorem 2.1 Suppose (H_1) holds and also assume there exists a constant $K_1 > 0$ such that (H_2)

$$\|g\|_0 \leq m - 3|c|M - \|p\|_0,$$

where $\|g\|_0 = \max_{\{t\in[0,T],|x_1|\leq K_1,\cdots,|x_n|\leq K_1\}} |g(t, x_1, x_2, \cdots, x_n)|$ and $\|p\|_0 = \max_{t\in[0,T]} |p(t)|$. Then Eq.(1.1) possesses a nontrivial $T-$periodic solution.

Theorem 2.2 Suppose (H_1) holds and also assume
(H_3)

$$|g(t, x_1, x_2, \cdots, x_n)| \leq \gamma \sum_{i=1}^{n} |x_i|.$$

Then Eq.(1.1) has at least one $T-$periodic solution as $0 < \gamma < \frac{1}{n}[(1-|c|)m - |c|M]$.

In order to prove the main theorems we need some preliminaries. Set

$$X := \{x | x \in C^2(\mathbf{R}, \mathbf{R}), x(t+T) = x(t), \forall t \in \mathbf{R}\}$$

and $x^{(0)}(t) = x(t)$ and define the norm on X as follows

$$||x|| = \max_{t\in[0,T]} |x(t)| + \max_{t\in[0,T]} |x^{'}(t)| + \max_{t\in[0,T]} |x^{''}(t)|.$$

Remark 2.3 If $x \in X$, then it follows that $x^{(i)}(0) = x^{(i)}(T)(i = 0,1,2)$.

In order to prove our main results, we need the following Lemma [10].

Lemma 2.4 ([10]). Suppose that M is a positive number and satisfies $0 < M < (\frac{\pi}{T})^2$. Then for any function φ defined in $[0,T]$, the following equation

$$\begin{cases} x^{''}(t) + Mx(t) = \varphi(t), \\ x(0) = x(T), x^{'}(0) = x^{'}(T) \end{cases}$$

has a unique solution

$$x(t) = \int_0^T G(t,s)\varphi(s)ds,$$

where

$$G(t,s)\begin{cases} w(t-s), & (k-1)T \le s \le t \le kT \\ w(T+t-s), & (k-1)T \le t \le s \le kT(k \in \mathbf{N}), \end{cases}$$

$$w(t) = \frac{\cos\alpha(t-\frac{T}{2})}{2\alpha\sin\frac{\alpha T}{2}}$$

and $\alpha = \sqrt{M}$. Here

$$\max_{t\in[0,T]} \int_0^T |G(t,s)|ds = \frac{1}{M}.$$

Proof of Theorem 2.1: For $\forall x \in X$, define the operators $U : X \longrightarrow X$ and $S : X \longrightarrow X$ respectively by

$$(Ux)(t) = -cx(t-\tau) \tag{2.1}$$

and

$$\begin{aligned} (Sx)(t) \quad &= cx(t-\tau) + \int_0^T G(t,s)[-cx^{''}(s-\tau)(M-a(s))x(s) + p(s) \\ &-g(s, x(s-\tau_1(s)), x(s-\tau_2(s))\cdots, x(s-\tau_n(s)))]ds. \end{aligned} \tag{2.2}$$

It is clear that a fixed point of $U+S$ is a $T-$periodic solution of Eq.(1.1).

We are going to demonstrate that U and S satisfy the conditions of Theorem A.

Let $x, y \in X$ and $|x| \le K_1, |y| \le K_1$(here K_1 is as in the statement of Theorem 2.1). Now we prove that $|Ux + Sy| \le K_1$ holds.

First, we have the following equality:

$$\int_0^T G(t,s)x^{''}(s-\tau)ds = M\int_0^T G(t,s)x(s-\tau)ds. \tag{2.3}$$

In fact, we have from Lemma 2.4

$$\begin{aligned}
\int_0^T G(t,s)x''(s-\tau)ds &= \int_0^t \frac{\cos\alpha(t-s-\frac{T}{2})}{2\alpha\sin\frac{T\alpha}{2}} d[x'(s-\tau)] + \int_t^T \frac{\cos\alpha(t-s+\frac{T}{2})}{2\alpha\sin\frac{T\alpha}{2}} d[x'(s-\tau)] \\
&= \frac{\cos\alpha(t-s-\frac{T}{2})}{2\alpha\sin\frac{T\alpha}{2}} x'(s-\tau)|_0^t - \alpha\int_0^t \frac{\sin\alpha(t-s-\frac{T}{2})}{2\alpha\sin\frac{T\alpha}{2}} d[x(s-\tau)] \\
&+ \frac{\cos\alpha(t-s+\frac{T}{2})}{2\alpha\sin\frac{T\alpha}{2}} x'(s-\tau)|_t^T - \alpha\int_t^T \frac{\sin\alpha(t-s+\frac{T}{2})}{2\alpha\sin\frac{T\alpha}{2}} d[x(s-\tau)] \\
&= -\alpha\Big[\frac{\sin\alpha(t-s-\frac{T}{2})}{2\alpha\sin\frac{T\alpha}{2}} x(s-\tau)|_0^t + \frac{\sin\alpha(t-s+\frac{T}{2})}{2\alpha\sin\frac{T\alpha}{2}} x(s-\tau)|_t^T\Big] \\
&+ \alpha^2\Big[\int_0^t \frac{\cos\alpha(t-s-\frac{T}{2})}{2\alpha\sin\frac{T\alpha}{2}} x(s-\tau)ds + \int_t^T \frac{\cos\alpha(t-s+\frac{T}{2})}{2\alpha\sin\frac{T\alpha}{2}} x(s-\tau)ds\Big] \\
&= M\int_0^T G(t,s)x(s-\tau)ds,
\end{aligned} \tag{2.4}$$

so (2.3) holds.
From (H_1), (H_2) and (2.1)-(2.3), we have

$$\begin{aligned}
|(Uy)(t)+(Sx)(t)| &\le |(Uy)(t)| + |(Sx)(t)| \\
&\le 2|c|K_1 + |\int_0^T G(t,s)(M-a(s))x(s) - cx''(s-\tau) + p(s) \\
&-g(s,x(s-\tau_1(s)),x(s-\tau_2(s))\cdots,x(s-\tau_n(s)))]ds| + |c|K_1 \\
&\le 2|c|K_1 + \frac{M-m}{M}K_1 + \frac{\|g\|_0}{M} + |c|M||\int_0^T G(t,s)x(s-\tau)ds| \\
&\le 3|c|K_1 + \frac{M-m}{M}K_1 + \frac{\|g\|_0+\|p\|_0}{M} \\
&\le K_1, \quad x,y\in X,
\end{aligned} \tag{2.5}$$

where $\|g\|_0$ and $\|p\|_0$ are given in (H_2).

Set

$$K_2 = \frac{\rho_0[(M-m)K_1+|c|K_3+\|g\|_0+\|p\|_0]}{1-2|c|}, \tag{2.6}$$

where $\rho_0 = \frac{T}{2\sin\frac{T\alpha}{2}}$,

$$K_3 = \frac{MK_1+\|g\|_0+\|p\|_0}{1-|c|} \tag{2.7}$$

and

$$G = \{x\in X : |x(t)|\le K_1, |x'(t)|\le K_2, |x''(t)|\le K_3\}.$$

It is clear that G is a bounded, convex and closed subset of X.

(1) For $\forall x,y\in G$, we will show that

$$|\frac{d}{dt}[(Uy)(t)+(Sx)(t)]| \le K_2 \tag{2.8}$$

and

$$|\frac{d^2[(Uy)(t)+(Sx)(t)]}{dt^2}| \le K_3. \tag{2.9}$$

From (2.1) we have

$$\frac{d}{dt}[(Ux)(t)] = -cx'(t-\tau) \tag{2.10}$$

and

$$\frac{d^2[(Ux)(t)]}{dt^2} = -cx''(t-\tau). \tag{2.11}$$

Also from Lemma 2.4 and (2.2) we have

$$\begin{aligned} \frac{d}{dt}[(Sx)(t)] &= \int_0^T G_t(t,s)[(M-a(s))x(s) - cx''(s-\tau) + p(s) \\ &-g(s,x(s-\tau_1(s)),x(s-\tau_2(s))\cdots,x(s-\tau_n(s)))]ds + cx''(t-\tau), \end{aligned} \tag{2.12}$$

where

$$G_t(t,s)\begin{cases} \widetilde{w}(t-s), & (k-1)T \le s \le t \le kT \\ \widetilde{w}(T+t-s), & (k-1)T \le t \le s \le kT(k \in \mathbf{N}) \end{cases}$$

and

$$\widetilde{w}(t) = \frac{\sin\alpha(t-\frac{T}{2})}{2\sin\frac{\alpha T}{2}},$$

since

$$\begin{aligned} \frac{d}{dt}[(Sx)(t)] &= \{\int_0^T G_t(t,s)[(M-a(s))x(s) - cx''(s-\tau) + p(s) \\ &-g(s,x(s-\tau_1(s)),x(s-\tau_2(s))\cdots,x(s-\tau_n(s)))]ds + cx''(t-\tau)\}' \\ &= \{\int_0^t \frac{\cos\alpha(t-s-\frac{T}{2})}{2\alpha\sin\frac{T\alpha}{2}}[(M-a(s))x(s) - cx''(s-\tau) + p(s) \\ &-g(s,x(s-\tau_1(s)),x(s-\tau_2(s))\cdots,x(s-\tau_n(s)))]ds + cx''(t-\tau)\}' \\ &+\{\int_t^s \frac{\cos\alpha(t-s+\frac{T}{2})}{2\alpha\sin\frac{T\alpha}{2}}[(M-a(s))x(s) - cx''(s-\tau) + p(s) \\ &-g(s,x(s-\tau_1(s)),x(s-\tau_2(s))\cdots,x(s-\tau_n(s)))]ds + cx''(t-\tau)\}' \\ &= \alpha\{\int_0^t \frac{\cos\alpha(t-s-\frac{T}{2})}{2\alpha\sin\frac{T\alpha}{2}}[(M-a(s))x(s) - cx''(s-\tau) + p(s) \\ &-g(s,x(s-\tau_1(s)),x(s-\tau_2(s))\cdots,x(s-\tau_n(s)))]ds + cx''(t-\tau)\} \\ &+\alpha\{\int_t^s \frac{\cos\alpha(t-s+\frac{T}{2})}{2\alpha\sin\frac{T\alpha}{2}}[(M-a(s))x(s) - cx''(s-\tau) + p(s) \\ &-g(s,x(s-\tau_1(s)),x(s-\tau_2(s))\cdots,x(s-\tau_n(s)))]ds + cx''(t-\tau)\}. \end{aligned}$$

Note

$$\int_0^T |G_t(t,s|ds \leq \frac{T}{2\sin\frac{\alpha T}{2}} = \rho_0$$

and

$$\frac{d^2[(Sx)(t)]}{dt^2} = p(t) - a(t)x(t) - g(t, x(t-\tau_1(t)), x(t-\tau_2(t)) \cdots, x(t-\tau_n(t))). \tag{2.13}$$

From (2.6),(2.7) and (2.10)-(2.13), we have

$$\begin{aligned} |\frac{d}{dt}[(Uy)(t) + (Sx)(t)]| &\leq |\frac{d}{dt}[(Uy)(t)]| + |\frac{d}{dt}[(Sx)(t)]| \\ &\leq 2|c|K_2 + \rho_0[(M-m)K_1 + |c|K_3 + \|g\|_0 + \|p\|_0] \\ &\leq K_2 \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} |\frac{d^2[(Uy)(t)+(Sx)(t)]}{dt^2}| &= |(M-a(t))x(t) - cy''(t-\tau) + p(t) \\ &-g(t, x(t-\tau_1(t)), x(t-\tau_2(t)) \cdots, x(t-\tau_n(t)))| \\ &\leq (M-m)K_1 + |c|K_3 + \|g\|_0 + \|p\|_0 \\ &\leq K_3. \end{aligned} \tag{2.15}$$

From (2.5), (2.14) and (2.15), we have $Ux + Sy \in G$ for $\forall x, y \in G$.

(2) U is a contraction mapping.

Let $x, y \in G$ and we from (2.1) that

$$\begin{aligned} \|Ux - Uy\| &= \max_{t\in[0,T]}|cx(t-\tau) - cy(t-\tau)| + \max_{t\in[0,T]}|cx'(t-\tau) - cy'(t-\tau)| \\ &+\max_{t\in[0,T]}|cx''(t-\tau) - cy''(t-\tau)| \\ &= |c|[\max_{t\in[0,T]}|x(t-\tau) - y(t-\tau)| + \max_{t\in[0,T]}|x'(t-\tau) - y'(t-\tau)| \\ &+\max_{t\in[0,T]}|x''(t-\tau) - y''(t-\tau)|] \\ &= |c|\|x-y\|. \end{aligned}$$

Since $|c| < 1$, U is a contraction mapping.

(3) S is completely continuous.

We can obtain the continuity of S from the continuity of $a(t), p(t)$ and $g(t, x(t-\tau_1(t)), x(t-\tau_2(t)) \cdots, x(t-\tau_n(t)))$ for $t \in [0,T], x \in G$. In fact, suppose that $x_k \in G$ and $\|x_k - s\| \to 0$ as

$k \to +\infty$. Since G is closed convex subset of X, we have $x \in G$. Then

$$\begin{aligned} |Sx_k - Sx| &= c[x_k(t-\tau) - x(t-\tau)] + c[x_k(t-\tau) - x(t-\tau)] \\ &+ \int_0^T G(t,s)\{(M-a(s))(x_k(s) - x(s)) - c[x_k''(s-\tau) - x''(s-\tau)] \\ &-[g(s, x_k(s-\tau_1(s)), x_k(s-\tau_2(s)) \cdots, x_k(s-\tau_n(s))) \\ &-g(s, x(s-\tau_1(s)), x(s-\tau_2(s)) \cdots, x(s-\tau_n(s)))]\}ds. \end{aligned} \tag{2.16}$$

Using the Lebesgue dominated convergence theorem, we have from (2.12), (2.13) and (2.16) that

$$\lim_{k\to+\infty} \|Sx_k - Sx\| = 0.$$

Then S is continuous.

Next, we prove that Sx is relatively compact. It suffices to show that the family of functions $\{Sx : x \in G\}$ is uniformly bounded and equicontinuous on $[0, T]$. From (2.2), (2.12) and(2.13), it is easy to see that $\{Sx : x \in G\}$ is uniformly bounded and equicontinuity. Since S is continuous and is relatively compact, S is completely continuous. By Theorem A (Kranoselskii fixed point theorem), we have a fixed point x of $U + S$. That means that x is a $T-$periodic solution of Eq.(1.1).

In order to prove Theorem 2.2, we need some preliminaries. Set

$$Z := \{x | x \in C^1(\mathbf{R}, \mathbf{R}), x(t+T) = x(t), \forall t \in \mathbf{R}\}$$

and $x^{(0)}(t) = x(t)$ and define the norm on Z as follows

$$||x|| = \max\{\max_{t\in[0,T]}|x(t)|, \max_{t\in[0,T]}|x'(t)|\},$$

and set

$$Y := \{y | y \in C(\mathbf{R}, \mathbf{R}), y(t+T) = y(t), \forall t \in \mathbf{R}\}.$$

We define the norm on Y as follow $||y||_0 = \max_{t\in[0,T]}|y(t)|$. Thus both $(Z, ||\cdot||)$ and $(Y, ||\cdot||_0)$ are Banach spaces.

Remark 2.5 If $x \in Z$, then it follows that $x^{(i)}(0) = x^{(i)}(T) (i = 0, 1)$.

Define the operators $L : Z \longrightarrow Y$ and $N : Z \longrightarrow Y$ respectively by

$$Lx(t) = x''(t), \quad t \in \mathbf{R}, \tag{2.17}$$

and

$$\begin{aligned} Nx(t) &= -cx''(t-\tau) - a(t)x(t) + p(t) \\ &\quad -g(t, x(t-\tau_1(t)), x(t-\tau_2(t)), \cdots, x(t-\tau_n(t))), \quad t \in \mathbf{R}. \end{aligned} \tag{2.18}$$

Clearly,

$$KerL = \{x \in Z : x(t) = c \in \mathbf{R}\} \tag{2.19}$$

and

$$ImL = \{y \in Y : \int_0^T y(t)dt = 0\} \tag{2.20}$$

is closed in Y. Thus L is a Fredholm mapping of index zero.

Let us define $P : Z \to Z$ and $Q : Y \to Y/Im(L)$ respectively by

$$Px(t) = \frac{1}{T}\int_0^T x(t)dt = x(0), \quad t \in \mathbf{R}, \tag{2.21}$$

for $x = x(t) \in X$ and

$$Qy(t) = \frac{1}{T}\int_0^T y(t)dt, \quad t \in \mathbf{R} \tag{2.22}$$

for $y = y(t) \in Y$. It is easy to see that $ImP = KerL$ and $ImL = KerQ = Im(I-Q)$. It follows that $L|_{DomL\cap KerP} : (I-P)Z \longrightarrow ImL$ has an inverse which will be denoted by K_P.

Let Ω be an open and bounded subset of Z, we can easily see that $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N(\overline{\Omega})$ is compact. Thus the mapping N is $L-$compact on $\overline{\Omega}$. That is, we have the following result.

Lemma 2.6. Let L, N, P and Q be defined by (2.17), (2.18), (2.21) and (2.22) respectively. Then L is a Fredholm mapping of index zero and N is $L-$compact on $\overline{\Omega}$, where Ω is any open and bounded subset of Z.

In order to prove Theorem 2.2, we need the following Lemma [12].

Lemma 2.7 ([12 and Remark 2.5]). Let $x(t) \in C^{(n)}(\mathbf{R},\mathbf{R}) \cap C_T$. Then

$$||x^{(i)}||_0 \le \frac{1}{2}\int_0^T |x^{(i+1)}(s)|ds, \quad i = 1, 2, \cdots, n-1,$$

where $n \ge 2$ and $C_T := \{x | x \in C(R,R), x(t+T) = x(t), \forall t \in \mathbf{R}\}$.

Now, we consider the following auxiliary equation

$$\begin{aligned} x''(t) &+ c\lambda x''(t-\tau) + a(t)\lambda x(t) = \lambda p(t) \\ &-\lambda g(t, x(t-\tau_1(t)), x(t-\tau_2(t)), \cdots, x(t-\tau_n(t))), \end{aligned} \tag{2.23}$$

where $0 < \lambda < 1$.

Lemma 2.8. Suppose that conditions of Theorem 2.2 are satisfied. If $x(t)$ is a $T-$periodic solution of Eq.(2.23), then there are positive constants $D_i (i = 0, 1)$, which are independent of λ, such that

$$||x^{(i)}||_0 \le D_i, \quad t \in [0, T], \quad i = 0, 1. \tag{2.24}$$

Proof: Suppose that $x(t)$ is a $T-$periodic solution of (2.23). We have from (H_3) and (2.23) that

$$|x^{''}(t)| \le \max_{t\in[0,T]} |c||x^{''}(t)| + M||x||_0 + ||p||_0 + \gamma n||x||_0. \tag{2.25}$$

From (2.25), we have

$$\max_{t\in[0,T]} |x^{''}(t)| \le \frac{1}{1-|c|}[(M + \gamma n)||x||_0 + ||p||_0]. \tag{2.26}$$

On the other hand, from Lemma 2.4 and (2.23), we get

$$\begin{aligned} x(t) \quad &= \int_0^T \widetilde{G}(t,s)\lambda[(M - a(s))x(s) + p(s) - cx^{''}(s-\tau) \\ &-g(s, x(s-\tau_1(s)), x(s-\tau_2(s)), \cdots, x(s-\tau_n(s))]ds, \end{aligned} \tag{2.27}$$

where

$$\widetilde{G}(t,s) \begin{cases} \widetilde{w}(t-s), \quad (k-1)T \le s \le t \le kT \\ \widetilde{w}(T+t-s), \quad (k-1)T \le t \le s \le kT (k \in \mathbf{N}), \end{cases} \tag{2.28}$$

$$\widetilde{w}(t) = \frac{\cos\alpha_1(t-\frac{T}{2})}{2\alpha_1 \sin\frac{\alpha_1 T}{2}}, \tag{2.29}$$

$\alpha_1 = \sqrt{\lambda M}$ and

$$\max_{t\in[0,T]} \int_0^T |\widetilde{G}(t,s)|ds = \frac{1}{\lambda M}. \tag{2.30}$$

From (H_3), (2.27) and (2.30), we have

$$\begin{aligned} \|x\|_0 \quad &= \max_{t\in[0,T]} |\int_0^T \widetilde{G}(t,s)\lambda[(M - a(s))x(s) + p(s) - cx^{''}(s-\tau) \\ &-g(s, x(s-\tau_1(s)), x(s-\tau_2(s)), \cdots, x(s-\tau_n(s))]ds| \\ &\le \frac{1}{M}[(M-m)\|x\|_0 + \|p\|_0 + |c|\max_{t\in[0,T]}|x^{''}(t)| + \gamma n\|x\|_0]. \end{aligned} \tag{2.31}$$

From (2.31), we have

$$\|x\|_0 \le \frac{|c|\max_{t\in[0,T]}|x^{''}(t)| + \|p\|_0}{m-\gamma n}. \tag{2.32}$$

Thus combining (2.26) and (2.32), we see that

$$\max_{t\in[0,T]} |x^{''}(t)| \le \frac{M+m}{m(1-|c|)-M|c|-\gamma n} = \xi \tag{2.33}$$

and

$$\|x\|_0 \le \frac{|c|\xi + \|p\|_0}{m-\gamma n} = D_0. \tag{2.34}$$

Finally from Lemma 2.4, (2.33) and (2.34), we get

$$||x^{'}||_0 \leq D_1. \tag{2.35}$$

The proof of Lemma 2.8 is complete.

Proof of Theorem 2.2: Suppose that $x(t)$ is a T-periodic solution of Eq.(2.23). By Lemma 2.8, there exist positive constants $D_i (i = 0, 1)$ which are independent of λ such that (2.24) is true. Consider any positive constant $\overline{D} > \max_{0\leq i\leq 1}\{D_i\} + \|p\|_0$.

Set

$$\Omega := \{x \in Z : ||x|| < \overline{D}\}.$$

We know that L is a Fredholm mapping of index zero and N is L-compact on $\overline{\Omega}$(see [3]).

Recall

$$Ker(L) = \{x \in Z : x(t) = c \in \mathbf{R}\}$$

and the norm on Z is

$$||x|| = \max\{\max_{t\in[0,T]}|x(t)|, \max_{t\in[0,T]}|x^{'}(t)|\}.$$

Then we have

$$x = \overline{D} \quad \text{or} \quad x = -\overline{D} \quad \text{for} \quad x \in \partial\Omega \cap Ker(L). \tag{2.36}$$

From (H_3) and (2.36), we have(if $\overline{D}$ is chosen large enough)

$$a(t)\overline{D} + g(t, \overline{D}, \overline{D}, \cdots, \overline{D}) - \|p\|_0 > 0 \quad \text{for} \quad t \in [0, T] \tag{2.37}$$

and

$$x^{'}(t) = 0 \quad \text{and} \quad x^{''}(t) = 0, \quad \text{for} \quad t \in [0, T]. \tag{2.38}$$

Finally from (2.18), (2.22), (2.37) and (2.38), we have

$$\begin{aligned}(QNx) \quad &= \tfrac{1}{T}\int_0^T[-cx^{''}(t-\tau) - a(t)x(t) + p(t)]dt \\ &-g(t, x(t-\tau_1(t)), x(t-\tau_2(t)), \cdots, x(t-\tau_n(t)))]dt \\ &\neq 0, \quad \forall x \in \partial\Omega \cap Ker(L).\end{aligned}$$

Then, for any $x \in KerL \cap \partial\Omega$ and $\eta \in [0, 1]$, we have

$$\begin{aligned}xH(x, \eta) \quad &= -\eta x^2 - \tfrac{x}{T}(1-\eta)\int_0^T[cx^{''}(t-\tau) + a(t)x(t) - p(t) \\ &+g(t, x(t-\tau_1(t)), x(t-\tau_2(t)), \cdots, x(t-\tau_n(t)))dt]dt \\ &\neq 0.\end{aligned}$$

Thus

$$
\begin{aligned}
deg\{QN, \quad & \Omega\cap Ker(L),0\} \\
&= deg\{-\frac{1}{T}\int_0^T[cx''(t-\tau)+a(t)x(t)-p(t) \\
&+g(t,x(t-\tau_1(t)),x(t-\tau_2(t)),\cdots,x(t-\tau_n(t)))]dt,\Omega\cap Ker(L),0\} \\
&= deg\{-x,\Omega\cap Ker(L),0\} \\
&\neq 0.
\end{aligned}
$$

From Lemma 2.8 for any $x\in\partial\Omega\cap Dom(L)$ and $\lambda\in(0,1)$ we have $Lx\neq\lambda Nx$. By Theorem B (Mawhin's continuation theorem), the equation $Lx = Nx$ has at least a solution in $Dom(L)\cap\overline{\Omega}$, so there exists a T-periodic solution of Eq.(1.1). The proof is complete.

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