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Differences of weighted composition operators between weighted Banach spaces of holomorphic functions and weighted Bloch type spaces

Elke Wolf
Mathematical Institute,
University of Paderborn,
D-33095 Paderborn, Germany.
email: `lichte@math.uni-paderborn.de`

ABSTRACT

We consider analytic self-maps ϕ_1, ϕ_2 of the open unit disk as well as analytic maps ψ_1, ψ_2. These maps induce differences of weighted composition operators acting between weighted Banach spaces of holomorphic functions and weighted Bloch type spaces. In this article we give necessary and sufficient conditions for such a difference to be bounded resp. compact.

RESUMEN

Nosotros consideramos auto aplicaciones ϕ_1, ϕ_2 del disco unitario abierto bien como aplicaciones analíticas ψ_1, ψ_2. Estas aplicaciones inducen diferencias de composición de operadores con peso actuando entre espacios de Banach pesados de funciones holomorfas y espacios de tipo Bloch con peso. En este artículo damos condiciones necesarias y suficientes para que tal diferencia sea acotada, respectivamente, compacta.

Key words and phrases: *weighted composition operators, weighted Bloch type spaces, weighted Banach spaces of holomorphic functions.*

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1 Introduction

For analytic self-maps ϕ_1, ϕ_2 of $\mathbb{D}$ and analytic maps ψ_1, ψ_2 the corresponding weighted composition operators $\psi_i C_{\phi_i}$ are defined by $\psi_i C_{\phi_i} f = \psi_i f \circ \phi_i$, $i = 1, 2$. Composition operators and weighted composition operators acting on various spaces of analytic functions have recently been of much interest, see for example [14], [8], [12], [2], [4], [13]. Differences of them have been studied e.g. in [3], [9], [16], [17], [18].
Let v and w be strictly positive, continuous and bounded functions *(weights)* on $\mathbb{D}$ and $H(\mathbb{D})$ be the set of all analytic functions on $\mathbb{D}$. In this article we are interested in differences $\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2}$ acting between weighted Banach spaces of holomorphic functions

$$H_v^\infty := \{f \in H(\mathbb{D});\ \|f\|_v := \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty\}$$

and the weighted Bloch type spaces B_w of functions $f \in H(\mathbb{D})$ satisfying
$\|f\|_{B_w} := \sup_{z\in\mathbb{D}} v(z)|f'(z)| < \infty$.
Our aim is to give necessary and sufficient conditions for a difference $\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2} : H_v^\infty \to B_w$ to be bounded resp. compact in terms of the involved weights and the analytic maps $\phi_1, \phi_2, \psi_1, \psi_2$.

2 Notation and auxiliary results

An introduction to the concept of composition operators can be found in the monographs [5] and [15]. In this article we are especially interested in radial weights (i.e. weights with $v(z) = v(|z|)$ for every $z \in \mathbb{D}$) which satisfy additionally the Lusky condition $(L1)$ (due to Lusky [10])

$$(L1) \qquad \inf_{k\in\mathbb{N}} \frac{v(1-2^{-k-1})}{v(1-2^{-k})} > 0.$$

When dealing with differences of weighted composition operators we need some geometric data. Recall that for any $z \in \mathbb{D}$, φ_z is the Möbius transformation which interchanges the origin and z, namely $\varphi_z(w) = \frac{z-w}{1-\bar{z}w}, w \in \mathbb{D}$. The pseudohyperbolic distance $\rho(z,w)$ for $z, w \in \mathbb{D}$ is defined by $\rho(z,w) = |\varphi_z(w)|$. Moreover, we have $\varphi_z'(w) = \frac{|z|^2-1}{(1-\bar{z}w)^2}$ for $z, w \in \mathbb{D}$. Let us recall some auxiliary results.

The next lemma is taken from [3], see also [6].

Lemma 1. *(Bonet-Lindström-Wolf [3]) Let v be a radial weight satisfying the Lusky condition $(L1)$ and let $f \in H_v^\infty$. Then there exists a constant $C_v > 0$ (depending on the weight v) such that*

$$|f(z) - f(p)| \leq C_v \|f\|_v \max\left\{\frac{1}{v(z)}, \frac{1}{v(p)}\right\} \rho(z,p)$$

for all $z, p \in \mathbb{D}$.

Theorem 2. *(Harutyunyan-Lusky, [7] Theorem 2.1) Let v and w be radial weights which are continuously differentiable with respect to $|z|$ with $\lim_{|z|\to 1} v(z) = \lim_{|z|\to 1} w(z) = 0$ and such that H_w^∞ is isomorphic to l_∞. If $\limsup_{r\to 1}\left(-\frac{w'(r)}{v(r)}\right) < \infty$, then $D : H_v^\infty \to H_w^\infty, f \to f'$ is bounded.*

For conditions when H_w^∞ is isomorphic to l_∞ we refer the reader to [11] and [7]. By [7] we know that the following weights have the desired properties:
$w(z) = (1 - |z|)^\alpha, \alpha > 0, w(z) = e^{-\frac{1}{1-|z|}}$, $z \in \mathbb{D}$.

For the study of the compactness of the difference $\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2}$ we need the following result.

Proposition 3. *(Cowen-MacCluer, [5] Proposition 3.11) Let X and Y be H_v^∞ or B_w. Then $\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2} : X \to Y$ is compact if and only if for every bounded sequence $(f_n)_{n\in\mathbb{N}}$ in X such that $f_n \to 0$ uniformly on the compact subsets of $\mathbb{D}$, then $(\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2}) f_n \to 0$ in Y.*

3 Main Result

In the sequel we consider weights v of the following type: Let ν be a holomorphic function on $\mathbb{D}$, non-vanishing and strictly positive on $[0, 1[$. Moreover we assume that ν is decreasing on $[0, 1[$ and satisfies $\lim_{r\to 1} \nu(r) = 0$. Then we define the corresponding weight v by $v(z) := \nu(|z|^2)$ for every $z \in \mathbb{D}$. Furthermore, we suppose the boundedness of the function ν' on $\mathbb{D}$. Next, we give some illustrating examples of weights of this type:

(i) Consider $\nu(z) = (1-z)^\alpha$, $\alpha \geq 1$. Then the corresponding weight is the so-called standard weight $v(z) = (1 - |z|^2)^\alpha$.

(ii) Selecting $\nu(z) = e^{-\frac{1}{(1-z)^\alpha}}$, $\alpha \geq 1$, we obtain the weight $v(z) = e^{-\frac{1}{(1-|z|^2)^\alpha}}$.

(iii) Choose $\nu(z) = \sin(1 - z)$ and the corresponding weight is given by $v(z) = \sin(1 - |z|^2)$.

Fix a point $p \in \mathbb{D}$ and an analytic self-map ϕ of $\mathbb{D}$. We introduce a function

$$v_{\phi(p)}(z) := \nu(\overline{\phi(p)} z) \text{ for every } z \in \mathbb{D}.$$

Since ν is holomorphic on $\mathbb{D}$, the function $v_{\phi(p)}$ is also holomorphic on $\mathbb{D}$. Furthermore, $v_{\phi(p)}(\phi(p)) = \nu(|\phi(p)|^2) = v(\phi(p))$ and $v'_{\phi(p)}(z) = \overline{\phi(p)}\nu'(\overline{\phi(p)} z)$ for every $z \in \mathbb{D}$, i.e. $v'_{\phi(p)}(\phi(p)) = \overline{\phi(p)}\nu'(|\phi(p)|^2)$.

We start with considering boundedness of operators $\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2} : H_v^\infty \to B_w$ and give first a necessary condition in terms of the involved weights and then a sufficient condition.

Proposition 4. *Let w be a weight and v be a weight as described in the beginning of this section. Let $\psi_1, \psi_2 \in H(\mathbb{D})$ and ϕ_1, ϕ_2 be analytic self-maps of $\mathbb{D}$. If $\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2} : H_v^\infty \to B_w$ is bounded, then the following conditions are satisfied*

(a) $\sup_{z\in\mathbb{D}} w(z) \left| \frac{\psi_1'(z)}{v(\phi_1(z))^{\frac{1}{2}}} \varphi^2_{\phi_2(z)}(\phi_1(z)) + 2\frac{\psi_1(z)}{v(\phi_1(z))^{\frac{1}{2}}} \varphi_{\phi_2(z)}(\phi_1(z))\varphi'_{\phi_2(z)}(\phi_1(z)) \right| < \infty,$

(b) $\sup_{z\in\mathbb{D}} w(z) \left| \frac{\psi_2'(z)}{v(\phi_2(z))^{\frac{1}{2}}} \varphi^2_{\phi_1(z)}(\phi_2(z)) + 2\frac{\psi_2(z)}{v(\phi_2(z))^{\frac{1}{2}}} \varphi_{\phi_1(z)}(\phi_2(z))\varphi'_{\phi_1(z)}(\phi_2(z)) \right| < \infty,$

(c) $\sup_{z\in\mathbb{D}} \left| \frac{\psi_1(z) w(z) \overline{\phi_1(z)} \nu'(|\phi_1(z)|^2)}{v(\phi_1(z))} \right| \rho(\phi_1(z), \phi_2(z)) < \infty,$

(d) $\sup_{z\in\mathbb{D}} \left| \frac{\psi_2(z) w(z) \overline{\phi_2(z)} \nu'(|\phi_2(z)|^2)}{v(\phi_2(z))} \right| \rho(\phi_1(z), \phi_2(z)) < \infty,$

Proof (a) Fix a point $p \in \mathbb{D}$ and put

$$f_{\phi_1(p)}(z) := \left(\frac{2}{v_{\phi_1(p)}(z)} - \frac{v_{\phi_1(p)}(\phi_1(p))}{v_{\phi_1(p)}(z)^2} \right)^{\frac{1}{2}}$$

and

$$g_{\phi_1(p)}(z) := f_{\phi_1(p)}(z)\varphi^2_{\phi_2(p)}(z) \text{ for every } z \in \mathbb{D}.$$

Next, we get

$$\|g_{\phi_1(p)}\|_v \le \sup_{z\in\mathbb{D}} \left| v(z)^2 \frac{2}{v_{\phi_1(p)}(z)} - v(z)^2 \frac{v_{\phi_1(p)}(\phi_1(p))}{v_{\phi_1(p)}(z)^2} \right|^{\frac{1}{2}} \le (3M)^{\frac{1}{2}}$$

where $M = \sup_{z\in\mathbb{D}} v(z)$ and therefore the constant does not depend on the choice of p. Thus, $g_{\phi_1(p)} \in H_v^\infty$ and $g'_{\phi_1(p)}(z) = f'_{\phi_1(p)}(z)\varphi^2_{\phi_2(p)}(z) + 2f_{\phi_1(p)}(z)\varphi'_{\phi_2(p)}(z)\varphi_{\phi_2(p)}(z)$ for every $z \in \mathbb{D}$, where $f'_{\phi_1(p)}(z) = \left(-\frac{v'_{\phi_1(p)}(z)}{v_{\phi_1(p)}(z)^2} + \frac{v'_{\phi_1(p)}(z)v_{\phi_1(p)}(\phi_1(p))}{v_{\phi_1(p)}(z)^3}\right)\left(\frac{2}{v_{\phi_1(p)}(z)} - \frac{v_{\phi_1(p)}(\phi_1(p))}{v_{\phi_1(p)}(z)^2}\right)^{-\frac{1}{2}}$ and $\varphi'_{\phi_2(p)}(z) = \frac{|\phi_2(p)|^2-1}{(1-\overline{\phi_2(p)}z)^2}$ for every $z \in \mathbb{D}$. We get $f_{\phi_1(p)}(\phi_1(p)) = \frac{1}{v(\phi_1(p))^{\frac{1}{2}}}$ and $f'_{\phi_1(p)}(\phi_1(p)) = 0$ and hence $g_{\phi_1(p)}(\phi_1(p)) = \frac{\varphi^2_{\phi_2(p)}(\phi_1(p))}{v(\phi_1(p))^{\frac{1}{2}}}$ as well as $g'_{\phi_1(p)}(\phi_1(p)) = 2\frac{\varphi'_{\phi_2(p)}(\phi_1(p))\varphi_{\phi_2(p)}(\phi_1(p))}{v(\phi_1(p))^{\frac{1}{2}}}$. Now,

$$\begin{aligned} & w(p) \left| \frac{\psi_1'(p)\varphi^2_{\phi_2(p)}(\phi_1(p))}{v(\phi_1(p)))^{\frac{1}{2}}} + 2\frac{\psi_1(p)\varphi_{\phi_2(p)}(\phi_1(p))\varphi'_{\phi_2(p)}(\phi_1(p))}{v(\phi_1(p)))^{\frac{1}{2}}} \right| \\ = {} & w(p) \left| \psi_1'(p) g_{\phi_1(p)}(\phi_1(p)) + \psi_1(p)\phi_1'(p) g'_{\phi_1(p)}(\phi_1(p)) - \psi_2'(p) g_{\phi_1(p)}(\phi_2(p)) - \psi_2(p)\phi_2'(p) g'_{\phi_1(p)}(\phi_2(p)) \right| \\ \le {} & \|\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2}\| \|g_{\phi_1(p)}\|_v < \infty. \end{aligned}$$

Thus, (a) follows, and we can show (b) analogously. For the proof of condition (c) we fix a point $p \in \mathbb{D}$ and put

$$f_{\phi_1(p)}(z) := \frac{v_{\phi_1(p)}(\phi_1(p))}{v_{\phi_1(p)}(z)} - \left(\frac{v_{\phi_1(p)}(\phi_1(p))}{v_{\phi_1(p)}(z)}\right)^{\frac{1}{2}} = \frac{v(\phi_1(p))}{v_{\phi_1(p)}(z)} - \left(\frac{v(\phi_1(p))}{v_{\phi_1(p)}(z)}\right)^{\frac{1}{2}}$$

and

$$g_{\phi_1(p)}(z) := f_{\phi_1(p)}(z)\varphi^2_{\phi_2(p)}(z) \text{ for every } z \in \mathbb{D}.$$

Hence $\|g_{\phi_1(p)}\|_v \le 2M$ and we get

$$g'_{\phi_1(p)}(z) = f'_{\phi_1(p)}(z)\varphi^2_{\phi_2(p)}(z) + 2f_{\phi_1(p)}(z)\varphi_{\phi_2(p)}(z)\varphi'_{\phi_2(p)}(z) \text{ for every } z \in \mathbb{D},$$

where

$$f'_{\phi_1(p)}(z) = -\frac{v_{\phi_1(p)}(\phi_1(p))v'_{\phi_1(p)}(z)}{v_{\phi_1(p)}(z)^2} + \frac{1}{2}\frac{v_{\phi_1(p)}(\phi_1(p))^{\frac{1}{2}}v'_{\phi_1(p)}(z)}{v_{\phi_1(p)}(z)^{\frac{3}{2}}}$$

Thus, we obtain $f_{\phi_1(p)}(\phi_1(p)) = 0$ and $f'_{\phi_1(p)}(\phi_1(p)) = -\frac{1}{2}\frac{\overline{\phi_1(p)}\nu'(|\phi_1(p)|^2)}{v(\phi_1(p))}$. Hence $g_{\phi_1(p)}(\phi_1(p)) = 0$ and $g'_{\phi_1(p)}(\phi_1(p)) = -\frac{1}{2}\frac{\overline{\phi_1(p)}\nu'(|\phi_1(p)|^2)\varphi^2_{\phi_2(p)}(\phi_1(p))}{v(\phi_1(p))}$. Finally,

$$\begin{aligned} & \frac{1}{2} w(p) \left| \frac{\overline{\phi_1(p)}\nu'(|\phi_1(p)|^2)\varphi^2_{\phi_2(p)}(\phi_1(p))}{v_{\phi_2(p)}(\phi_1(p))} \right| \\ = {} & w(p) \left| \psi_1'(p) g_{\phi_1(p)}(\phi_1(p)) + \psi_1(p)\phi_1'(p) g'_{\phi_1(p)}(\phi_1(p)) - \psi_2'(p) g_{\phi_1(p)}(\phi_2(p)) - \psi_2(p)\phi_2'(p) g'_{\phi_1(p)}(\phi_2(p)) \right| \\ \le {} & \|\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2}\| \|g_{\phi_1(p)}\|_v < \infty. \end{aligned}$$

The claim follows. We can show (d) analogously.

Proposition 5. *Let v and w be weights. If*

(a) *there is a weight u such that the operator $D : H_v^\infty \to H_u^\infty, f \to f'$ is bounded and additionally* $\sup_{z\in\mathbb{D}} \max\left\{\frac{w(z)}{u(\phi_1(z))}, \frac{w(z)}{u(\phi_2(z))}\right\} \rho(\phi_1(z), \phi_2(z)) < \infty$ *as well as* $\sup_{z\in\mathbb{D}} \max\left\{\frac{w(z)}{u(\phi_1(z))}, \frac{w(z)}{u(\phi_2(z))}\right\} |\phi_1'(z)\psi_1(z) - \phi_2'(z)\psi_2(z)| < \infty$,

(b) $\sup_{z\in\mathbb{D}} \max\left\{\frac{1}{v(\phi_1(z))}, \frac{1}{v(\phi_2(z))}\right\} w(z)|\psi_1'(z) - \psi_2'(z)| < \infty$,

(c) $\sup_{z\in\mathbb{D}} \max\left\{\frac{1}{v(\phi_1(z))}, \frac{1}{v(\phi_2(z))}\right\} w(z) \max\{|\psi_1'(z)|, |\psi_2'(z)|\}\rho(\phi_1(z), \phi_2(z)) < \infty$

then $\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2} : H_v^\infty \to B_w$ is bounded.

Proof Let $f \in H_v^\infty$. Using Lemma 1 we obtain

$$\begin{aligned}
\sup_{z\in\mathbb{D}} w(z)|((\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2})f)'(z)| &\le \sup_{z\in\mathbb{D}} w(z)|\psi_1'(z) - \psi_2'(z)||f(\phi_1(z))| \\
&+ \sup_{z\in\mathbb{D}} w(z)|\psi_2'(z)||f(\phi_1(z)) - f(\phi_2(z))| \\
&+ \sup_{z\in\mathbb{D}} w(z)|f'(\phi_1(z))||\phi_1'(z)\psi_1(z) - \phi_2'(z)\psi_2(z)| \\
&+ \sup_{z\in\mathbb{D}} w(z)|\phi_2'(z)\psi_2(z)||f'(\phi_1(z)) - f'(\phi_2(z))| \\
&\le \sup_{z\in\mathbb{D}} w(z)|\psi_1'(z) - \psi_2'(z)| \max\left\{\frac{1}{v(\phi_1(z))}, \frac{1}{v(\phi_2(z))}\right\} \|f\|_v \\
&+ \sup_{z\in\mathbb{D}} \max\{|\psi_1'(z)|, |\psi_2'(z)|\} \max\left\{\frac{1}{v(\phi_1(z))}, \frac{1}{v(\phi_2(z))}\right\} \rho(\phi_1(z), \phi_2(z))\|f\|_v \\
&+ \sup_{z\in\mathbb{D}} \frac{w(z)}{u(\phi_1(z))}\|f'\|_u|\phi_1'(z)\psi_1(z) - \phi_2'(z)\psi_2(z)| \\
&+ \sup_{z\in\mathbb{D}} \max\left\{\frac{w(z)}{u(\phi_1(z))}, \frac{w(z)}{u(\phi_2(z))}\right\} \rho(\phi_1(z), \phi_2(z))\|f'\|_u \\
&\le \sup_{z\in\mathbb{D}} w(z)|\psi_1'(z) - \psi_2'(z)| \max\left\{\frac{1}{v(\phi_1(z))}, \frac{1}{v(\phi_2(z))}\right\} \|f\|_v \\
&+ \sup_{z\in\mathbb{D}} \max\{|\psi_1'(z)|, |\psi_2'(z)|\} \max\left\{\frac{1}{v(\phi_1(z))}, \frac{1}{v(\phi_2(z))}\right\} \rho(\phi_1(z), \phi_2(z))\|f\|_v \\
&+ \sup_{z\in\mathbb{D}} \max\left\{\frac{w(z)}{u(\phi_1(z))}, \frac{w(z)}{u(\phi_2(z))}\right\} \|D\|\|f\|_v|\phi_1'(z)\psi_1(z) - \phi_2'(z)\psi_2(z)| \\
&+ \sup_{z\in\mathbb{D}} \max\left\{\frac{w(z)}{u(\phi_1(z))}, \frac{w(z)}{u(\phi_2(z))}\right\} \rho(\phi_1(z), \phi_2(z))\|D\|\|f\|_v
\end{aligned}$$

and the claim follows.

Next, we turn our attention to compactness of $\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2} : H_v^\infty \to B_w$.

Proposition 6. *Let w be a weight and v be a weight as described in the beginning of this section. Let $\psi_1, \psi_2 \in H(\mathbb{D})$ and ϕ_1, ϕ_2 be analytic self-maps of $\mathbb{D}$. If $\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2} : H_v^\infty \to B_w$ is bounded, then the following conditions are satisfied* *(a)* $\limsup_{|\phi_1(z)|\to 1} w(z)\left|\frac{\psi_1'(z)}{v(\phi_1(z))^{\frac{1}{2}}}\varphi^2_{\phi_2(z)}(\phi_1(z))+2\frac{\psi_1(z)}{v(\phi_1(z))^{\frac{1}{2}}}\varphi_{\phi_2(z)}(\phi_1(z))\varphi'_{\phi_2(z)}(\phi_1(z))\right|=0,$

(b) $\limsup_{|\phi_2(z)|\to 1} w(z)\left|\frac{\psi_2'(z)}{v(\phi_2(z))^{\frac{1}{2}}}\varphi^2_{\phi_1(z)}(\phi_2(z))+2\frac{\psi_2(z)}{v(\phi_2(z))^{\frac{1}{2}}}\varphi_{\phi_1(z)}(\phi_2(z))\varphi'_{\phi_1(z)}(\phi_2(z))\right|=0,$

(c) $\limsup_{|\phi_1(z)|\to 1}\left|\frac{\psi_1(z)w(z)\overline{\phi_1(z)}\nu'(|\phi_1(z)|^2)}{v(\phi_1(z))}\right|\rho(\phi_1(z),\phi_2(z))=0,$

(d) $\limsup_{|\phi_2(z)|\to 1}\left|\frac{\psi_2(z)w(z)\overline{\phi_2(z)}\nu'(|\phi_2(z)|^2)}{v(\phi_2(z))}\right|\rho(\phi_1(z),\phi_2(z))=0,$

Proof (a) Consider a sequence $(z_n)_n\subset\mathbb{D}$ such that $|\phi_1(z_n)|\to 1$ if $n\to\infty$. We set

$$f_{\phi_1(z_n)}(z):=v_{\phi_1(z_n)}(\phi_1(z_n))^{\frac{1}{6}}\left(\frac{3}{2}\frac{1}{v_{\phi_1(z_n)}(z)^2}-\frac{v_{\phi_1(z_n)}(\phi_1(z_n))}{v_{\phi_1(z_n)}(z)^3}\right)^{\frac{1}{3}}$$

and

$$g_{\phi_1(z_n)}(z):=f_{\phi_1(z_n)}(z)\varphi^2_{\phi_2(z_n)}(z)\text{ for every }z\in\mathbb{D}.$$

Thus $\|g_{\phi_1(z_n)}\|_v\leq\sup_{z\in\mathbb{D}}v_{\phi_1(z_n)}(\phi_1(z_n))^{\frac{1}{6}}\left|\frac{3}{2}\frac{v(z)^3}{v_{\phi_1(z_n)}(z)^2}-\frac{v(z)^3v_{\phi_1(z_n)}(\phi_1(z_n))}{v_{\phi_1(z_n)}(z)^3}\right|^{\frac{1}{3}}\leq M^{\frac{1}{6}}\left(\frac{5}{2}M\right)^{\frac{1}{3}}$ for every $n\in\mathbb{N}$, where $M:=\sup_{z\in\mathbb{D}}v(z)$. Thus, $(g_{\phi_1(z_n)})_{n\in\mathbb{N}}$ is a bounded sequence in H_v^∞ which converges to zero uniformly on the compact subsets of $\mathbb{D}$. Moreover,

$$g'_{\phi_1(z_n)}(z)=f'_{\phi_1(z_n)}(z)\varphi^2_{\phi_2(z_n)}(z)+2f_{\phi_1(z_n)}(z)\varphi_{\phi_2(z_n)}(z)\varphi'_{\phi_2(z_n)}(z)\text{ for every }z\in\mathbb{D},$$

where

$$f'_{\phi_1(z_n)}(z)=v_{\phi_1(z_n)}(\phi_1(z_n))^{\frac{1}{6}}\left(\frac{3}{2}\frac{1}{v_{\phi_1(z_n)}(z)^2}-\frac{v_{\phi_1(z_n)}(\phi_1(z_n))}{v_{\phi_1(z_n)}(z)^3}\right)^{-\frac{2}{3}}\cdot\\ \cdot\left(-\frac{v'_{\phi_1(z_n)}(z)}{v_{\phi_1(z_n)}(z)^3}+\frac{v_{\phi_1(z_n)}(\phi_1(z_n))}{v_{\phi_1(z_n)}(z)^4}v'_{\phi_1(z_n)}(z)\right)$$

for every $n\in\mathbb{N}$. By Proposition 3, the fact that $\psi_1C_{\phi_1}-\psi_2C_{\phi_2}$ is compact yields

$$\|(\psi_1C_{\phi_1}-\psi_2C_{\phi_2})g_{\phi_1(z_n)}\|_{B_w}\to 0\text{ if }n\to\infty.$$

Finally,

$$\|(\psi_1C_{\phi_1}-\psi_2C_{\phi_2})g_{\phi_1(z_n)}\|_{B_w}\geq\\ w(z_n)\left|\frac{\psi_1'(z_n)}{v(\phi_1(z_n))^{\frac{1}{2}}}\varphi^2_{\phi_2(z_n)}(\phi_1(z_n))+2\frac{\psi_1(z_n)\varphi_{\phi_2(z_n)}(\phi_1(z_n))\varphi'_{\phi_2(z_n)}(\phi_1(z_n))}{v(\phi_1(z_n))^{\frac{1}{2}}}\right|$$

Thus, (a) follows and we can show (b) analogously. Consider now

$$f_{\phi_1(z_n)}(z):=\frac{v_{\phi_1(z_n)}(\phi_1(z_n))}{v_{\phi_1(z_n)}(z)}-\left(\frac{v_{\phi_1(z_n)}(\phi_1(z_n))}{v_{\phi_1(z_n)}(z)}\right)^{\frac{1}{2}}=\frac{v(\phi_1(z_n))}{v_{\phi_1(z_n)}(z)}-\left(\frac{v(\phi_1(z_n))}{v_{\phi_1(z_n)}(z)}\right)^{\frac{1}{2}}$$

and

$$g_{\phi_1(z_n)}(z):=f_{\phi_1(z_n)}(z)\varphi^2_{\phi_2(z_n)}(z)\text{ for every }z\in\mathbb{D}.$$

Then $\|g_{\phi_1(z_n)}\|_v \le \sup_{z\in\mathbb{D}} v(z)\left|\frac{v_{\phi_1(z_n)}(\phi_1(z_n))}{v_{\phi_1(z_n)}(z)} - \left(\frac{v_{\phi_1(z_n)}(\phi_1(z_n))}{v_{\phi_1(z_n)}(z)}\right)^{\frac{1}{2}}\right| \le 2M$ for every $n \in \mathbb{N}$. Thus $(g_{\phi_1(z_n)})_n$ is a bounded sequence in H_v^∞ and $g_{\phi_1(z_n)} \to 0$ uniformly on every compact subset of $\mathbb{D}$. Moreover $g_{\phi_1(z_n)}(\phi_1(z_n)) = 0$ and $g'_{\phi_1(z_n)}(\phi_1(z_n)) = -\frac{1}{2}\frac{v'_{\phi_1(z_n)}(\phi_1(z_n))}{v(\phi_1(z_n))}\varphi^2_{\phi_2(z_n)}(\phi_1(z_n))$. Since $\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2}$ is compact, by Proposition 3 we have $\|(\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2})g_n\|_{B_w} \to 0$ if $n \to \infty$. Thus,

$$\begin{aligned}\|(\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2})g_{\phi_1(z_n)}\|_{B_w} &= \sup_{z\in\mathbb{D}} w(z)|((\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2})g_{\phi_1(z_n)})'(z)| \\ &\ge \frac{1}{2} w(z_n)|\psi_1(z_n)\phi_1'(z_n)\overline{\phi_1(z_n)}|\rho(\phi_1(z_n),\phi_2(z_n))^2\frac{|\nu'(|\phi_1(z_n)|^2)|}{v(\phi_1(z_n))}.\end{aligned}$$

Finally, $\limsup_{|\phi_1(z)|\to 1} w(z)|\psi(z)||\phi_1'(z)||\overline{\phi_1(z)}|\frac{|\nu'(|\phi_1(z)|^2)|}{v(\phi_1(z))} = 0$, and (c) holds. (d) follows analogously.

Proposition 7. *Let v and w be weights. If*

(a) there is a weight u such that the operator $D : H_v^\infty \to H_u^\infty, f \to f'$ is bounded and additionally $\limsup_{\max\{|\phi_1(z)|,|\phi_2(z)|\}\to 1} \max\left\{\frac{w(z)}{u(\phi_1(z))}, \frac{w(z)}{u(\phi_2(z))}\right\}\rho(\phi_1(z),\phi_2(z)) = 0$ as well as $\limsup_{\max\{|\phi_1(z)|,|\phi_2(z)|\}\to 1} \max\left\{\frac{w(z)}{u(\phi_1(z))}, \frac{w(z)}{u(\phi_2(z))}\right\}|\phi_1'(z)\psi_1(z) - \phi_2'(z)\psi_2(z)| = 0,$

(b) $\limsup_{\max\{|\phi_1(z)|,\phi_2(z)|\}\to 1} \max\left\{\frac{1}{v(\phi_1(z))}, \frac{1}{v(\phi_2(z))}\right\} w(z)|\psi_1'(z) - \psi_2'(z)| = 0,$

(c) $\limsup_{\max\{|\phi_1(z)|,|\phi_2(z)|\}\to 1} \max\left\{\frac{1}{v(\phi_1(z))}, \frac{1}{v(\phi_2(z))}\right\} w(z)\max\{|\psi_1'(z)|,|\psi_2'(z)|\}\rho(\phi_1(z),\phi_2(z)) = 0$

then $\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2} : H_v^\infty \to B_w$ is compact.

Proof Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in H_v^∞ with $\|f_n\|_v \le 1$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. By the assumption, for any $\varepsilon > 0$ there is $0 < \delta < 1$ such that $\delta < \max\{|\phi_1(z)|,|\phi_2(z)|\} < 1$ implies

$$\max\left\{\frac{w(z)}{u(\phi_1(z))}, \frac{w(z)}{u(\phi_2(z))}\right\}\rho(\phi_1(z),\phi_2(z)) < \varepsilon$$
$$\max\left\{\frac{w(z)}{u(\phi_1(z))}, \frac{w(z)}{u(\phi_2(z))}\right\}|\phi_1'(z)\psi_1(z) - \phi_2'(z)\psi_2(z)| < \varepsilon$$
$$\max\left\{\frac{1}{v(\phi_1(z))}, \frac{1}{v(\phi_2(z))}\right\} w(z)|\psi_1'(z) - \psi_2'(z)| < \varepsilon$$
$$\max\left\{\frac{1}{v(\phi_1(z))}, \frac{1}{v(\phi_2(z))}\right\} w(z)\max\{|\psi_1'(z)|,|\psi_2'(z)|\}\rho(\phi_1(z),\phi_2(z)) < \varepsilon$$

Then applying Lemma 1

$$\begin{aligned}
\sup_{z\in\mathbb{D}} w(z)|((\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2})f_n)'(z)| &\le \sup_{z\in\mathbb{D}} w(z)|\psi_1'(z) - \psi_2'(z)||f_n(\phi_1(z))| \\
&+ \sup_{z\in\mathbb{D}} w(z)|\psi_2'(z)||f_n(\phi_1(z)) - f_n(\phi_2(z))| \\
&+ \sup_{z\in\mathbb{D}} w(z)|f_n'(\phi_1(z))||\phi_1'(z)\psi_1(z) - \phi_2'(z)\psi_2(z)| \\
&+ \sup_{z\in\mathbb{D}} w(z)|\phi_2'(z)\psi_2(z)||f_n'(\phi_1(z)) - f_n'(\phi_2(z))| \\
\le \sup_{\{z;\ \max\{|\phi_1(z)|,|\phi_2(z)|\}>\delta\}} & w(z)|\psi_1'(z) - \psi_2'(z)| \max\left\{\frac{1}{v(\phi_1(z))}, \frac{1}{v(\phi_2(z))}\right\} \|f_n\|_v \\
+ \sup_{\{z;\ \max\{|\phi_1(z)|,|\phi_2(z)|\}>\delta\}} & \max\{|\psi_1'(z)|, |\psi_2'(z)|\} \max\left\{\frac{1}{v(\phi_1(z))}, \frac{1}{v(\phi_2(z))}\right\} \rho(\phi_1(z), \phi_2(z))\|f_n\|_v \\
&+ \sup_{\{z;\ \max\{|\phi_1(z)|,|\phi_2(z)|\}>\delta\}} \frac{w(z)}{u(\phi_1(z))}\|f'\|_u|\phi_1'(z)\psi_1(z) - \phi_2'(z)\psi_2(z)| \\
&+ \sup_{\{z;\ \max\{|\phi_1(z)|,|\phi_2(z)|\}>\delta\}} \max\left\{\frac{w(z)}{u(\phi_1(z))}, \frac{w(z)}{u(\phi_2(z))}\right\} \rho(\phi_1(z), \phi_2(z))\|f_n'\|_u \\
&+ \sup_{\{z;\ \max\{|\phi_1(z)|,\phi_2(z)|\}\le\delta\}} w(z)|\psi_1'(z) - \psi_2'(z)||f_n(\phi_1(z))|
\end{aligned}$$

$$\begin{aligned}
&+ \sup_{\{z;\ \max\{|\phi_1(z)|,\phi_2(z)|\}\le\delta\}} w(z)|\psi_2'(z)||f_n(\phi_1(z))| \\
&+ \sup_{\{z;\ \max\{|\phi_1(z)|,\phi_2(z)|\}\le\delta\}} w(z)|\psi_2'(z)||f_n(\phi_2(z))| \\
&+ \sup_{\{z;\ \max\{|\phi_1(z)|,\phi_2(z)|\}\le\delta\}} w(z)|f_n'(\phi_1(z))||\phi_1'(z)\psi_1(z) - \phi_2'(z)\psi_2(z)| \\
&+ \sup_{\{z;\ \max\{|\phi_1(z)|,\phi_2(z)|\}\le\delta\}} w(z)|\phi_2'(z)\psi_2(z)||f_n'(\phi_1(z))| \\
&+ \sup_{\{z;\ \max\{|\phi_1(z)|,\phi_2(z)|\}\le\delta\}} w(z)|\phi_2'(z)\psi_2(z)||f_n'(\phi_2(z))|.
\end{aligned}$$

The claim follows.

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