CUBO A Mathematical Journal
Vol.12, $N^{\underline{o}}$02, (217–234). June 2010

Fredholm Property of Matrix Wiener-Hopf plus and minus Hankel Operators with Semi-Almost Periodic Symbols

L. P. Castro[1] and A. S. Silva[2]
Department of Mathematics,
Aveiro University,
3810-193 Aveiro, Portugal
email: `castro@ua.pt`
email: `anabela.silva@ua.pt`

ABSTRACT

We will present sufficient conditions for the Fredholm property of Wiener-Hopf plus and minus Hankel operators with different Fourier matrix symbols in the C^*-algebra of semi-almost periodic elements. In addition, under such conditions, we will derive a formula for the sum of the Fredholm indices of these Wiener-Hopf plus Hankel and Wiener-Hopf minus Hankel operators. Some examples are provided to illustrate the results of the paper.

RESUMEN

Presentaremos condiciones suficientes para garantizar la propiedad de Fredholm de operadores de tipo Wiener-Hopf más y menos Hankel con diferentes símbolos de Fourier matriciales en la C*-álgebra de elementos semi-casi periódicos. Además, bajo tales condiciones, obtendremos una fórmula para la suma de los índices de Fredholm de estos operadores Wiener-Hopf más Hankel y Wiener-Hopf menos Hankel. Algunos ejemplos son dados para ilustrar los resultados del artículo.

Key words and phrases: *Fredholm property, Fredholm index, Wiener-Hopf operator, Hankel operator, semi-almost periodic matrix-valued function*

[1]Corresponding author: castro@ua.pt
[2]Sponsored by *Fundação para a Ciência e a Tecnologia* (Portugal) under grant number SFRH/BD/38698/2007.

Math. Subj. Class.: *47B35, 47A05, 47A12, 47A20, 42A75.*

1 Introduction

One of the objectives of the present paper is to obtain sufficient conditions for the Fredholm property of matrix Wiener-Hopf plus and minus Hankel operators of the form

$$W_{\Phi_1} \pm H_{\Phi_2} \ : \ [L^2_+(\mathbb{R})]^N \to [L^2(\mathbb{R}_+)]^N \qquad (N \in \mathbb{N}) \tag{1}$$

with W_{Φ_1} and H_{Φ_2} being matrix Wiener-Hopf and Hankel operators defined by

$$W_{\Phi_1} = r_+\mathcal{F}^{-1}\Phi_1 \cdot \mathcal{F} \quad \text{and} \quad H_{\Phi_2} = r_+\mathcal{F}^{-1}\Phi_2 \cdot \mathcal{F}J\,,$$

respectively. We denote by $B^{N\times N}$ the Banach algebra of all $N \times N$ matrices with entries in a Banach algebra B, and B^N will denote the Banach space of all N dimensional vectors with entries in a Banach space B. Let $L^2(\mathbb{R})$ be the usual space of square-integrable Lebesgue measurable functions on the real line $\mathbb{R}$, and $L^2(\mathbb{R}_+)$ the corresponding one in the positive half-line $\mathbb{R}_+ = (0, +\infty)$. We are using the notation $L^2_+(\mathbb{R})$ for the subspace of $L^2(\mathbb{R})$ formed by all the functions supported on the closure of $\mathbb{R}_+$. In addition, r_+ represents the operator of restriction from $[L^2_+(\mathbb{R})]^N$ into $[L^2(\mathbb{R}_+)]^N$, $\mathcal{F}$ denotes the Fourier transformation, J is the reflection operator given by the rule $J\varphi(x) = \widetilde{\varphi}(x) = \varphi(-x)$, $x \in \mathbb{R}$, and (in general) $\Phi_1,\ \Phi_2 \in [L^\infty(\mathbb{R})]^{N\times N}$ are the so-called Fourier matrix symbols. It is well-known that for such Fourier matrix symbols (with Lebesgue measurable and essentially bounded entries) the operators in (1) are bounded.

We would like to point out that the operators presented in (1) have been central objects in several recent research programmes (cf. e.g. [1]–[8]). One of the reasons for such interest is related to the fact that eventual additional knowledge about *regularity properties* of (1) have direct consequences in different types of applications (see [9]–[12]).

In the present work, the main purpose is to obtain conditions which will characterize the situation when $W_{\Phi_1} + H_{\Phi_2}$ and $W_{\Phi_1} - H_{\Phi_2}$ are at the same time Fredholm operators, and to present a formula for the sum of their Fredholm indices. All these will be done for matrices Φ_1 and Φ_2 in the class of semi-almost periodic elements (cf. Definition 2.1). Therefore, the present work deals with a more general situation than what was under consideration in [1, 7, 13], and some of the present results can be seen as a generalization of part of the results of the just mentioned works. However, the most general situation of considering the operators $W_{\Phi_1} + H_{\Phi_2}$ and $W_{\Phi_1} - H_{\Phi_2}$ independent of each other (with semi-almost periodic symbols) is not considered in the present paper and remains open.

2 Preliminary results and notions

The smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains all the functions e_λ $(\lambda \in \mathbb{R})$, where $e_\lambda(x) = e^{i\lambda x}$, $x \in \mathbb{R}$, is denoted by AP and called the algebra of *almost periodic functions*:

$$AP := alg_{L^\infty(\mathbb{R})}\{e_\lambda : \lambda \in \mathbb{R}\}.$$

In addition, we will also use the notation

$$AP_+ := alg_{L^\infty(\mathbb{R})}\{e_\lambda : \lambda \geq 0\}, \qquad AP_- := alg_{L^\infty(\mathbb{R})}\{e_\lambda : \lambda \leq 0\}$$

for these two subclasses of AP (which are still closed subalgebras of $L^\infty(\mathbb{R})$).

We will likewise make use of the Wiener subclass of AP (denoted by APW) formed by all those elements from AP which allow a representation by an absolutely convergent series. Therefore, APW is precisely the (proper) subclass of all functions $\varphi \in AP$ which can be written in an absolutely convergent series of the form:

$$\varphi = \sum_j \varphi_j \, e_{\lambda_j} \ , \qquad \lambda_j \in \mathbb{R} \ , \quad \sum_j |\varphi_j| < \infty \ .$$

We recall that all AP functions have a well-known mean value. The existence of such a number is provided in the next proposition.

Proposition 2.1. *(cf., e.g., [14, Proposition 2.22]) Let $A \subset (0,\infty)$ be an unbounded set and let*

$$\{I_\alpha\}_{\alpha\in A} = \{(x_\alpha, y_\alpha)\}_{\alpha\in A}$$

be a family of intervals $I_\alpha \subset \mathbb{R}$ such that $|I_\alpha| = y_\alpha - x_\alpha \to \infty$ as $\alpha \to \infty$. If $\varphi \in AP$, then the limit

$$M(\varphi) := \lim_{\alpha\to\infty} \frac{1}{|I_\alpha|} \int_{I_\alpha} \varphi(x)\, dx$$

exists, is finite, and is independent of the particular choice of the family $\{I_\alpha\}$.

For any $\varphi \in AP$, the number that has just been introduced $M(\varphi)$ is called the *Bohr mean value* or simply the *mean value* of φ. In the matrix case the *mean value* is defined entry-wise.

Let $\dot{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$. We will denote by $C(\dot{\mathbb{R}})$ the set of all continuous functions φ on the real line for which the two limits

$$\varphi(-\infty) := \lim_{x\to-\infty} \varphi(x), \qquad \varphi(+\infty) := \lim_{x\to+\infty} \varphi(x)$$

exist and coincide. The common value of these two limits will be denoted by $\varphi(\infty)$. Furthermore, $C_0(\dot{\mathbb{R}})$ will represent the collection of all $\varphi \in C(\dot{\mathbb{R}})$ for which $\varphi(\infty) = 0$.

Let $C(\overline{\mathbb{R}}) := C(\mathbb{R}) \cap PC(\dot{\mathbb{R}})$, where $C(\mathbb{R})$ is the usual set of continuous functions on the real line and $PC(\dot{\mathbb{R}})$ is the set of all bounded piecewise continuous functions on $\dot{\mathbb{R}}$.

As mentioned above, we will deal with Fourier symbols from the C^*-algebra of semi-almost periodic elements which is defined as follows.

Definition 2.1. The C^*-algebra SAP of all *semi-almost periodic functions* on $\mathbb{R}$ is the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains AP and $C(\overline{\mathbb{R}})$:

$$SAP = alg_{L^\infty(\mathbb{R})}\{AP, C(\overline{\mathbb{R}})\}.$$

In addition, it is possible to interpret the SAP functions in a different form due to the following characterization of D. Sarason [15].

Theorem 2.1. *Let $u \in C(\overline{\mathbb{R}})$ be any function for which $u(-\infty) = 0$ and $u(+\infty) = 1$. If $\varphi \in SAP$, then there is φ_l, $\varphi_r \in AP$ and $\varphi_0 \in C_0(\dot{\mathbb{R}})$ such that*

$$\varphi = (1-u)\varphi_l + u\varphi_r + \varphi_0. \tag{2}$$

The functions φ_l, φ_r are uniquely determined by φ, and independent of the particular choice of u. The maps

$$\varphi \mapsto \varphi_l, \qquad \varphi \mapsto \varphi_r$$

are C^*-algebra homomorphisms of SAP onto AP.

This theorem is also valid in the matrix case.

Let us now recall the so-called right and left AP factorizations. In such notions, we will use the notation $\mathcal{G}B$ for the group of all invertible elements of a Banach algebra B.

Definition 2.2. A matrix function $\Phi \in \mathcal{G}AP^{N\times N}$ is said to admit a *right AP factorization* if it can be represented in the form

$$\Phi(x) = \Phi_-(x)D(x)\Phi_+(x) \tag{3}$$

for all $x \in \mathbb{R}$, with

$$\Phi_- \in \mathcal{G}AP_-^{N\times N}, \;\; \Phi_+ \in \mathcal{G}AP_+^{N\times N}, \tag{4}$$

and D is a diagonal matrix of the form

$$D(x) = \operatorname{diag}\left[e^{i\lambda_1 x}, \dots, e^{i\lambda_N x}\right], \qquad \lambda_j \in \mathbb{R}.$$

The numbers λ_j are called the *right AP indices* of the factorization. A right AP factorization with $D = I_{N\times N}$ is referred to be a *canonical right AP factorization*.

If in a right AP factorization besides condition (4) the factors $\Phi_\pm$ belong to APW, then we say that Φ admits a *right APW factorization* (it being clear in such a case that $\Phi \in APW$).

It is said that a matrix function $\Phi \in \mathcal{G}AP^{N\times N}$ admits a *left AP factorization* if instead of (3) we have

$$\Phi(x) = \Phi_+(x)\, D(x)\, \Phi_-(x)$$

for all $x \in \mathbb{R}$, and $\Phi_\pm$ and D having the same property as above.

Note that from the last definition it follows that if an invertible almost periodic matrix function Φ admits a right AP factorization, then $\widetilde{\Phi}$ admits a left AP factorization, and also Φ^{-1} admits a left AP factorization.

The vector containing the right AP indices will be denoted by $k(\Phi)$, i.e., in the above case $k(\Phi) := (\lambda_1, \dots, \lambda_N)$. If we consider the case with equal right AP indices ($k(\Phi) := (\lambda_1, \lambda_1, \dots, \lambda_1)$), then the matrix

$$\mathbf{d}(\Phi) := M(\Phi_-)M(\Phi_+)$$

is independent of the particular choice of the right AP factorization. In this case, this matrix $\mathbf{d}(\Phi)$ is called the *geometric mean* of Φ.

In order to relate operators and to transfer certain operator properties between the related operators, we will also be using the known notion of equivalence after extension relation between bounded linear operators.

Definition 2.3. Consider two bounded linear operators $T : X_1 \to X_2$ and $S : Y_1 \to Y_2$, acting between Banach spaces. We say that T is *equivalent after extension to* S if there are Banach spaces Z_1 and Z_2 and invertible bounded linear operators E and F such that

$$\begin{bmatrix} T & 0 \\ 0 & I_{Z_1} \end{bmatrix} = E \begin{bmatrix} S & 0 \\ 0 & I_{Z_2} \end{bmatrix} F, \tag{5}$$

where I_{Z_1}, I_{Z_2} represent the identity operators in Z_1 and Z_2, respectively. This relation between T and S will be denoted by $T \overset{*}{\sim} S$.

Note that such operator relation between two operators T and S, if obtained, allows several consequences on the properties of these two operators. Namely, T and S will have the same Fredholm regularity properties (i.e., the properties that directly depend on the kernel and on the image of the operator). As we will realize in the next result, such kind of operator relation is valid for a diagonal operator constructed with our Wiener-Hopf plus and minus Hankel operators and a corresponding pure Wiener-Hopf operator.

Lemma 2.1. *Let* $\Phi_1, \Phi_2 \in \mathcal{G}[L^\infty(\mathbb{R})]^{N\times N}$. *Then*

$$\mathfrak{D}_{\Phi_{1,2}} := \begin{bmatrix} W_{\Phi_1} + H_{\Phi_2} & 0 \\ 0 & W_{\Phi_1} - H_{\Phi_2} \end{bmatrix} : [L^2_+(\mathbb{R})]^{2N} \to [L^2(\mathbb{R}_+)]^{2N} \tag{6}$$

is equivalent after extension to the Wiener-Hopf operator $W_\Psi : [L^2_+(\mathbb{R})]^{2N} \to [L^2(\mathbb{R}_+)]^{2N}$ *with Fourier symbol*

$$\Psi = \begin{bmatrix} \Phi_1 - \Phi_2 \widetilde{\Phi_1^{-1}} \widetilde{\Phi_2} & -\Phi_2 \widetilde{\Phi_1^{-1}} \\ \widetilde{\Phi_1^{-1}} \widetilde{\Phi_2} & \widetilde{\Phi_1^{-1}} \end{bmatrix}. \tag{7}$$

We refer to [4, Theorem 2.1] for a detailed proof of this lemma (where all the elements in the corresponding operator relation are given in explicit form and within the context of a so-called Δ-*relation after extension*; see [16]).

3 The Fredholm property

In the present section we will work out characterizations for the Fredholm property of $W_{\Phi_1} + H_{\Phi_2}$ and $W_{\Phi_1} - H_{\Phi_2}$. We start with the general case (where no dependence between the SAP matrices Φ_1 and Φ_2 is imposed), and in the last subsection (of the present section) we will consider a particular case where some relation between Φ_1 and Φ_2 will allow extra detailed descriptions.

3.1 General Case

We start by recalling a known Fredholm characterization for Wiener-Hopf operators with SAP matrix Fourier symbols having lateral almost periodic representatives admitting right AP factorizations.

Theorem 3.1. *(cf. e.g., [14, Theorem 10.11]) Let $\Phi \in SAP^{N\times N}$ and assume that the almost periodic representatives Φ_ℓ and Φ_r admit a right AP factorization. Then the Wiener-Hopf operator W_Φ is Fredholm if and only if:*

(i) $\Phi \in \mathcal{G}SAP^{N\times N}$;

(ii) *The almost periodic representatives Φ_ℓ and Φ_r admit canonical right AP factorizations (and therefore with $k(\Phi_\ell) = k(\Phi_r) = (0, \dots, 0)$);*

(iii) $\mathrm{sp}(\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell)) \cap (-\infty, 0] = \emptyset$, *where* $\mathrm{sp}(\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell))$ *stands for the set of the eigenvalues of the matrix*

$$\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell) := [\mathbf{d}(\Phi_r)]^{-1}\mathbf{d}(\Phi_\ell).$$

The matrix version of Sarason's Theorem (cf. Theorem 2.1) applied to Ψ in (7) says that if $\Psi \in \mathcal{G}SAP^{2N\times 2N}$ then this matrix function admits the following representation

$$\Psi = (1-u)\Psi_\ell + u\Psi_r + \Psi_0, \tag{8}$$

where $\Psi_{\ell,r} \in \mathcal{G}AP^{2N\times 2N}$ are defined for the particular Ψ in (7) by

$$\Psi_\ell = \begin{bmatrix} \Phi_{1\ell} - \Phi_{2\ell}\widetilde{\Phi_{1r}^{-1}}\widetilde{\Phi_{2r}} & -\Phi_{2\ell}\widetilde{\Phi_{1r}^{-1}} \\ \widetilde{\Phi_{1r}^{-1}}\widetilde{\Phi_{2r}} & \widetilde{\Phi_{1r}^{-1}} \end{bmatrix} \tag{9}$$

and

$$\Psi_r = \begin{bmatrix} \Phi_{1r} - \Phi_{2r}\widetilde{\Phi_{1\ell}^{-1}}\widetilde{\Phi_{2\ell}} & -\Phi_{2r}\widetilde{\Phi_{1\ell}^{-1}} \\ \widetilde{\Phi_{1\ell}^{-1}}\widetilde{\Phi_{2\ell}} & \widetilde{\Phi_{1\ell}^{-1}} \end{bmatrix} \tag{10}$$

(with $\Phi_{1\ell}$, Φ_{1r} and $\Phi_{2\ell}$, Φ_{2r} being the local representatives at $\mp\infty$ of Φ_1 and Φ_2, respectively), $u \in C(\overline{\mathbb{R}})$, $u(-\infty) = 0$, $u(+\infty) = 1$, $\Psi_0 \in [C_0(\dot{\mathbb{R}})]^{2N\times 2N}$.

From (9) it follows that

$$\widetilde{\Psi_\ell^{-1}} = \begin{bmatrix} \widetilde{\Phi_{1\ell}^{-1}} & \widetilde{\Phi_{1\ell}^{-1}}\widetilde{\Phi_{2\ell}} \\ -\Phi_{2r}\widetilde{\Phi_{1\ell}^{-1}} & \Phi_{1r} - \Phi_{2r}\widetilde{\Phi_{1\ell}^{-1}}\widetilde{\Phi_{2\ell}} \end{bmatrix}. \tag{11}$$

Therefore, we obtain that

$$\Psi_r = \begin{bmatrix} 0 & I_N \\ I_N & 0 \end{bmatrix} \widetilde{\Psi_\ell^{-1}} \begin{bmatrix} 0 & I_N \\ I_N & 0 \end{bmatrix}. \tag{12}$$

These representations, and the above relation between the operator (6) and the pure Wiener-Hopf operator, lead to the following characterization in the case when Ψ_ℓ admits a right AP factorization.

Theorem 3.2. *Let $\Psi \in SAP^{2N\times 2N}$ and assume that Ψ_ℓ admits a right AP factorization. In this case, the Wiener-Hopf plus and minus Hankel operators $W_{\Phi_1} + H_{\Phi_2}$ and $W_{\Phi_1} - H_{\Phi_2}$ are both Fredholm if and only if the following three conditions are satisfied:*

(c1) *$\Psi \in \mathcal{G}SAP^{2N\times 2N}$;*

(c2) *Ψ_ℓ admits a canonical right AP factorization;*

(c3) $\mathrm{sp}[H\mathbf{d}(\Psi_\ell)] \cap i\mathbb{R} = \emptyset$*, where* $H = \begin{bmatrix} 0 & I_N \\ I_N & 0 \end{bmatrix}$.

Proof. (i) Let us assume that the Wiener-Hopf plus and minus Hankel operators $W_{\Phi_1} + H_{\Phi_2}$ and $W_{\Phi_1} - H_{\Phi_2}$ are both Fredholm operators. Then, W_Ψ is also Fredholm due to the above presented equivalence after extension relation. Therefore, using Theorem 3.1 we obtain that $\Psi \in \mathcal{G}SAP^{2N\times 2N}$, Ψ_ℓ and Ψ_r admit canonical right AP factorizations and

$$\mathrm{sp}(\mathbf{d}^{-1}(\Psi_r)\mathbf{d}(\Psi_\ell)) \cap (-\infty, 0] = \emptyset. \tag{13}$$

In particular, we realize that propositions (c1) and (c2) are already fulfilled. Additionally, the canonical right AP factorization of Ψ_ℓ can be normalized into

$$\Psi_\ell = \theta_- \Lambda \theta_+, \tag{14}$$

where $\theta_\pm$ have the same factorization properties as the original lateral factors of the canonical factorization but with $M(\theta_\pm) = I$, and where $\Lambda := \mathbf{d}(\Psi_\ell)$. Let

$$H = \begin{bmatrix} 0 & I_N \\ I_N & 0 \end{bmatrix}. \tag{15}$$

From (12) and (14) we derive that

$$\Psi_r = H\widetilde{\Psi_\ell^{-1}}H = H\widetilde{\theta_+^{-1}}\Lambda^{-1}\widetilde{\theta_-^{-1}}H$$

which shows that

$$\mathbf{d}(\Psi_r) = H\Lambda^{-1}H \tag{16}$$

and therefore

$$\mathbf{d}^{-1}(\Psi_r) = H\Lambda H. \tag{17}$$

In this way, we conclude that

$$\begin{aligned} \mathrm{sp}[\mathbf{d}^{-1}(\Psi_r)\mathbf{d}(\Psi_\ell)] &= \mathrm{sp}[H\Lambda H\Lambda] \\ &= \mathrm{sp}[(H\Lambda)^2]. \end{aligned}$$

Thus, (13) turns out to be equivalent to

$$\mathrm{sp}[(H\Lambda)^2] \cap (-\infty, 0] = \emptyset$$

which leads to

$$\mathrm{sp}[H\Lambda] \cap i\mathbb{R} = \emptyset\,.$$

Therefore, the proposition (c3) is also satisfied.

(ii) Let us now assume that (c1), (c2) and (c3) hold true. From condition (c1) we have $\Psi \in \mathcal{G}SAP^{2N\times 2N}$. The left and right representatives of Ψ are given by (9) and (10). Due to the fact that Ψ_ℓ admits a canonical right AP factorization, it follows that Ψ_ℓ^{-1} admits a canonical left AP factorization and $\widetilde{\Psi_\ell^{-1}}$ admits a canonical right AP factorization. Therefore,

$$\begin{bmatrix} 0 & I_N \\ I_N & 0 \end{bmatrix} \widetilde{\Psi_\ell^{-1}} \begin{bmatrix} 0 & I_N \\ I_N & 0 \end{bmatrix} = \Psi_r \tag{18}$$

admits a canonical right AP factorization. These two canonical right AP factorizations and condition (c3) imply that

$$\mathrm{sp}(\mathbf{d}^{-1}(\Psi_r)\mathbf{d}(\Psi_\ell)) \cap (-\infty, 0] = \emptyset.$$

All these facts together with Theorem 3.1 give us that W_Ψ is a Fredholm operator. Using the equivalence after extension relation, we obtain that the Wiener-Hopf plus and minus Hankel operators $W_{\Phi_1} + H_{\Phi_2}$ and $W_{\Phi_1} - H_{\Phi_2}$ are both Fredholm operators. □

Let us now think about the case of $\Psi \in SAPW^{2N\times 2N}$, where $SAPW$ denotes the algebra of all semi-almost periodic functions φ whose almost periodic representatives φ_ℓ and φ_r (cf. (2)) belong to APW.

If $\Psi \in SAPW^{2N\times 2N}$, then in Theorem 3.2 we can drop the assumption which states that Ψ_ℓ admits an AP factorization and also simplify the corresponding conditions (c1) and (c2):

Corollary 3.1. *Let $\Psi \in SAPW^{2N\times 2N}$. The Wiener-Hopf plus and minus Hankel operators $W_{\Phi_1} + H_{\Phi_2}$ and $W_{\Phi_1} - H_{\Phi_2}$ are both Fredholm if and only if the following three conditions are satisfied:*

(c1′) $\Psi \in \mathcal{G}SAPW^{2N\times 2N}$;

(c2′) Ψ_ℓ *admits a canonical right APW factorization;*

(c3′) $\mathrm{sp}[H\mathbf{d}(\Psi_\ell)] \cap i\mathbb{R} = \emptyset$, *where* $H = \begin{bmatrix} 0 & I_N \\ I_N & 0 \end{bmatrix}$.

Proof. The result is derived from Theorem 3.2 and from the following known facts which apply to any $\Phi \in \mathcal{G}APW^{2N\times 2N}$: (j) Φ has a canonical right AP factorization if and only if Φ has a canonical right APW factorization; (jj) Φ has a canonical right APW factorization if and only if W_Φ is invertible.

In fact, for our $\Psi \in SAPW^{2N\times 2N}$, note that if both operators $W_{\Phi_1} + H_{\Phi_2}$ and $W_{\Phi_1} - H_{\Phi_2}$ have the Fredholm property, then by the above equivalence after extension relation we also have that the Wiener-Hopf operator W_Ψ is a Fredholm operator. Therefore, W_{Ψ_ℓ} and W_{Ψ_r} are invertible operators and from (jj) this is equivalent to Ψ_ℓ and Ψ_r to admit canonical right APW factorizations. Thus, the assertion now follows from Theorem 3.2 and proposition (j). □

3.2 The case of $\Phi_1 = \widetilde{\Phi_2}$

For some particular cases where Φ_1 and Φ_2 are dependent on each other, we can simplify the statement of Theorem 3.2 by making use of consequent equivalence after extension operator relations. In the present subsection we will analyze the case of $\Phi_1 = \widetilde{\Phi_2}$.

Let $\Phi_2 \in \mathcal{G}SAP^{N\times N}$ and consider $\Phi_1 = \widetilde{\Phi_2}$. In this case, the matrix Ψ takes the form

$$\Psi = \begin{bmatrix} 0 & -I_N \\ \Phi_2^{-1}\widetilde{\Phi_2} & \widetilde{\Phi_2^{-1}} \end{bmatrix}$$

and the Wiener-Hopf operator W_Ψ is equivalent after extension to the operator $W_{\Phi_2^{-1}\widetilde{\Phi_2}}$. In fact, we have in this case:

$$W_\Psi = r_+\mathcal{F}^{-1}\begin{bmatrix} 0 & -I_N \\ I_N & \widetilde{\Phi_2^{-1}} \end{bmatrix}\mathcal{F}\ell_0 r_+\mathcal{F}^{-1}\begin{bmatrix} \Phi_2^{-1}\widetilde{\Phi_2} & 0 \\ 0 & I_N \end{bmatrix}\mathcal{F}$$

(where $\ell_0 : [L^2(\mathbb{R}_+)]^{2N} \to [L^2_+(\mathbb{R})]^{2N}$ denotes the zero extension operator). This together with the equivalence after extension relation between the operator (6) and W_Ψ shows that

$$\mathfrak{D}_{\Phi_{1,2}} \overset{*}{\sim} W_{\Phi_2^{-1}\widetilde{\Phi_2}} \tag{19}$$

(due to the transitivity of the equivalence after extension relation).

From Theorem 2.1 we conclude that $\Phi_2 \in \mathcal{G}SAP^{N\times N}$ admits the following representation

$$\Phi_2 = (1-u)\Phi_{2\ell} + u\Phi_{2r} + \Phi_{20} \tag{20}$$

(with $\Phi_{20} \in [C_0(\dot{\mathbb{R}})]^{N\times N}$) and

$$\Phi_2^{-1}\widetilde{\Phi_2} = [(1-u)\Phi_{2\ell} + u\Phi_{2r} + \Phi_{20}]^{-1}[(1-\widetilde{u})\widetilde{\Phi_{2\ell}} + \widetilde{u}\widetilde{\Phi_{2r}} + \widetilde{\Phi_{20}}]. \tag{21}$$

Therefore, from (21), we obtain that

$$(\Phi_2^{-1}\widetilde{\Phi_2})_\ell = \Phi_{2\ell}^{-1}\widetilde{\Phi_{2r}}, \qquad (\Phi_2^{-1}\widetilde{\Phi_2})_r = \Phi_{2r}^{-1}\widetilde{\Phi_{2\ell}}. \tag{22}$$

These representations and the above relation between W_Ψ and $W_{\Phi_2^{-1}\widetilde{\Phi_2}}$ (when $\Phi_1 = \widetilde{\Phi_2}$), allow us to construct the following result.

Theorem 3.3. *Let $\Phi_2 \in SAP^{N\times N}$ and assume that $\Phi_{2\ell}^{-1}\widetilde{\Phi_{2r}}$ admits a right AP factorization. In this case, the Wiener-Hopf plus and minus Hankel operators $W_{\widetilde{\Phi_2}} + H_{\Phi_2}$ and $W_{\widetilde{\Phi_2}} - H_{\Phi_2}$ are both Fredholm operators if and only if the following three conditions are satisfied:*

(I) $\Phi_2 \in \mathcal{G}SAP^{N\times N}$*;*

(II) $\Phi_{2\ell}^{-1}\widetilde{\Phi_{2r}}$ *admits a canonical right AP factorization;*

(III) $\mathrm{sp}[\mathbf{d}(\Phi_{2\ell}^{-1}\widetilde{\Phi_{2r}})] \cap i\mathbb{R} = \emptyset$*.*

Proof. (i) If $W_{\widetilde{\Phi_2}} \pm H_{\Phi_2}$ are both Fredholm operators, then from a similar reasoning as in [5, Proposition 2.6] it follows that $\Phi_2 \in \mathcal{G}[L^\infty(\mathbf{R})]^{N\times N}$ and therefore $\Phi_2 \in \mathcal{G}SAP^{N\times N}$.

The Fredholm property of the Wiener-Hopf plus and minus Hankel operators $W_{\widetilde{\Phi_2}} + H_{\Phi_2}$ and $W_{\widetilde{\Phi_2}} - H_{\Phi_2}$ implies that the operator W_Ψ is Fredholm and due to the transitivity of equivalence after extension relations, it follows that the operator $W_{\Phi_2^{-1}\widetilde{\Phi_2}}$ has also the Fredholm property (cf. (19)). Employing Theorem 3.1 we obtain that $\Phi_2^{-1}\widetilde{\Phi_2} \in \mathcal{G}SAP^{N\times N}$, $(\Phi_2^{-1}\widetilde{\Phi_2})_\ell$ and $(\Phi_2^{-1}\widetilde{\Phi_2})_r$ admit canonical right AP factorizations and

$$\mathrm{sp}[\mathbf{d}^{-1}((\Phi_2^{-1}\widetilde{\Phi_2})_r)\mathbf{d}((\Phi_2^{-1}\widetilde{\Phi_2})_\ell)] \cap (-\infty, 0] = \emptyset. \tag{23}$$

Due to (22) we conclude that $\Phi_{2\ell}^{-1}\widetilde{\Phi_{2r}}$ admits a canonical right AP factorization and we derive from (23) that

$$\mathrm{sp}[\mathbf{d}^{-1}(\Phi_{2r}^{-1}\widetilde{\Phi_{2\ell}})\mathbf{d}(\Phi_{2\ell}^{-1}\widetilde{\Phi_{2r}})] \cap (-\infty, 0] = \emptyset. \tag{24}$$

A canonical right AP factorization of $\Phi_{2\ell}^{-1}\widetilde{\Phi_{2r}}$ can be normalized into

$$\Phi_{2\ell}^{-1}\widetilde{\Phi_{2r}} = \Theta_-\Lambda\Theta_+, \tag{25}$$

where $\Theta_\pm$ have the same factorization properties as the original lateral factors of the canonical factorization but with $M(\Theta_\pm) = I$, and where $\Lambda := \mathbf{d}(\Phi_{2\ell}^{-1}\widetilde{\Phi_{2r}})$. Thus, (25) allows

$$\Phi_{2r}^{-1}\widetilde{\Phi_{2\ell}} = (\widetilde{\Phi_{2\ell}^{-1}\widetilde{\Phi_{2r}}})^{-1} = \widetilde{\Theta_+^{-1}}\Lambda^{-1}\widetilde{\Theta_-^{-1}}$$

which shows that

$$\mathbf{d}(\Phi_{2r}^{-1}\widetilde{\Phi_{2\ell}}) = \Lambda^{-1}$$

and therefore (24) turns out to be equivalent to

$$\mathrm{sp}[\Lambda^2] \cap (-\infty, 0] = \emptyset.$$

From the eigenvalue definition, it therefore results in

$$\mathrm{sp}[\Lambda] \cap i\mathbb{R} = \emptyset$$

which proves proposition (lll).

(ii) Let us now consider that (l)–(lll) hold true. The property (l) implies that $\Phi_2^{-1}\widetilde{\Phi_2}$ is also invertible in $SAP^{N\times N}$. Since $\Phi_{2\ell}^{-1}\widetilde{\Phi_{2r}} = (\Phi_2^{-1}\widetilde{\Phi_2})_\ell$ admits a canonical right AP factorization, then

$$(\widetilde{\Phi_2^{-1}\widetilde{\Phi_2}})_\ell = \widetilde{\Phi_{2\ell}^{-1}}\Phi_{2r}$$

admits a canonical left AP factorization and

$$[(\widetilde{\Phi_2^{-1}\widetilde{\Phi_2}})_\ell]^{-1} = \Phi_{2r}^{-1}\widetilde{\Phi_{2\ell}}$$

admits a canonical right AP factorization. These last two canonical right AP factorizations and condition (lll) imply that

$$\begin{aligned}\mathrm{sp}[\mathbf{d}^{-1}((\Phi_2^{-1}\widetilde{\Phi_2})_r)\mathbf{d}((\Phi_2^{-1}\widetilde{\Phi_2})_\ell)] \cap (-\infty, 0] &= \mathrm{sp}[\mathbf{d}^{-1}(\Phi_{2r}^{-1}\widetilde{\Phi_{2\ell}})\mathbf{d}(\Phi_{2\ell}^{-1}\widetilde{\Phi_{2r}})] \cap (-\infty, 0] \\ &= \emptyset.\end{aligned}$$

All these facts together with Theorem 3.1 show that $W_{\Phi_2^{-1}\widetilde{\Phi_2}}$ is a Fredholm operator. Using the equivalence after extension relations of (19) and Lemma 2.1, we obtain that the Wiener-Hopf plus and minus Hankel operators $W_{\widetilde{\Phi_2}} + H_{\Phi_2}$ and $W_{\widetilde{\Phi_2}} - H_{\Phi_2}$ have the Fredholm property. □

4 Index formula

In the present section we will be concentrated in obtaining a Fredholm index formula for $\mathfrak{D}_{\Phi_{1,2}}$, i.e., for the sum of Wiener-Hopf plus and minus Hankel operators $W_{\Phi_1} \pm H_{\Phi_2}$ with Fourier symbols $\Phi_1, \Phi_2 \in \mathcal{G}SAP^{N\times N}$ such that Ψ_ℓ admits a right AP factorization. Within this context, let us now assume that $W_{\Phi_1} + H_{\Phi_2}$ and $W_{\Phi_1} - H_{\Phi_2}$ are Fredholm operators.

4.1 General situation

Let $\mathcal{G}SAP_{0,0}$ denote the set of all functions $\varphi \in \mathcal{G}SAP$ for which $k(\varphi_\ell) = k(\varphi_r) = 0$. To define the Cauchy index of $\varphi \in \mathcal{G}SAP_{0,0}$ we need the lemma presented below.

Lemma 4.1. *(See e.g. [14, Lemma 3.12]) Let $A \subset (0,\infty)$ be an unbounded set and let*

$$\{I_\alpha\}_{\alpha\in A} = \{(x_\alpha, y_\alpha)\}_{\alpha\in A}$$

be a family of intervals such that $x_\alpha \geq 0$ and $|I_\alpha| = y_\alpha - x_\alpha \to \infty$, as $\alpha \to \infty$. If $\varphi \in \mathcal{G}SAP_{0,0}$ and $\arg\varphi$ is any continuous argument of φ, then the limit

$$\frac{1}{2\pi}\lim_{\alpha\to\infty}\frac{1}{|I_\alpha|}\int_{I_\alpha}((arg\varphi)(x) - (arg\varphi)(-x))dx \tag{26}$$

exists, is finite, and is independent of the particular choices of $\{(x_\alpha, y_\alpha)\}_{\alpha\in A}$ and $\arg\varphi$.

The limit (26) is denoted by $\mathrm{ind}\varphi$ and is usually called the *Cauchy index* of φ. Moreover, following [7, Section 4.3] we can generalize the notion of Cauchy index for SAP presented in Lemma 4.1 for functions with $k(\varphi_\ell) + k(\varphi_r) = 0$.

The following theorem provides a formula for the Fredholm index of matrix Wiener-Hopf operators with SAP Fourier symbols.

Theorem 4.1. *(Cf. e.g. [14, Theorem 10.12]) Let $\Phi \in SAP^{N\times N}$. If the almost periodic representatives Φ_ℓ, Φ_r admit right AP factorizations, and if W_Φ is a Fredholm operator, then*

$$\mathrm{Ind}\, W_\Phi = -\mathrm{ind}[\det\Phi] - \sum_{k=1}^{N}\left(\frac{1}{2} - \left\{\frac{1}{2} - \frac{1}{2\pi}arg\ \xi_k\right\}\right) \tag{27}$$

where $\xi_1, \ldots, \xi_N \in \mathbb{C}\backslash(-\infty, 0]$ are the eigenvalues of the matrix $\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell)$ and $\{\cdot\}$ stands for the fractional part of a real number. Additionally, when choosing $\arg\xi_k$ in $(-\pi,\pi)$, we have

$$\mathrm{Ind}\, W_\Phi = -\mathrm{ind}\ [\det\Phi] - \frac{1}{2\pi}\sum_{k=1}^{N}\arg\xi_k.$$

We will now be concerned with the question of finding a formula for the sum of the Fredholm indices of $W_{\Phi_1} + H_{\Phi_2}$ and $W_{\Phi_1} - H_{\Phi_2}$ (i.e., $\mathrm{Ind}[W_{\Phi_1} + H_{\Phi_2}] + \mathrm{Ind}[W_{\Phi_1} - H_{\Phi_2}]$). Using the equivalence after extension relation presented in Lemma 2.1, we conclude that

$$\mathrm{Ind}[W_{\Phi_1} + H_{\Phi_2}] + \mathrm{Ind}[W_{\Phi_1} - H_{\Phi_2}] = \mathrm{Ind}\, W_\Psi.$$

Observing that W_Ψ is a Fredholm operator and using (27), we obtain

$$\mathrm{Ind} W_\Psi = -\mathrm{ind}[\det \Psi] - \sum_{k=1}^{2N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{2\pi} \arg \eta_k \right\} \right) \tag{28}$$

where $\eta_k \in \mathbb{C}\backslash(-\infty, 0]$ are the eigenvalues of the matrix of $\mathbf{d}^{-1}(\Psi_r)\mathbf{d}(\Psi_\ell) = (H\mathbf{d}(\Psi_\ell))^2$, with $H = \begin{bmatrix} 0 & I_N \\ I_N & 0 \end{bmatrix}$ (cf. (16)–(17)). Therefore, (28) can be rewritten as

$$\mathrm{Ind} W_\Psi = -\mathrm{ind}[\det \Psi] - \sum_{n=1}^{2N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{\pi} \arg \zeta_k \right\} \right) \tag{29}$$

where $\zeta_k \in \mathbb{C}\backslash i\mathbb{R}$ are the eigenvalues of the matrix $H\mathbf{d}(\Psi_\ell)$. Moreover, formula (28) is reduced to

$$\mathrm{Ind} W_\Psi = -\mathrm{ind}[\det \Psi] - \frac{1}{\pi} \sum_{k=1}^{2N} \beta(\zeta_k) \tag{30}$$

where

$$\beta(\zeta_k) := \begin{cases} arg(\zeta_k) & \text{if } \Re e\ \zeta_k > 0 \\ arg(-\zeta_k) & \text{if } \Re e\ \zeta_k < 0 \end{cases} \tag{31}$$

when choosing the argument in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

These conclusions are assembled in the following corollary.

Corollary 4.1. *Let $\Psi \in \mathcal{G}SAP^{2N\times 2N}$ and assume that Ψ_ℓ admits a right AP factorization. If $W_{\Phi_1} \pm H_{\Phi_2}$ are Fredholm operators, then*

$$\mathrm{Ind}[W_{\Phi_1} + H_{\Phi_2}] + \mathrm{Ind}[W_{\Phi_1} - H_{\Phi_2}] = -\mathrm{ind}[\det \Psi] - \sum_{k=1}^{2N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{\pi} \arg \zeta_k \right\} \right) \tag{32}$$

where $\zeta_k \in \mathbb{C}\backslash i\mathbb{R}$ are the eigenvalues of the matrix $H\mathbf{d}(\Psi_\ell)$. Moreover, making use of (31)*, formula* (32) *simplifies to the following one:*

$$\mathrm{Ind}[W_{\Phi_1} + H_{\Phi_2}] + \mathrm{Ind}[W_{\Phi_1} - H_{\Phi_2}] = -\mathrm{ind}[\det \Psi] - \frac{1}{\pi} \sum_{k=1}^{2N} \beta(\zeta_k). \tag{33}$$

4.2 The case of $\Phi_1 = \widetilde{\Phi_2}$

For the particular case where $\Phi_1 = \widetilde{\Phi_2}$ we can simplify formula (33) even further. In fact, when $\Phi_1 = \widetilde{\Phi_2}$, employing the equivalence after extension relation (19), we deduce that

$$\mathrm{Ind}[W_{\Phi_1} + H_{\Phi_2}] + \mathrm{Ind}[W_{\Phi_1} - H_{\Phi_2}] = -\mathrm{ind}[\det(\Phi_2^{-1}\widetilde{\Phi_2})] - \frac{1}{\pi} \sum_{k=1}^{N} \beta(\delta_k), \tag{34}$$

where $\delta_k \in \mathbb{C}\backslash i\mathbb{R}$ are the eigenvalues of the matrix $\mathbf{d}(\Phi_{2\ell}^{-1}\widetilde{\Phi_{2r}})$ and

$$\beta(\delta_k) = \begin{cases} \arg(\delta_k) & \text{if } \Re e\ \delta_k > 0 \\ \arg(-\delta_k) & \text{if } \Re e\ \delta_k < 0 \end{cases}$$

with the argument in both cases being chosen in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

In addition, let us now simplify the form of $\mathrm{ind}[\det(\Phi_2^{-1}\widetilde{\Phi_2})]$. Observing that the matrix $\Phi_{2\ell}^{-1}\widetilde{\Phi_{2r}}$ has a canonical right AP factorization, it holds $k(\Phi_{2\ell}^{-1}\widetilde{\Phi_{2r}}) = (0, \dots, 0)$ and consequently

$$k(\det(\Phi_{2\ell}^{-1}\widetilde{\Phi_{2r}})) = 0.$$

Taking this into consideration, it follows that

$$\begin{aligned} k((\det \Phi_2^{-1})_\ell) + k((\det \Phi_2^{-1})_r) &= k(\det(\Phi_{2\ell}^{-1})) + k(\det(\Phi_{2r}^{-1})) \\ &= k(\det(\Phi_{2\ell}^{-1})) + k(\det(\Phi_{2r})^{-1}) \\ &= k(\det(\Phi_{2\ell}^{-1})) + k[(\widetilde{\det(\Phi_{2r})^{-1}})^{-1}] \\ &= k(\det(\Phi_{2\ell}^{-1})) + k(\widetilde{\det(\Phi_{2r})}) \\ &= k(\det(\Phi_{2\ell}^{-1})) + k(\det(\widetilde{\Phi_{2r}})) \\ &= k(\det(\Phi_{2\ell}^{-1}) \det(\widetilde{\Phi_{2r}})) \\ &= k(\det(\Phi_{2\ell}^{-1}\widetilde{\Phi_{2r}})) \\ &= 0 \end{aligned} \tag{35}$$

also because for any $f \in \mathcal{G}AP$ we have $k(f) = k(\widetilde{f^{-1}})$ and $[\det \Phi]_\ell = \det \Phi_\ell$. Applying a similar reasoning to $\widetilde{\Phi_2}$, we obtain

$$\begin{aligned} k((\det \widetilde{\Phi_2})_\ell) + k((\det \widetilde{\Phi_2})_r) &= k(\det(\widetilde{\Phi_{2\ell}})) + k(\det(\widetilde{\Phi_{2r}})) \\ &= k(\det(\widetilde{\widetilde{\Phi_{2\ell}})^{-1}}) + k(\det(\widetilde{\Phi_{2r}})) \\ &= k(\det(\Phi_{2\ell}^{-1})) + k(\det(\widetilde{\Phi_{2r}})) \\ &= k(\det(\Phi_{2\ell}^{-1}) \det(\widetilde{\Phi_{2r}})) \\ &= k(\det(\Phi_{2\ell}^{-1}\widetilde{\Phi_{2r}})) \\ &= 0. \end{aligned} \tag{36}$$

Employing now (26), (35) and (36), the following computation holds true:

$$\begin{aligned} \mathrm{ind}[\det(\Phi_2^{-1}\widetilde{\Phi_2})] &= \mathrm{ind}[\det \Phi_2^{-1} \det \widetilde{\Phi_2}] \\ &= \mathrm{ind}[\det \Phi_2^{-1}] + \mathrm{ind}[\det \widetilde{\Phi_2}] \\ &= \mathrm{ind}[\det \Phi_2]^{-1} + \mathrm{ind}[\widetilde{\det \Phi_2}] \\ &= \mathrm{ind}[\det \Phi_2]^{-1} - \mathrm{ind}[\det \Phi_2] \\ &= -\mathrm{ind}[\det \Phi_2] - \mathrm{ind}[\det \Phi_2] \\ &= -2\,\mathrm{ind}[\det \Phi_2]. \end{aligned}$$

Thus, we have just concluded the following corollary.

Corollary 4.2. *Let $\Phi_1,\ \Phi_2 \in \mathcal{G}SAP^{N\times N}$ such that $\Phi_1 = \widetilde{\Phi_2}$ and assume that $\Phi_{2\ell}^{-1}\widetilde{\Phi_{2r}}$ admits a right AP factorization. If $W_{\Phi_1} \pm H_{\Phi_2}$ are Fredholm operators, then*

$$\mathrm{Ind}[W_{\Phi_1} + H_{\Phi_2}] + \mathrm{Ind}[W_{\Phi_1} - H_{\Phi_2}] = 2\,\mathrm{ind}[\det \Phi_2] - \frac{1}{\pi}\sum_{k=1}^{N} \beta(\delta_k) \tag{37}$$

where $\delta_k \in \mathbb{C}\backslash i\mathbb{R}$ *are the eigenvalues of the matrix* $\mathbf{d}(\Phi_{2\ell}^{-1}\widetilde{\Phi_{2r}})$ *and*

$$\beta(\delta_k) = \begin{cases} \arg(\delta_k) & \text{if } \Re e\ \delta_k > 0 \\ \arg(-\delta_k) & \text{if } \Re e\ \delta_k < 0 \end{cases} \tag{38}$$

with the argument in both cases being chosen in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

5 Examples

In the present section we exemplify the above theory with two particular cases of corresponding Fourier symbol matrices Φ_1 and Φ_2.

5.1 First example

Let $\Phi_1 = \widetilde{\Phi_2}$, with

$$\Phi_2(x) = (1-u(x))\begin{bmatrix} e^{ix} & 0 \\ 0 & e^{-ix} \end{bmatrix} + u(x)\begin{bmatrix} e^{-ix} & 0 \\ 0 & e^{ix} \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{x-i} \\ \frac{1}{x+i} & 0 \end{bmatrix} \tag{39}$$

and where u is the real-valued function defined by

$$u(x) = \begin{cases} \frac{1}{2}e^{x} & \text{if } x < 0 \\ 1 - \frac{1}{2}e^{-x} & \text{if } x \geq 0. \end{cases} \tag{40}$$

From (39) and Theorem 2.1, it becomes clear that $\Phi_2 \in SAP^{2\times 2}$. In addition, we will show that $\Phi_2 \in \mathcal{G}SAP^{2\times 2}$. To this purpose, let us compute the determinant of Φ_2:

$$\begin{aligned} \det \Phi_2(x) &= \det \begin{bmatrix} (1-u(x))e^{ix} + u(x)e^{-ix} & -\frac{1}{x-i} \\ \frac{1}{x+i} & (1-u(x))e^{-ix} + u(x)e^{ix} \end{bmatrix} \\ &= 1 + (2u(x) - 2u^2(x))(\cos(2x) - 1) + \tfrac{1}{x^2+1}. \end{aligned}$$

Recalling that u is a real-valued function given by (40), we obtain

$$\det \Phi_2(x) = \begin{cases} 1 + (e^{x} - e^{2x})(\cos(2x) - 1) + \frac{1}{x^2+1} & \text{if } x < 0 \\ 1 + (e^{-2x} - e^{-x})(\cos(2x) - 1) + \frac{1}{x^2+1} & \text{if } x \geq 0 \end{cases} \tag{41}$$

Let us first show that $\det \Phi_2(x) \neq 0$ for $x \in (-\infty, 0)$.

In this domain $e^{x} - e^{2x}$ belongs to $(0, \frac{1}{4}]$ and $\cos(2x) - 1 \in [-2, 0]$. Therefore,

$$-\frac{1}{2} < (e^{x} - e^{2x})(\cos(2x) - 1) \leq 0$$

and hence,

$$\frac{1}{2} < 1 + (e^{x} - e^{2x})(\cos(2x) - 1) \leq 1$$

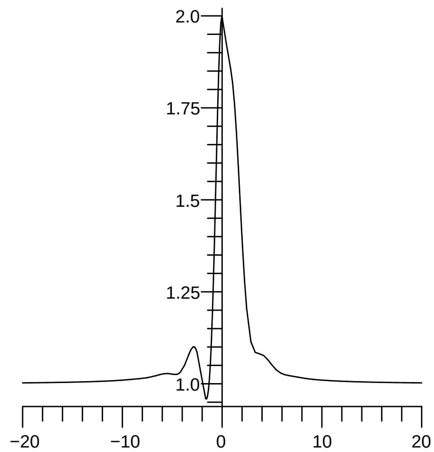

Figure 1: The range of $\det \Phi_2$ in the first example.

(cf. Figure 1). Observing that $\frac{1}{x^2+1} \in (0,1)$ (when $x<0$), we conclude that for $x<0$:

$$\det \Phi_2 > \frac{1}{2}\,. \tag{42}$$

Let us now consider $x \in [0,+\infty)$. In this case, we have $e^{-2x} - e^{-x} \in [-\frac{1}{4}, 0]$. This implies that

$$0 \leq (e^{-2x} - e^{-x})(\cos(2x) - 1) < \frac{1}{2}.$$

Hence,

$$1 \leq 1 + (e^{-2x} - e^{-x})(\cos(2x) - 1) < \frac{3}{2}.$$

Observing that $\frac{1}{x^2+1} \in (0,1]$ $(x \geq 0)$ we conclude that for $x \geq 0$:

$$\det\ \Phi_2 > 1. \tag{43}$$

From (42) and (43), it follows that $\Phi_2 \in \mathcal{G}SAP^{2\times 2}$.

Now, a direct computation yields that

$$\Phi_{2\ell}^{-1}\widetilde{\Phi_{2r}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which obviously admits a canonical right AP factorization and

$$\mathbf{d}(\Phi_{2\ell}^{-1}\widetilde{\Phi_{2r}}) = I_{2\times 2}.$$

Hence,

$$\mathrm{sp}[\mathbf{d}(\Phi_{2\ell}^{-1}\widetilde{\Phi_{2r}})] \cap i\mathbb{R} = \{1\} \cap i\mathbb{R} = \emptyset.$$

This allows us to conclude that the operators $W_{\widetilde{\Phi_2}} \pm H_{\Phi_2}$ have the Fredholm property. Thus, by using the above theory (cf. Corollary 4.2) we are now in a position to compute the sum of their Fredholm indices. For this case, we have

$$\mathrm{Ind}[W_{\widetilde{\Phi_2}} + H_{\Phi_2}] + \mathrm{Ind}[W_{\widetilde{\Phi_2}} - H_{\Phi_2}] - 2\,\mathrm{ind}\det(\Phi_2) - \frac{1}{\pi}\sum_{k=1}^{2}\beta(\delta_k)$$

where $\delta_k \in \mathbb{C}\backslash i\mathbb{R}$ are the eigenvalues of the matrix $\mathbf{d}(\Phi_{2\ell}^{-1}\widetilde{\Phi_{2r}})$ and β is given by (38). In addition, we have already seen that $\det\Phi_2$ is a real-valued positive function, and therefore its argument is zero. Altogether, we have:

$$\mathrm{Ind}[W_{\widetilde{\Phi_2}} + H_{\Phi_2}] + \mathrm{Ind}[W_{\widetilde{\Phi_2}} - H_{\Phi_2}] = 0$$

(since the eigenvalues of $\mathbf{d}(\Phi_{2\ell}^{-1}\widetilde{\Phi_{2r}})$ are also real and positive, and therefore their arguments are also zero).

5.2 Second example

Let $\Phi_1 = 1 + e^{-x^2}$ and $\Phi_2 = -(1-u(x))e^{-ix} + u(x)e^{-2ix}$, where u is the real-valued function defined by

$$u(x) = \frac{1}{2} + \frac{1}{2}\tanh(x).$$

Consequently, observing that $\widetilde{u}(x) = 1 - u(x)$ we have (cf. (7))

$$\Psi = \begin{bmatrix} 1 + e^{-x^2} + \frac{\left(u(x)e^{-\frac{ix}{2}} - (1-u(x))e^{\frac{ix}{2}}\right)^2}{1+e^{-x^2}} & \frac{(1-u(x))e^{-ix} - u(x)e^{-2ix}}{1+e^{-x^2}} \\ \frac{-u(x)e^{ix} + (1-u(x))e^{2ix}}{1+e^{-x^2}} & \frac{1}{1+e^{-x^2}} \end{bmatrix}.$$

From Theorem 2.1, it becomes clear that Φ_1 and $\Phi_2 \in SAP$ and thus, the matrix Ψ belongs to $SAP^{2\times 2}$. Since $\det\Psi = 1$, we conclude that $\Psi \in \mathcal{G}SAP^{2\times 2}$.

Following (9), we obtain

$$\Psi_\ell = \begin{bmatrix} 1 + e^{ix} & e^{-ix} \\ e^{2ix} & 1 \end{bmatrix}.$$

Moreover, observing that

$$\Psi_\ell = \begin{bmatrix} e^{-ix} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{2ix} & 1 \\ 1 & 0 \end{bmatrix},$$

we conclude that Ψ_ℓ admits a canonical right AP factorization and

$$\mathbf{d}(\Psi_\ell) = \begin{bmatrix} M(e^{-ix}) & M(1) \\ M(1) & 0 \end{bmatrix} \begin{bmatrix} M(e^{2ix}) & M(1) \\ M(1) & 0 \end{bmatrix}.$$

Since $M(e^{-ix}) = M(e^{2ix}) = 0$ and $M(1) = 1$, we obtain that $\mathbf{d}(\Psi_\ell) = I_{2\times 2}$ and therefore,

$$H\mathbf{d}(\Psi_\ell) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Hence,

$$\mathrm{sp}[H\mathbf{d}(\Psi_\ell)] \cap i\mathbb{R} = \{-1, 1\} \cap i\mathbb{R} = \emptyset.$$

These are sufficient conditions for these operators $W_{\Phi_1} \pm H_{\Phi_2}$ to have the Fredholm property (cf. Theorem 3.1).

Let us now calculate the sum of their Fredholm indices. For this case, we have

$$\mathrm{Ind}[W_{\Phi_1} + H_{\Phi_2}] + \mathrm{Ind}[W_{\Phi_1} - H_{\Phi_2}] - \mathrm{ind}[\det \Psi] - \frac{1}{\pi} \sum_{k=1}^{2} \beta(\zeta_k)$$

where $\zeta_k \in \mathbb{C} \backslash i\mathbb{R}$ are the eigenvalues of the matrix $H\mathbf{d}(\Psi_\ell)$ and β is given by (31). In addition, we have previously seen that $\det \Psi = 1$, therefore having a zero argument. Altogether, we have $\mathrm{Ind}[W_{\Phi_1} + H_{\Phi_2}] + \mathrm{Ind}[W_{\Phi_1} - H_{\Phi_2}] = 0$.

Acknowledgement. This work was supported in part by *Unidade de Investigação Matemática e Aplicações* of Universidade de Aveiro through the Portuguese Science Foundation (*FCT–Fundação para a Ciência e a Tecnologia*).

Received: March 2009. Revised: May 2009.

References

[1] G. Bogveradze, *Fredholm Theory for Wiener-Hopf plus Hankel Operators*, PhD Thesis, University of Aveiro, Aveiro, 2008.

[2] G. Bogveradze and L. P. Castro, *Wiener-Hopf plus Hankel operators on the real line with unitary and sectorial symbols*, Contemp. Math. **414** (2006), 77–85.

[3] L. P. Castro, F.-O. Speck and F. S. Teixeira, *A direct approach to convolution type operators with symmetry*, Math. Nachr. **269-270** (2004), 73–85.

[4] L. P. Castro and A. S. Silva, *Invertibility of matrix Wiener-Hopf plus Hankel operators with symbols producing a positive numerical range*, Z. Anal. Anwend. **28** (2009), 119–127.

[5] T. Ehrhardt, *Factorization Theory for Toeplitz plus Hankel Operators and Singular Integral Operators with Flip*, Habilitation Thesis, Technischen Universitität Chemnitz, Chemnitz, 2004.

[6] A. B. Lebre, E. Meister, F. S. Teixeira, *Some results on the invertibility of Wiener-Hopf-Hankel Operators*, Z. Anal. Anwend. **11** (1992), 57–76.

[7] A. P. Nolasco, *Regularity Properties of Wiener-Hopf-Hankel Operators*. PhD Thesis, University of Aveiro, Aveiro, 2007.

[8] A. P. Nolasco and L. P. Castro, *A Duduchava-Saginashvili's type theory for Wiener-Hopf plus Hankel operators*, J. Math. Anal. Appl. **331** (2007), 329–341.

[9] L. P. Castro and D. Kapanadze, *Exterior wedge diffraction problems with Dirichlet, Neumann and Impedance boundary conditions*, Acta Appl. Math., 110 (2010), 289-311.

[10] L. P. Castro, F.-O. Speck and F. S. Teixeira, *Explicit solution of a Dirichlet-Neumann wedge diffraction problem with a strip*, J. Integral Equations Appl. **15** (2003), 359–383.

[11] L. P. Castro, F.-O. Speck and F. S. Teixeira, *On a class of wedge diffraction problems posted by Erhard Meister*, Oper. Theory Adv. Appl. **147** (2004), 211–238.

[12] E. Meister, F.-O. Speck and F. S. Teixeira, *Wiener-Hopf-Hankel operators for some wedge diffraction problems with mixed boundary conditions*, J. Integral Equations Appl. **4** (1992), 229–255.

[13] G. Bogveradze and L. P. Castro, *On the Fredholm index of matrix Wiener-Hopf plus/minus Hankel operators with semi-almost periodic symbols*, Oper. Theory Adv. Appl. **181** (2008), 143–158.

[14] A. Böttcher, Yu. I. Karlovich and I. M. Spitkovsky, *Convolution Operators and Factorization of Almost Periodic Matrix Functions*, Birkhäuser, Basel, 2002.

[15] D. Sarason, *Toeplitz operators with semi-almost periodic symbols*, Duke Math. J. **44** (1977), 357–364.

[16] L. P. Castro and F.-O. Speck, *Regularity properties and generalized inverses of delta-related operators*, Z. Anal. Anwend. **17** (1998), 577–598.