

Uniform Spectral Estimates for Families of Schrödinger Operators with Magnetic Field of Constant Intensity and Applications

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ABSTRACT

The aim of this paper is to establish uniform estimates of the bottom of the spectrum of the Neumann realization of $(i\nabla + q\mathbf{A})^2$ on a bounded open set Ω with smooth boundary when $|\nabla \times \mathbf{A}| = 1$ and $q \rightarrow +\infty$. This problem was motivated by a question occurring in the theory of liquid crystals and appears also in superconductivity questions in large domains.

RESUMEN

El objetivo de este artículo es establecer estimativas uniformes del espectro inferior de la realización de Neumann de $(i\nabla + q\mathbf{A})^2$ sobre un conjunto Ω abierto acotado con frontera suave cuando $|\nabla \times \mathbf{A}| = 1$ y $q \rightarrow +\infty$. Este problema fue motivado por una cuestión que ocurre en teoría de cristales líquidos y aparece también en cuestiones de superconductividad en dominios grandes.

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Math. Subj. Class.: *35P, 35J10, 35Q, 81Q20, 82D30.*

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with C^3 boundary and $\mathbf{A} \in C^2(\overline{\Omega})$.

We will study the quadratic form $q_{\mathbf{A}}$ defined by:

$$q_{\mathbf{A}}(u) = \int_{\Omega} |(i\nabla + q\mathbf{A})u|^2 dx, \quad \forall u \in H^1(\Omega)$$

and consider the associated selfadjoint operator, i.e the Neumann realization of $(i\nabla + q\mathbf{A})^2$ on Ω . We denote $\mu_{\Omega}(q, \mathbf{A})$ or $\mu(q, \mathbf{A})$ the lowest eigenvalue of the previous operator. Our purpose is to study the behavior of this eigenvalue as q tends to infinity and to control the uniformity of the estimates with respect to the magnetic field.

We let

$$\mathcal{A} = \{\mathbf{A} \in C^3(\overline{\Omega}) : |\mathbf{B}| = 1 \text{ where } \mathbf{B} = \nabla \times \mathbf{A}\}. \quad (1.1)$$

The main estimates obtained in this paper are summarized in the two following theorems:

Theorem 1.1 (Uniform lower bound). *For all $\epsilon \in]0, \frac{1}{2}[$, there exists $C = C(\Omega, \epsilon) > 0$ and $q_0 > 0$, such that, for all $q \geq q_0$ and for all $\mathbf{A} \in \mathcal{A}$,*

$$\mu(q, \mathbf{A}) \geq \Theta_0 q - C \left(q^{1-2\epsilon} + (1 + |\nabla \mathbf{B}|_{\infty}) q^{1/2+2\epsilon} \right).$$

Theorem 1.2 (Uniform upper bound). *For all $\delta \in]0, 1/2[$, there exists $C = C(\Omega, \delta) > 0$ and $q_0 = q_0(\Omega; \delta) > 0$, such that for all $q \geq q_0$ and all $\mathbf{A} \in \mathcal{A}$:*

$$\mu(q, \mathbf{A}) \leq \Theta_0 q + C(q^{2\delta} + |\mathbf{B}|_{C^1}^2 q^{2-4\delta} + |\mathbf{B}|_{C^1} q^{1-\delta} + |\mathbf{B}|_{C^2} q^{3/2-3\delta} + |\mathbf{B}|_{C^2}^2 q^{2-6\delta}),$$

where $|\mathbf{B}|_{C^1}^2 = |\mathbf{B}|_{\infty} + |\nabla \mathbf{B}|_{\infty}^2$ and $|\mathbf{B}|_{C^2}^2 = |\mathbf{B}|_{C^1}^2 + |\nabla^2 \mathbf{B}|_{\infty}^2$.

Remark 1.3.

In those theorems, ϵ and δ are left undefined because they will play a role in the application to families of vector potentials where the semi-norms of \mathbf{B} could become large. If the magnetic field is fixed and if we are not interested in uniformity, we take $\epsilon = \frac{1}{8}$ and $\delta = \frac{1}{3}$ to have the optimal estimates (relatively to the method), leading to the remainder $O(q^{3/4})$ in the first case (in fact, one can hope a remainder $O(q^{2/3})$) and $O(q^{2/3})$ in the second case (cf. [HM04]). Let us also mention that, in dimension 2, the remainder is $O(q^{1/2})$ (see [FH08]).

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Let us briefly recall the motivation of those estimates.

Liquid crystals

The first one occurs in the theory of liquid crystals. The asymptotic properties of the Landau-de Gennes functional (cf. [BCLP02, dG95, HP07, Pan03, Pan06]) lead to the analysis of the minimizers (or local minimizers) of the reduced functional:

$$\mathcal{F}(\psi, \mathbf{n}) = \int_{\Omega} |(i\nabla + q\mathbf{n})\psi|^2 + r|\psi|^2 + \frac{g}{2}|\psi|^4 dx,$$

where $r > 0$, $\psi \in H^1(\Omega, \mathbb{C})$ and $\mathbf{n} \in \mathcal{C}(\tau)$, with $\mathcal{C}(\tau)$ defined for $\tau > 0$ by:

$$\mathcal{C}(\tau) = \{Q\mathbf{n}_{\tau}Q^t, Q \in SO_3\}, \tag{1.2}$$

where SO_3 denotes the set of the rotations in \mathbb{R}^3 and

$$\mathbf{n}_{\tau} = (\cos(\tau x_3), \sin(\tau x_3), 0). \tag{1.3}$$

Let us notice that, if $\mathbf{n} \in \mathcal{C}(\tau)$,

$$\nabla \times \mathbf{n} + \tau\mathbf{n} = 0$$

and consequently

$$|\nabla \times \mathbf{n}|_{\infty} = \tau.$$

Then, the analysis of the positivity of the Hessian of the functional at $\psi = 0$ leads to a spectral problem and we are led to study the asymptotic properties of

$$\mu^*(q, \tau) = \inf_{\mathbf{n} \in \mathcal{C}(\tau)} \mu(q, \mathbf{n}), \tag{1.4}$$

as $q\tau \rightarrow +\infty$.

In this context, X-B. Pan has given estimates (cf. [Pan06]) as:

$$q\tau \rightarrow +\infty \quad \text{and} \quad \tau \rightarrow 0$$

and Helffer and Pan give some extensions in [HP07] including the case:

$$q\tau \rightarrow +\infty \quad \text{and} \quad \tau \text{ bounded.}$$

In this paper, we treat the case where:

$$q\tau \rightarrow +\infty \quad \text{and} \quad \tau \rightarrow +\infty.$$

Superconductivity

The second one occurs in the theory of superconductivity in large domains. In [Alm02, Alm08], Y. Almog has analyzed properties of minimizers of Ginzburg-Landau's functional when the size of Ω tends to infinity; Theorems 1.1 and 1.2 permit to treat another regime (for the linear problem); in fact, q will be allowed to tend to infinity.

Organization of the paper

The paper is organized as follows. First, we prove Theorem 1.1 in Section 2 and Theorem 1.2 in Section 3, then we will prove an Agmon estimate in Section 4 in order to study the localization of first eigenfunctions in the considered asymptotic regime. Finally, Section 5 will present the applications to the theory of liquid crystals and to the superconductivity in large domains. In each case we will show that the eigenfunctions become localized at the boundary. This corresponds to what is called surface smecticity in the first case and surface superconductivity in the second case (see [Pan04]).

2 Lower Bound

In this section, we give the proof of Theorem 1.1. It is based on a localization technique through a partition of unity and the analysis of simplified models.

2.1 Partition of unity

For each $r > 0$, we consider a partition of unity (cf. [HM04]) with the property that there exists $C = C(\Omega) > 0$ such that:

$$\sum_j |\chi_j^r|^2 = 1 \quad \text{on } \Omega; \quad (2.5)$$

$$\sum_j |\nabla \chi_j^r|^2 \leq \frac{C}{r^2} \quad \text{on } \Omega. \quad (2.6)$$

Each χ_j^r is a C^∞ -cutoff function with support in the ball of center x_j and radius r (denoted by B_j). We will choose r later for optimizing the error. We will use the IMS formula (cf. [CFKS86]):

Lemma 2.1.

$$q_{\mathbf{A}}(u) = \sum_j q_{\mathbf{A}}(\chi_j u) - \sum_j \|\nabla \chi_j^r u\|^2, \quad \forall u \in H^1(\Omega). \quad (2.7)$$

So, in order to minimize $q_{\mathbf{A}}(u)$, we will be reduced to the minimization of $q_{\mathbf{A}}(v)$, with v supported in some B_j , the price to pay being an error of order $\frac{C}{r^2}$.

2.2 Approximation by the constant magnetic field in a ball or a semi-ball

We want to have estimates depending only on the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$, that's why we look for a canonical choice of \mathbf{A} depending only on \mathbf{B} ; it is the aim of the following lemmas. Let B a ball (or semi-ball) of center 0 and radius $r > 0$.

Lemma 2.2. *Let $\mathbf{F} \in \mathcal{C}^2(\overline{B}, \mathbb{R}^3)$.*

We assume the existence a constant $C > 0$ such that:

$$|\nabla \times \mathbf{F}| \leq C|x|,$$

for $x \in \overline{B}$. Then, there exists $u \in \mathcal{C}^3(\overline{B})$ and $\alpha > 0$ a constant such that:

$$|\mathbf{F}(x) - \nabla u(x)| \leq \alpha C|x|^2,$$

for all $x \in \overline{B}$.

Proof.

The proof is similar to the one of Poincaré's theorem.

Let us define, for all $x \in \overline{B}$:

$$u(x) = \int_0^1 \mathbf{F}(tx) \cdot x dt.$$

Let us verify that u is suitable. As $\mathbf{F} \in \mathcal{C}^2(\overline{B}, \mathbb{R}^3)$, we can extend \mathbf{F} in a \mathcal{C}^2 function on \mathbb{R}^3 , so by computing, we have, for $x \in \overline{B}$:

$$\partial_i u(x) = F_i(x) + \sum_{j=1, j \neq i}^3 \int_0^1 (\partial_i F_j - \partial_j F_i)(tx) t x_j dt.$$

■

As a consequence, we immediately deduce the lemma which will give us uniformity in our further estimates :

Lemma 2.3. *There exists $C > 0$ such that, for all $\mathbf{A} \in \mathcal{C}^2(\overline{B})$, there exists $\phi \in \mathcal{C}^3(\overline{B})$ verifying:*

$$|\mathbf{A}(x) - \mathbf{A}^{lin}(x) - \nabla \phi(x)| \leq C|\nabla \mathbf{B}|_\infty |x|^2,$$

for $x \in \overline{B}$ and where \mathbf{A}^{lin} is defined by:

$$\mathbf{A}^{lin}(x) = \frac{1}{2} \mathbf{B}(0) \wedge x.$$

2.3 Local estimates for the lower bound

We now distinguish two cases: the balls inside Ω and those which intersect the boundary. For the balls inside Ω , we are reduced to the problem of Dirichlet with constant magnetic field, and for the other balls, to the problem of Neumann on an half plane with constant magnetic field.

2.3.1 Study inside Ω

Let j such that B_j does not intersect the boundary.

We recall the inequality (problem of Dirichlet with constant magnetic field), for all $\psi \in C_0^\infty(\Omega)$:

$$\int_{\Omega} |(i\nabla + q\mathbf{A}^{lin})\psi|^2 dx \geq q \int_{\Omega} |\psi|^2 dx. \quad (2.8)$$

Using Lemma 2.3 we make the change of gauge $v \mapsto e^{-i\phi}v$ and with the classical inequality:

$$|a + b|^2 \geq (1 - \lambda^2)|a|^2 - \frac{1}{\lambda^2}|b|^2,$$

for $\lambda > 0$, we get:

$$\begin{aligned} \int_{\Omega} |(i\nabla + q(\mathbf{A} - \nabla\phi))(\chi_j u e^{-i\phi})|^2 dx &\geq \left((1 - \lambda^2) \int_{\Omega} |(i\nabla + q\mathbf{A}^{lin})(\chi_j u e^{-i\phi})|^2 dx \right. \\ &\quad \left. - C^2 q^2 \frac{|\nabla\mathbf{B}|_\infty^2 r^4}{\lambda^2} \right) \int_{\Omega} |\chi_j u|^2 dx, \end{aligned}$$

where \mathbf{A}^{lin} is defined in Lemma 2.3. Thus, we find with (2.8) applied to $\psi = \chi_j u e^{-i\phi}$:

$$\int_{\Omega} |(i\nabla + q\mathbf{A})\chi_j u|^2 dx \geq \left((1 - \lambda^2)q - C^2 q^2 \frac{|\nabla\mathbf{B}|_\infty^2 r^4}{\lambda^2} \right) \int_{\Omega} |\chi_j u|^2 dx.$$

2.3.2 Study near the boundary

We refer to [HM02, HM04], but we will control carefully the uniformity. We first recall some properties of the harmonic oscillator on an half axis (see [DH93, HM01]).

Harmonic oscillator on \mathbb{R}_+

For $\xi \in \mathbb{R}$, we consider the Neumann realization $\mathfrak{h}^{N,\xi}$ in $L^2(\mathbb{R}_+)$ associated with the operator

$$-\frac{d^2}{dt^2} + (t + \xi)^2, \quad \mathcal{D}(\mathfrak{h}^{N,\xi}) = \{u \in B^2(\mathbb{R}_+) : u'(0) = 0\}. \quad (2.9)$$

One knows that it has compact resolvent and its lowest eigenvalue is denoted $\mu(\xi)$; the associated L^2 -normalized and positive eigenstate is denoted u_ξ and is in the Schwartz class. The function $\xi \mapsto \mu(\xi)$ admits a unique minimum in $\xi = \xi_0$ and we let: $\Theta_0 = \mu(\xi_0)$.

We now introduce local coordinates near the boundary in order to compare with the harmonic oscillator on \mathbb{R}_+ :

Local coordinates near the boundary

Let's assume that $0 \in \partial\Omega$. In a neighborhood V of 0, we take local coordinates (y_1, y_2) on $\partial\Omega$ (via a \mathcal{C}^3 map ϕ). We denote $\mathbf{N}(\phi(y_1, y_2))$ the interior unit normal to the boundary at the point $\phi(y_1, y_2)$ and define local coordinates in V :

$$\Phi(y_1, y_2, y_3) = \phi(y_1, y_2) + y_3 \mathbf{N}(\phi(y_1, y_2)).$$

More precisely, for a point $x \in V$, $\phi(y_1, y_2)$ is the projection of x on $\partial\Omega \cap V$ and $y_3 = d(x, \partial\Omega)$. Taking a convenient map ϕ , we can assume

$$\Phi(0) = 0 \text{ and } D_0\Phi = Id.$$

Let j such that $B_j \cap \partial\Omega \neq \emptyset$. We can assume that $x_j \in \partial\Omega$ and $x_j = 0$ without loss of generality. After a change of variables, we have:

$$\int_{\Omega} |(i\nabla + q\mathbf{A})\chi_j u|^2 dx = \int_{y_3 > 0} |(i\nabla_y + q\tilde{\mathbf{A}})\widetilde{\chi_j u}|_{(D\Phi)^{-1}((D\Phi)^{-1})^t}^2 |\det(D\Phi)| dy,$$

where the tilde denotes the functions in the new coordinates and

$$\tilde{\mathbf{A}} = D_y \Phi(\mathbf{A}(\Phi(y))).$$

There exists $C > 0$ (uniform in j) such that:

$$\int_{y_3 > 0} |(i\nabla_y + q\tilde{\mathbf{A}})\widetilde{\chi_j u}|_{(D\Phi)^{-1}((D\Phi)^{-1})^t}^2 |\det(D\Phi)| dy \geq (1 - Cr) \int_{y_3 > 0} |(i\nabla_y + q\tilde{\mathbf{A}})\widetilde{\chi_j u}|^2 dy.$$

We use again the approximation by the constant magnetic field (for semi-balls) on the support of $\widetilde{\chi_j}$.

More precisely, there exists $\alpha > 0$ uniform in j , such that

$$\text{supp}(\widetilde{\chi_j}) \subset B(x_j, \alpha r).$$

Then, we change our partition of unity: we replace the balls which intersect the boundary by $\Phi(B(x_j, \alpha r))$. There exists $C > 0$ such that for all j , there exists $\tilde{\mathbf{A}}^{lin}$ (defined in Lemma 2.3) satisfying:

$$\begin{aligned} \int_{y_3 > 0} |(i\nabla_y + q\tilde{\mathbf{A}})\widetilde{\chi_j u}|^2 dy &\geq (1 - \lambda^2) \int_{y_3 > 0} |(i\nabla_y + q\tilde{\mathbf{A}}^{lin})\widetilde{\chi_j u}|^2 dy \\ &\quad - \frac{C^2 q^2}{\lambda^2} |\nabla \tilde{\mathbf{B}}|_{\infty}^2 r^4 \int_{\Omega} |\widetilde{\chi_j u}|^2 dy, \end{aligned}$$

where $\tilde{\mathbf{B}} = \nabla_y \times \tilde{\mathbf{A}}$. In order to express $\tilde{\mathbf{B}}$ as a function of \mathbf{B} , we need the following lemma:

Lemma 2.4. *With the previous notations, we have:*

$$\tilde{\mathbf{B}} = \det(D\Phi)((D\Phi)^{-1})^t \mathbf{B}.$$

Proof.

The result is standard. Let us recall the proof for completeness. Let us introduce the 1-form ω :

$$\omega = A_1 dx_1 + A_2 dx_2 + A_3 dx_3.$$

In the new coordinates $x = \Phi(y)$, we have, with the previous notations:

$$\omega = \tilde{A}_1 dy_1 + \tilde{A}_2 dy_2 + \tilde{A}_3 dy_3.$$

Then, it remains to write:

$$d\omega = (\nabla \times \mathbf{A})_1 dx_2 \wedge dx_3 + (\nabla \times \mathbf{A})_2 dx_1 \wedge dx_3 + (\nabla \times \mathbf{A})_3 dx_1 \wedge dx_2,$$

and to express dx_i as a function of (dy_j) .

The comatrix formula gives the conclusion. ■

We are reduced to the case of constant magnetic field (of intensity q) on $\mathbb{R}_+^3 = \{y_3 > 0\}$ (see [HM02, HM04, LP00]) and we get:

$$\int_{y_3 > 0} |(i\nabla_y + q\tilde{\mathbf{A}})\widetilde{\chi_j u}|^2 dy \geq \Theta_0 q \int_{y_3 > 0} |\widetilde{\chi_j u}|^2 dy.$$

Thus, we find:

$$\int_{y_3 > 0} |(i\nabla_y + q\tilde{\mathbf{A}})\widetilde{\chi_j u}|^2 dy \geq \left((1 - \lambda^2)q\Theta_0 - \frac{C^2 q^2}{\lambda^2} (1 + |\nabla \mathbf{B}|_\infty^2) r^4 \right) \int_{y_3 > 0} |\widetilde{\chi_j u}|^2 dy.$$

2.3.3 End of the proof

We take, for $0 < \epsilon < \frac{1}{2}$, $r = \frac{1}{q^{1/2-\epsilon}}$. We divide by q and we choose λ such that:

$$\lambda^2 = \frac{q}{\lambda^2} (1 + |\nabla \mathbf{B}|_\infty^2) r^4.$$

Then, the previous estimates lead to the existence of $C > 0$ and $q_0 > 0$ depending only on Ω such that for all $q \geq q_0$:

$$\frac{q_{\mathbf{A}}(u)}{q} \geq \left(q\Theta_0 - C \left(r + \lambda^2 + \frac{1}{qr^2} \right) \right) \int_{\Omega} |u|^2 dx.$$

We finally find, by the minimax principle:

$$\frac{\mu(q, \mathbf{A})}{q} \geq \Theta_0 - C \left(\frac{1}{q^{1/2-2\epsilon}} + \frac{1}{q^{2\epsilon}} + \frac{|\nabla \mathbf{B}|_\infty}{q^{1/2-2\epsilon}} \right).$$

3 Upper Bound

In this section, we give a proof of Theorem 1.2. We refer to [HM04] and, in the next lines, we emphasize the crucial points where uniformity is concerned. In the case of the constant magnetic field on \mathbb{R}_+^3 , we know (cf. [HM02, LP00]) that the bottom of the spectrum is minimal when the magnetic field is tangent to the boundary. So, we will look for a quasimode localized near a point where the magnetic field is tangent. Then, we will take as trial function some truncation of u_{ξ_0} . So, we fix $x_0 \in \partial\Omega$ such that: $\mathbf{B}(x_0) \cdot \nu = 0$. Such a x_0 exists; indeed, noticing that $\operatorname{div}(\mathbf{B}) = 0$, the Stokes formula gives:

$$\int_{\partial\Omega} \mathbf{B} \cdot \nu d\sigma = \int_{\Omega} \operatorname{div}(\mathbf{B}) dx = 0.$$

We take $\delta \in]0, 1/2[$ and we suppose that u is such that

$$\text{supp}(u) \subset B(x_0, \alpha r),$$

with $r = \frac{1}{q^\delta}$.

We assume that the support of u is small enough and after a change of coordinates, we can use the same arguments as in Lemma 2.3 and take a gauge in which \mathbf{A} satisfies:

$$|\mathbf{A}(y) - \mathbf{A}^0(y)| \leq C|\mathbf{B}|_{C^2}|y|^3,$$

where $\mathbf{A}^0 = \mathbf{A}^{lin} + R$, with $R = (R_1, R_2, R_3)$ and R_j homogeneous polynomial of order 2 and where $\nabla^2 \mathbf{B}$ denotes the hessian matrix of \mathbf{B} .

We find:

$$q_{\mathbf{A}}(u) \leq (1 + Cr)(q_{\mathbf{A}^0}(u) + C(|\mathbf{B}|_{C^2}^2 r^6 q^2 \|u\|^2 + qr^3 |\mathbf{B}|_{C^2} \|u\| q_{\mathbf{A}^0}(u)^{1/2})).$$

We let:

$$u(y) = q^{1/4+\delta} e^{-i\xi_0 y_2 q^{1/2}} u_{\xi_0}(q^{1/2} y_3) \chi(4q^\delta y_3) \chi(4q^\delta (y_1^2 + y_2^2)^{1/2}).$$

We have to compare: $q_{\mathbf{A}^0}(u)$ and $q_{\mathbf{A}^{lin}}(u)$. We get:

$$\begin{aligned} q_{\mathbf{A}^0}(u) &\leq q_{\mathbf{A}^{lin}}(u) + Cq^2 r^4 (1 + |\nabla \mathbf{B}|_\infty^2) \|u\|^2 \\ &\quad + 2\Re \left\{ \int_{|y| \leq \alpha r, y_3 > 0} (i\nabla + q\mathbf{A}^{lin})u \cdot (q\mathbf{A}^0 - q\mathbf{A}^{lin})udy \right\}. \end{aligned}$$

We have to estimate the double product (cf. [HM04, section 6, p. 120]):

$$\left| \Re \left\{ \int_{|y| \leq \alpha r, y_3 > 0} (i\nabla + q\mathbf{A}^{lin})u \cdot (q\mathbf{A}^0 - q\mathbf{A}^{lin})udy \right\} \right| \leq C(1 + |\nabla \mathbf{B}|_\infty) q^{1-\delta}.$$

Moreover, using the exponential decrease of u_{ξ_0} , we have:

$$q_{\mathbf{A}^{lin}}(u) \leq \Theta_0 q + Cq^{2\delta}.$$

4 Agmon's Estimates

In order to estimate the asymptotic localization of the first eigenfunctions in the applications, we will need Agmon's estimates to have some exponential decrease inside Ω ; that is the aim of this section. Let us introduce some notations (see [Agm82, Alm08, FH08]). For $\gamma > 0$ small enough, let η_γ be a smooth cutoff function such that:

$$\eta_\gamma = \begin{cases} 1 & \text{if } d(y) = d(y, \partial\Omega) \geq \gamma \\ 0 & \text{if } y \notin \Omega \end{cases},$$

with

$$|\nabla \eta_\gamma| \leq \frac{C}{\gamma}.$$

We let: $\Omega_\gamma = \{y \in \Omega : d(y, \partial\Omega) \geq \gamma\}$. For $\alpha > 0$, we let

$$\xi(y) = \eta_\gamma e^{\alpha d(y)}.$$

We finally denote $\mu_0(q, \mathbf{A})$ the lowest eigenvalue of the Dirichlet's realization of $(i\nabla + q\mathbf{A})^2$ on Ω . We have the following localization property:

Proposition 4.1. *There exists $C > 0$ and $\gamma_0 > 0$, depending only on Ω such that for all $0 < \epsilon \leq 1$ and α verifying:*

$$0 < \alpha < \left(\frac{1}{1+\epsilon}\right)^{1/2} (\mu_0 - \mu)^{1/2},$$

and for all $0 < \gamma \leq \gamma_0$, if u is a normalized mode associated with the lowest eigenvalue $\mu = \mu(q, \mathbf{A})$ of the Neumann realization of $(i\nabla + q\mathbf{A})^2$, then:

$$\|\eta_\gamma e^{\alpha d(y)} u\|_{H^1(\Omega)} \leq \frac{C}{\sqrt{\epsilon\gamma}} \left(\frac{\mu_0 + 1}{\mu_0 - \mu - (1+\epsilon)\alpha^2}\right)^{1/2} e^{\alpha\gamma}.$$

Proof.

We consider the equation verified by u :

$$(i\nabla + q\mathbf{A})^2 u = \mu u.$$

One multiplies by $\xi^2 \bar{u}$ and integrate by parts (using $(i\nabla + q\mathbf{A})u \cdot \nu = 0$ on $\partial\Omega$) to get:

$$|(i\nabla + q\mathbf{A})(\xi u)|_2^2 = \mu |\xi u|_2^2 + |(\nabla \xi)u|_2^2.$$

We have, for all $\epsilon > 0$:

$$|(\nabla \xi)u|^2 \leq \left(1 + \frac{1}{\epsilon}\right) \int_{\Omega \setminus \Omega_\gamma} |\nabla \eta|^2 e^{2\alpha d(y)} |u|^2 + (1+\epsilon)\alpha^2 \int_{\Omega} |\xi u|^2.$$

We use that $|u|_2 = 1$ to find:

$$|\xi u|^2 \leq \frac{C}{\gamma} \left(1 + \frac{1}{\epsilon}\right) \frac{e^{2\alpha\gamma}}{\mu_0 - \mu - (1+\epsilon)\alpha^2}.$$

Moreover, the diamagnetic inequality gives:

$$|\nabla |\xi u||^2 \leq |(i\nabla + q\mathbf{A})(\xi u)|^2.$$

It follows that:

$$\|\xi u\|_{H^1(\Omega)}^2 \leq \frac{C}{\gamma} \left(1 + \frac{1}{\epsilon}\right) e^{2\alpha\gamma} \frac{\mu_0 + 1}{\mu_0 - \mu - (1+\epsilon)\alpha^2}.$$

■

5 Applications

We now describe two applications of our main results.

5.1 Application to an helical vector field

In this section, we study $\mu^*(q, \tau)$ defined in (1.4) as $q\tau \rightarrow +\infty$ and $\tau \rightarrow +\infty$. Due to the definition (1.4), we will use the uniform analysis of $\mu(q, \mathbf{n})$ with $\mathbf{n} \in \mathcal{C}(\tau)$.

5.1.1 Estimate of the first eigenvalue

The main theorem in this section is the following:

Theorem 5.1. *Let $c_0 > 0$ and $0 \leq x < \frac{1}{2}$. There exists $C > 0$ and $q_0 > 0$ depending only on Ω , c_0 and x such that, if (q, τ) verifies $q\tau \geq q_0$ and*

$$\tau \leq c_0(q\tau)^x, \tag{5.10}$$

then:

$$\Theta_0 - \frac{C}{(q\tau)^{1/4-x/2}} \leq \frac{\mu^*(q, \tau)}{q\tau} \leq \Theta_0 + \frac{C}{(q\tau)^{1/3-2x/3}}. \tag{5.11}$$

Remark 5.2.

This statement was obtained for $x = 0$ in [HP07] and rough estimates were given in [BCLP02] as $\frac{\tau}{q} \rightarrow 0$.

■

Proof.

Let us notice that:

$$\mu(q\tau, \frac{Q\mathbf{n}_\tau Q^t}{\tau}) = \mu(q, Q\mathbf{n}_\tau Q^t).$$

Moreover, there exists $C > 0$ such that for all $\tau > 0$ and $\mathbf{n} \in \mathcal{C}(\tau)$, if $\mathbf{A} = \frac{\mathbf{n}}{\tau}$ and $\mathbf{B} = \nabla \times \mathbf{A}$, then:

$$\begin{aligned} |\mathbf{B}|_\infty &= 1, \\ |\nabla \mathbf{B}|_\infty &\leq C\tau, \\ |\nabla^2 \mathbf{B}|_\infty &\leq C\tau^2. \end{aligned}$$

For the lower bound, we apply Theorem 1.1 to the subfamily $\mathcal{C}(\tau)$ of \mathcal{A} and, using (5.10), we get:

$$\frac{\mu^*(q, \tau)}{q\tau} \geq \Theta_0 - C \left(\frac{1}{(q\tau)^{1/2-2\epsilon}} + \frac{1}{(q\tau)^{2\epsilon}} + c_0 \frac{(q\tau)^x}{(q\tau)^{1/2-2\epsilon}} \right).$$

We choose ϵ such that $\frac{1}{2} - 2\epsilon - x = 2\epsilon$, i.e: $2\epsilon = \frac{1}{4} - \frac{x}{2}$.

For the upper bound, we apply Theorem 1.2 with $\mathbf{A} = \frac{\mathbf{n}_\tau}{\tau}$. Then, we get:

$$\mu^*(q, \tau) \leq \Theta_0 q\tau + C((q\tau)^{2\delta} + (q\tau)^{2-4\delta+2x} + (q\tau)^{1-\delta+x} + (q\tau)^{3/2-3\delta+2x} + (q\tau)^{2-6\delta+4x}).$$

We choose δ such that:

$$2\delta = 1 - \delta + x.$$

Thus, we take $\delta = \frac{1+x}{3}$ and the upper bound follows.

■

5.1.2 Localization of the ground state near the boundary as $\tau \rightarrow +\infty$

We first state a proposition:

Proposition 5.3. *For $\mathbf{A} \in \mathcal{A}$ (cf. (1.1)), we denote $\mu_0(q, \mathbf{A})$ the bottom of the spectrum of the Dirichlet realization of $(i\nabla + q\mathbf{A})^2$.*

Then, for all $\epsilon \in]0, 1/2[$, there exists $C = C(\Omega, \epsilon) > 0$ and $q_0 = q_0(\Omega, \epsilon)$ s.t. if $q \geq q_0$, for all $\mathbf{A} \in \mathcal{A}$:

$$\frac{\mu_0(q, \mathbf{A})}{q} \geq 1 - C \left(\frac{1}{q^{2\epsilon}} + \frac{|\nabla \mathbf{B}|_\infty}{q^{1/2-2\epsilon}} \right).$$

Proof.

We again use partition (2.5), formula (2.7) and the proof is the same as for Theorem 1.1. ■

We deduce:

Corollary 5.4. *Let $c_0 > 0$. For all $x \in [0, 1/2[$, there exists $C = C(\Omega, x, c_0) > 0$, such that, for all $\mathbf{n} \in \mathcal{C}(\tau)$ and (q, τ) such that $\tau \leq c_0(q\tau)^x$:*

$$\frac{\mu_0(q\tau, \frac{\mathbf{n}}{\tau})}{q\tau} \geq 1 - C \left(\frac{1}{q\tau} \right)^{1/4-x/2}.$$

As an immediate consequence of Proposition 4.1, we have the following theorem:

Theorem 5.5. *For all $x \in [0, 1/2[$, there exists $\delta_0 > 0$, $C > 0$, $c > 0$ such that if (q, τ) verifies $q\tau \geq \delta_0$ and $\tau \leq c_0(q\tau)^x$, then for all $\mathbf{n} \in \mathcal{C}(\tau)$ and u a L^2 -normalized solution of*

$$\begin{aligned} (i\nabla + q\mathbf{n})^2 u &= \mu(q, \mathbf{n})u, & \text{in } \Omega \\ (i\nabla + q\mathbf{n})u \cdot \nu &= 0, & \text{on } \partial\Omega \end{aligned}$$

we have:

$$\left\| \eta_{\frac{c}{\sqrt{q\tau}}} e^{((1-\Theta_0)^{1/2} \sqrt{q\tau} - r(q\tau))d(\cdot, \partial\Omega)} |u| \right\|_{H^1(\Omega)} \leq C,$$

where $r(q\tau) = (q\tau)^{3/8+x/4}$.

Proof.

We apply Proposition 4.1 with $\epsilon = (q\tau)^{-1/4+x/2}$, $\alpha = (1 - \Theta_0)^{1/2}(\sqrt{q\tau} - \lambda)$, $\gamma = \frac{1}{\sqrt{q\tau}}$, $\lambda = (q\tau)^{1/4+x_1/2}$, with $x < x_1 < \frac{1}{2}$ and we notice, by taking a truncated gaussian, that

$$\mu_0(q\tau, \frac{\mathbf{n}}{\tau}) \leq Cq\tau.$$

We finally take $x_1 = \frac{1}{2}(\frac{1}{2} + x)$ and apply the estimate (5.11) of Theorem 5.1. ■

5.2 Surface superconductivity in large domains

5.2.1 The problem

Let $R > 0$ and $x_0 \in \Omega$. We denote $\Omega_R = \{x_0 + R(x - x_0), x \in \Omega\}$. The aim of this part is to study the behaviour of $\mu_{\Omega_R}(q, \mathbf{A})$, where $\mathbf{A} \in \mathcal{A}$ (cf.(1.1)) as $q \rightarrow +\infty$ and $R \rightarrow +\infty$. In his work [Alm08], Almgren studies the regime q fixed and $R \rightarrow +\infty$ (see Theorem 1.1 in [Alm08]) and then makes q to tend to infinity (see Section 3 of [Alm08]). In this section, we give a theorem which treats another regime : $q \rightarrow +\infty$ and R with polynomial increase in q . We first observe the following scaling invariance:

Lemma 5.6. *Let $R > 0$. We have:*

$$\mu_{\Omega_R}(q, \mathbf{A}) = \frac{1}{R^2} \mu_{\Omega} \left(qR^2, \frac{\mathbf{A}(R \cdot)}{R} \right).$$

5.2.2 Estimate of the lowest eigenvalue

Theorem 5.7. *Let $c_0 > 0$ and $y \geq 0$. There exists $C > 0$, $q_0 > 0$ and $R_0 > 0$ depending only on Ω , c_0 and y such that, if (q, R) satisfies $q \geq q_0$, $R \geq R_0$ and $R \leq c_0 q^y$, then, for $\mathbf{A} \in \mathcal{A}$:*

$$\Theta_0 - \frac{C}{(qR^2)^{\frac{1}{4(1+2y)}}} \leq \frac{\mu_{\Omega_R}(q, \mathbf{A})}{q} \leq \Theta_0 - \frac{C}{(qR^2)^{\frac{1}{3(1+2y)}}}.$$

Proof.

We use Lemma 5.6. We let $x = \frac{y}{1+2y}$ and notice that $R \leq c(y)(qR^2)^x$ with $c(y) = c_0^{\frac{1}{1+2y}}$ and $\frac{\mathbf{A}(R \cdot)}{R} \in \mathcal{A}$. Then by the Theorems 1.1 and 1.2, we have the wished conclusion by using the same arguments as for Theorem 5.1. ■

5.2.3 Localization of the groundstate near the boundary in large domains

In the case of large domains, we prove a quite analogous theorem with Theorem 5.5:

Theorem 5.8. *For all $y \geq 0$, there exists $\delta_0 > 0$, $\delta_1 > 0$, $C > 0$, $c > 0$ such that if (q, R) verifies $q \geq \delta_0$, $R \geq \delta_1$ and $R \leq c_0 q^y$, then for all $\mathbf{A} \in \mathcal{A}$ and u a L^2 -normalized solution of*

$$\begin{aligned} (i\nabla + q\mathbf{A})^2 u &= \mu_{\Omega_R}(q, \mathbf{A})u, & \text{in } \Omega_R \\ (i\nabla + q\mathbf{A})u \cdot \nu &= 0, & \text{on } \partial\Omega_R \end{aligned}$$

we have,

$$\left\| \eta_{\frac{c}{\sqrt{q}}} e^{(1-\Theta_0)^{1/2}(\sqrt{q}-r(q,R))d(\cdot, \partial\Omega_R)} |u| \right\|_{H^1(\Omega_R)} \leq C,$$

where $r(q, R) = q^{1/2 - \frac{1}{8(1+2y)}} R^{-\frac{1}{4(1+2y)}}$.

Proof.

After a rescaling, the proof is the same as the one of Theorem 5.5.

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