

A Fixed Point Theorem of Reich in G -Metric Spaces

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ABSTRACT

In this paper we prove some fixed point results for mapping satisfying sufficient contractive conditions on a complete G -metric space, also we showed that if the G -metric space (X, G) is symmetric, then the existence and uniqueness of these fixed point results follows from Reich theorems in usual metric space (X, d_G) , where (X, d_G) the metric induced by the G -metric space (X, G) .

RESUMEN

En este artículo nosotros provamos algunos resultados de punto fijo para aplicaciones satisfaciendo condiciones suficientes de contractividad sobre un espacio G -métrico completo, también mostramos que si el espacio G -métrico (X, G) es simétrico, entonces la existencia y unicidad de estos resultados de punto fijo siguen de teoremas de Reich en espacios métricos usuales (X, d_G) , donde (X, d_G) es la métrica inducida por el espacio G -métrico (X, G) .

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1 Introduction

The study of fixed points of a functions satisfying certain contractive conditions has been at the center of vigorous research activity, because it has a wide range of applications in different areas such as, variational, linear inequalities, optimization and parameterize estimation problems.

In 2005, Z. Mustafa and B. Sims introduced a new class of generalized metric spaces (see [2], [3]), which are called G -metric spaces as generalization of metric space (X, d) , to develop and to introduce a new fixed point theory for a variety of mappings in this new setting, also to extend known metric space theorems to a more general setting. The G -metric space is as follows.

Definition 1. Let X be a nonempty set, and let $G : X \times X \times X \rightarrow \mathbf{R}^+$, be a function satisfying the following properties:

$$(G1) \quad G(x, y, z) = 0 \text{ if } x = y = z;$$

$$(G2) \quad 0 < G(x, x, y); \text{ for all } x, y \in X, \text{ with } x \neq y;$$

$$(G3) \quad G(x, x, y) \leq G(x, y, z), \text{ for all } x, y, z \in X, \text{ with } z \neq y;$$

$$(G4) \quad G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots, \text{ (symmetry in all three variables); and}$$

$$(G5) \quad G(x, y, z) \leq G(x, a, a) + G(a, y, z), \text{ for all } x, y, z, a \in X, \text{ (rectangle inequality).}$$

Then the function G is called a generalized metric, or, more specifically a G -metric on X , and the pair (X, G) is called a G -metric space.

Example 1. ([2]) Let (X, d) be a usual metric space, and define G_s and G_m on $X \times X \times X$ to \mathbf{R}^+ by

$$G_s(x, y, z) = d(x, y) + d(y, z) + d(x, z), \text{ and}$$

$$G_m(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}$$

for all $x, y, z \in X$. Then (X, G_s) and (X, G_m) are G -metric spaces.

Definition 2. ([3]) Let (X, G) be a G -metric space, and let (x_n) be a sequence of points of X . A point $x \in X$ is said to be the limit of the sequence (x_n) if $\lim_{n, m \rightarrow \infty} G(x, x_n, x_m) = 0$, and one say that the sequence (x_n) is G -convergent to x .

Thus, that if $x_n \rightarrow 0$ in a G -metric space (X, G) , then for any $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq N$, (we mean by \mathbf{N} the Natural numbers).

Proposition 1. ([3]) Let (X, G) be G -metric space. Then the following are equivalent.

$$(1) \quad (x_n) \text{ is } G\text{-convergent to } x.$$

$$(3) \quad G(x_n, x_n, x) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

$$(4) \quad G(x_n, x, x) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

$$(5) \quad G(x_m, x_n, x) \rightarrow 0, \text{ as } m, n \rightarrow \infty.$$

Definition 3. ([3]) Let (X, G) be a G -metric space, a sequence (x_n) is called G -Cauchy if given $\epsilon > 0$, there is $N \in \mathbf{N}$ such that $G(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \geq N$. That is $G(x_n, x_m, x_l) \rightarrow 0$ as $n, m, l \rightarrow \infty$.

Proposition 2. ([3]) In a G -metric space, (X, G) , the following are equivalent.

1. The sequence (x_n) is G -Cauchy.
2. For every $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq N$.

Definition 4. ([3]) Let (X, G) and (X', G') be two G -metric spaces, and let $f : (X, G) \rightarrow (X', G')$ be a function, then f is said to be G -continuous at a point $a \in X$ if and only if, given $\epsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$; and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \epsilon$. A function f is G -continuous at X if and only if it is G -continuous at all $a \in X$.

Proposition 3. ([3]) Let (X, G) , and (X', G') be two G -metric spaces. Then a function $f : X \rightarrow X'$ is G -continuous at a point $x \in X$ if and only if it is G -sequentially continuous at x ; that is, whenever (x_n) is G -convergent to x we have $(f(x_n))$ is G -convergent to $f(x)$.

Definition 5. ([3]) A G -metric space (X, G) is called symmetric G -metric space if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

It is clear that, any G -metric space where G derives from an underlying metric via G_s or G_m in Example 1 is symmetric.

The following example presents the simplest instance of a nonsymmetric G -metric and so also one which does not arise from any metric in the above ways.

Example 2. ([3]) Let $X = \{a, b\}$, and let,

$$\begin{aligned} G(a, a, a) &= G(b, b, b) = 0, \\ G(a, a, b) &= 1, G(a, b, b) = 2 \end{aligned}$$

and extend G to $X \times X \times X$ by symmetry in the variables. Then it is easily verified that G is a G -metric, but $G(a, b, b) \neq G(a, a, b)$.

Proposition 4. ([3]) Let (X, G) be a G -metric space, then the function $G(x, y, z)$ is jointly continuous in all three of its variables.

Proposition 5. ([3]) Every G -metric space (X, G) induces a metric space (X, d_G) defined by

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \forall x, y \in X.$$

Note that if (X, G) is symmetric, then

$$d_G(x, y) = 2G(x, y, y), \forall x, y \in X. \quad (1.1)$$

However, if (X, G) is not symmetric then it holds by the G -metric properties that

$$\frac{3}{2}G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y), \forall x, y \in X. \quad (1.2)$$

Definition 6. ([3]) A G -metric space (X, G) is said to be G -complete (or complete G -metric) if every G -Cauchy sequence in (X, G) is G -convergent in (X, G) .

Proposition 6. ([3]) A G -metric space (X, G) is G -complete if and only if (X, d_G) is a complete metric space.

Theorem 1.1 (Reich,[4]). Let (X, d) be a complete metric space, and T be a function mapping X into it self, satisfy the following condition,

$$d(T(x), T(y)) \leq ad(x, T(x)) + bd(y, T(y)) + cd(x, y), \forall x, y \in X. \quad (1.3)$$

where a, b, c are nonnegative numbers satisfying $a + b + c < 1$.

Then, T has a unique fixed point (i.e., there exists $u \in X$; $Tu = u$).

2 Main Results

In this section, we will present several fixed point results on a complete G -metric space.

Theorem 2.1. Let (X, G) be a complete G -metric space, and let $T : X \rightarrow X$ be a mapping satisfies the following condition

$$G(T(x), T(y), T(z)) \leq k\{G(x, T(x), T(x)) + G(y, T(y), T(y)) + G(z, T(z), T(z))\} \quad (2.1)$$

for all $x, y, z \in X$, where $k \in [0, 1/3)$. Then T has a unique fixed point (say u), and T is G -continuous at u .

Proof. Suppose that T satisfies condition (2.1), then for all $x, y \in X$, we have

$$G(Tx, Ty, Ty) \leq k[G(x, Tx, Tx) + 2G(y, Ty, Ty)], \text{ and} \quad (2.2)$$

$$G(Ty, Tx, Tx) \leq k[G(y, Ty, Ty) + 2G(x, Tx, Tx)]. \quad (2.3)$$

Suppose that (X, G) is symmetric. Then from the definition of metric (X, d_G) and (1.1), we have

$$d_G(Tx, Ty) \leq kd_G(x, Tx) + 2kd_G(y, Ty), \forall x, y \in X. \quad (2.4)$$

In this line, since $0 < k + 2k < 1$, then the metric condition (2.4) will be a special case of the Reich condition (1.3), so the existence and uniqueness of the fixed point follows from Theorem (1.1).

However, if (X, G) is not symmetric then we can conclude that
 $d_G(Tx, Ty) = G(Tx, Ty, Ty) + G(Ty, Tx, Tx) \leq 3kG(x, Tx, Tx) + 3kG(y, Ty, Ty)$,
 $\forall x, y \in X$.

So, by the definition of the metric (X, d_G) and (1.2), we get

$$d_G(Tx, Ty) \leq 2kd_G(x, Tx) + 2kd_G(y, Ty), \forall x, y \in X,$$

and, the metric condition gives no information about this map since $0 < 2k + 2k$ need not be less than 1. But the existence of a fixed point can be proved using properties of a G -metric.

Let $x_0 \in X$, be an arbitrary point, and define the sequence (x_n) by $x_n = T^n(x_0)$, then the condition (2.1) implies that

$$G(x_n, x_{n+1}, x_{n+1}) \leq kG(x_{n-1}, x_n, x_n) + 2kG(x_n, x_{n+1}, x_{n+1}), \text{ hence}$$

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{k}{1 - 2k} G(x_{n-1}, x_n, x_n).$$

Let $q = \frac{k}{1 - 2k}$, then $0 < q < 1$ since $0 \leq k < 1/3$.

So,

$$G(x_n, x_{n+1}, x_{n+1}) \leq q G(x_{n-1}, x_n, x_n).$$

Continuing in the same argument, we will find

$$G(x_n, x_{n+1}, x_{n+1}) \leq q^n G(x_0, x_1, x_1). \tag{2.5}$$

Moreover, for all $n, m \in \mathbf{N}$; $n < m$ we have by repeated use the rectangle inequality and using equation (2.5) that

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &+ G(x_{n+2}, x_{n+3}, x_{n+3}) + \dots + G(x_{m-1}, x_m, x_m) \\ &\leq (q^n + q^{n+1} + \dots + q^{m-1})G(x_0, x_1, x_1) \leq \frac{q^n}{1 - q} G(x_0, x_1, x_1), \end{aligned}$$

and so, $\lim G(x_n, x_m, x_m) = 0$, as $n, m \rightarrow \infty$. Thus (x_n) is G -Cauchy sequence, then by completeness of (X, G) , there exists $u \in X$ such that (x_n) is G -convergent to u .

Assume on the contrary that $T(u) \neq u$. Then

$$G(x_{n+1}, T(u), T(u)) \leq k \{G(x_n, x_{n+1}, x_{n+1}) + 2G(u, T(u), T(u))\}.$$

Taking the limit as $n \rightarrow \infty$, and using the fact that the function G is continuous on its variable, this leads to $G(u, T(u), T(u)) \leq 2k G(u, T(u), T(u))$. This contradiction implies that $u = T(u)$.

To prove uniqueness, suppose that u and v are two fixed points for T , then
 $G(u, v, v) \leq k G(u, T(u), T(u)) + 2k G(v, T(v), T(v)) = 0$, which implies that $u = v$.

To show that T is G -continuous at u , let $(y_n) \subseteq X$ be a sequence converges to u in (X, G) , then we can deduce that

$$G(u, T(y_n), T(y_n)) \leq k \{G(u, T(u), T(u)) + 2G(y_n, T(y_n), T(y_n))\}. \quad (2.6)$$

Moreover, from G -metric axioms we have,

$$G(y_n, T(y_n), T(y_n)) \leq G(y_n, u, u) + G(u, T(y_n), T(y_n)),$$

so, equation (2.6) implies that $G(u, T(y_n), T(y_n)) \leq \frac{2k}{1-2k} G(y_n, u, u)$. Taking the limit as $n \rightarrow \infty$, from which we see that $G(y_n, T(y_n), T(y_n)) \rightarrow 0$ and so, by Proposition 3, $T(y_n) \rightarrow u = Tu$, therefore T is G -continuous at u . This completes the prove of Theorem (2.1). \square

Corollary 1. *Let (X, G) be a complete G -metric spaces, and let $T : X \rightarrow X$ be a mapping satisfying the following condition for some $m \in \mathbf{N}$*

$$G(T^m(x), T^m(y), T^m(z)) \leq k \left\{ \begin{array}{l} G(x, T^m(x), T^m(x)) + G(y, T^m(y), T^m(y)) + \\ G(z, T^m(z), T^m(z)) \end{array} \right\} \quad (2.7)$$

for all $x, y, z \in X$, where $k \in [0, 1/3)$. Then T has unique fixed point (say u), and T^m is G -continuous at u .

Proof. From previous theorem we see that T^m has a unique fixed point (say u), that is, $T^m(u) = u$, and $T^m(u)$ is G -continuous at u . But, $T(u) = T(T^m(u)) = T^{m+1}(u) = T^m(T(u))$, so $T(u)$ is another fixed point for T^m and by uniqueness $Tu = u$. \square

Theorem 2.2. *Let (X, G) be a complete G -metric space, and let $T : X \rightarrow X$, be a mapping satisfying the following condition*

$$G(T(x), T(y), T(z)) \leq \alpha G(x, y, z) + \beta \left\{ \begin{array}{l} G(y, T(y), T(y)) + G(z, T(z), T(z)) + \\ G(x, T(x), T(x)) \end{array} \right\} \quad (2.8)$$

for all $x, y, z \in X$, where $0 \leq \alpha + 3\beta < 1$. Then T has unique fixed point (say u), and T is G -continuous at u .

Proof. Suppose that T satisfies condition (2.8). Then for all $x, y \in X$

$$G(Tx, Ty, Ty) \leq \alpha G(x, y, y) + \beta [G(x, Tx, Tx) + 2G(y, Ty, Ty)], \text{ and} \quad (2.9)$$

$$G(Ty, Tx, Tx) \leq \alpha G(y, x, x) + \beta [G(y, Ty, Ty) + 2G(x, Tx, Tx)]. \quad (2.10)$$

Suppose that (X, G) is symmetric. Then from the definition of metric (X, d_G) and (1.1) we get

$$d_G(Tx, Ty) \leq \alpha d_G(x, y) + \beta d_G(x, Tx) + 2\beta d_G(y, Ty), \forall x, y \in X. \quad (2.11)$$

Since $0 < \alpha + 3\beta < 1$, then the metric condition (2.11) becomes the same as Reich condition (1.3), so the existence and uniqueness of the fixed point follows from Theorem (1.1).

However, if (X, G) is not symmetric, then we conclude that
 $d_G(Tx, Ty) = G(Tx, Ty, Ty) + G(Ty, Tx, Tx) \leq$
 $\alpha[G(x, y, y) + G(y, x, x)] + 3\beta G(x, Tx, Tx) + 3\beta G(y, Ty, Ty), \forall x, y \in X.$

So, by the definition of the metric (X, d_G) and (1.2) we get

$$d_G(Tx, Ty) \leq \alpha d_G(x, y) + 2\beta d_G(x, Tx) + 2\beta d_G(y, Ty), \forall x, y \in X.$$

The metric condition gives no information about this map since $0 < \alpha + 2\beta + 2\beta$ need not be less 1, but this can be proved by G -metric.

Let $x_0 \in X$, be an arbitrary point, and define the sequence (x_n) by $x_n = T^n(x_0)$, then by (2.8) we can verify that

$$G(x_n, x_{n+1}, x_{n+1}) \leq \alpha G(x_{n-1}, x_n, x_n) + \beta \{G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1})\} \text{ then}$$

$$(1 - 2\beta)G(x_n, x_{n+1}, x_{n+1}) \leq (\alpha + \beta)G(x_{n-1}, x_n, x_n)$$

therefore,

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{\alpha + \beta}{1 - 2\beta} G(x_{n-1}, x_n, x_n).$$

Let $q = \frac{\alpha + \beta}{1 - 2\beta}$, then $0 \leq q < 1$ since $0 \leq \alpha + 3\beta < 1$.

So,

$$G(x_n, x_{n+1}, x_{n+1}) \leq qG(x_{n-1}, x_n, x_n).$$

Continuing in the same argument, we will find

$$G(x_n, x_{n+1}, x_{n+1}) \leq q^n G(x_0, x_1, x_1).$$

For all $n, m \in \mathbf{N}; n < m$, we have by repeated use the rectangle inequality that

$$G(x_n, x_m, x_m) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2})$$

$$+ G(x_{n+2}, x_{n+3}, x_{n+3}) + \dots + G(x_{m-1}, x_m, x_m)$$

$$\leq (q^n + q^{n+1} + \dots + q^{m-1})G(x_0, x_1, x_1) \leq \frac{q^n}{1 - q} G(x_0, x_1, x_1).$$

Then, $\lim G(x_n, x_m, x_m) = 0$, as $n, m \rightarrow \infty$, thus (x_n) is G -Cauchy sequence. Due to the completeness of (X, G) , there exists $u \in X$ such that (x_n) is G -convergent to u in (X, G) .

Suppose that $T(u) \neq u$. Then

$$G(x_n, T(u), T(u)) \leq \alpha G(x_{n-1}, u, u) + \beta \{G(x_{n-1}, x_n, x_n) + 2G(u, T(u), T(u))\},$$

and by taking the limit as $n \rightarrow \infty$, and using the fact that the function G is continuous, we get that $G(u, T(u), T(u)) \leq 2\beta G(u, T(u), T(u))$. This contradiction implies that $u = T(u)$.

To prove uniqueness, suppose that u and v are two fixed points for T . Then

$$G(u, v, v) \leq \alpha G(u, v, v) + \beta \{G(u, T(u), T(u)) + 2\beta G(v, T(v), T(v))\} = 0 + \alpha G(u, v, v), \text{ which is}$$

implies that $u = v$, since $0 < \alpha < 1$.

To show that T is G -continuous at u , let $(y_n) \subseteq X$ be a sequence converges to u in (X, G) , then we deduce that

$$\begin{aligned} G(u, T(y_n), T(y_n)) &\leq \alpha G(u, y_n, y_n) + \beta \{G(u, T(u), T(u)) + 2G(y_n, T(y_n), T(y_n))\} \\ &= \alpha G(u, y_n, y_n) + 2\beta G(y_n, T(y_n), T(y_n)). \end{aligned} \quad (2.12)$$

But, by G -metric axioms we have

$$G(y_n, T(y_n), T(y_n)) \leq G(y_n, u, u) + G(u, T(y_n), T(y_n)),$$

so equation (2.12) implies that

$G(u, T(y_n), T(y_n)) \leq \frac{\alpha}{1-2\beta} G(u, y_n, y_n) + \frac{2\beta}{1-2\beta} G(y_n, u, u)$. Taking the limit as $n \rightarrow \infty$, from which we see that $G(y_n, T(y_n), T(y_n)) \rightarrow 0$, and so by Proposition 3, $T(y_n) \rightarrow u = Tu$. So, T is G -continuous at u . This completes the proof of Theorem (2.2). \square

Corollary 2. Let (X, G) be a complete G -metric spaces, and let $T : X \rightarrow X$ be a mapping satisfying, the following condition for some $m \in \mathbf{N}$

$$G(T^m(x), T^m(y), T^m(z)) \leq \alpha G(x, y, z) + \beta \left\{ \begin{array}{l} G(x, T^m(x), T^m(x)) + \\ G(y, T^m(y), T^m(y)) + \\ G(z, T^m(z), T^m(z)) \end{array} \right\} \quad (2.13)$$

for all $x, y, z \in X$, where $0 \leq \alpha + 3\beta < 1$. Then T has unique fixed point (say u), and T^m is G -continuous at u .

Proof. We use the same argument in Corollary 1. \square

Theorem 2.3. Let (X, G) be complete G -metric space, and let $T : X \rightarrow X$ be a mapping satisfying the condition

$$G(T(x), T(y), T(z)) \leq \alpha G(x, y, z) + \beta \max \left\{ \begin{array}{l} G(x, T(x), T(x)), G(y, T(y), T(y)), \\ G(z, T(z), T(z)) \end{array} \right\} \quad (2.14)$$

for all $x, y, z \in X$, where $0 \leq \alpha + \beta < 1$. Then T has unique fixed point (say u), and T is G -continuous at u .

Proof. Suppose that T satisfies condition (2.14). Then for all $x, y \in X$

$$G(Tx, Ty, Ty) \leq \alpha G(x, y, y) + \beta \max\{G(x, Tx, Tx), G(y, Ty, Ty)\}, \text{ and} \quad (2.15)$$

$$G(Ty, Tx, Tx) \leq \alpha G(y, x, x) + \beta \max\{G(y, Ty, Ty), G(x, Tx, Tx)\}. \quad (2.16)$$

Suppose that (X, G) is symmetric. Then from the definition of metric (X, d_G) and (1.1) we get.

$$d_G(Tx, Ty) \leq \alpha d_G(x, y) + \beta \max\{d_G(x, Tx), d_G(y, Ty)\}, \forall x, y \in X. \quad (2.17)$$

In this line since $0 \leq \alpha + \beta < 1$, then the metric condition (2.17) will be a special case of the Reich condition (1.3). Therefor the existence and uniqueness of the fixed point follows from Theorem (1.1).

However, if (X, G) is not symmetric then

$$d_G(x, y) = G(Tx, Ty, Ty) + G(Ty, Tx, Tx) \leq \alpha[G(x, y, y) + G(y, x, x)] + 2\beta \max\{G(x, Tx, Tx), G(y, Ty, Ty)\}.$$

So, by definition of the metric (X, d_G) and (1.2), we will have

$$d_G(Tx, Ty) \leq \alpha d_G(x, y) + 2\beta \max\{\frac{2}{3}d_G(x, Tx), \frac{2}{3}d_G(y, Ty)\}, \forall x, y \in X.$$

The metric condition gives no information about this map since $\alpha + \frac{4\beta}{3}$ need not be less than 1. But the existence of a fixed point can be proved using properties of a G -metric.

Let $x_0 \in X$, be arbitrary point, and define the sequence (x_n) by $x_n = T^n(x_0)$, then by (2.14) we get.

$$G(x_n, x_{n+1}, x_{n+1}) \leq \alpha G(x_{n-1}, x_n, x_n) + \beta \max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\}. \quad (2.18)$$

We see that there are two cases :

(1) Suppose $\max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\} = G(x_{n-1}, x_n, x_n)$. Then in this case, equation (2.18) implies that

$$G(x_n, x_{n+1}, x_{n+1}) \leq (\alpha + \beta)G(x_{n-1}, x_n, x_n).$$

(2) Suppose $\max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\} = G(x_n, x_{n+1}, x_{n+1})$. Then in this case, equation (2.18) implies that

$$G(x_n, x_{n+1}, x_{n+1}) \leq \alpha G(x_{n-1}, x_n, x_n) + \beta G(x_n, x_{n+1}, x_{n+1}), \text{ therefore}$$

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{\alpha}{1 - \beta} G(x_{n-1}, x_n, x_n),$$

but in this case we have $G(x_{n-1}, x_n, x_n) \leq G(x_n, x_{n+1}, x_{n+1})$, hence

$$G(x_{n-1}, x_n, x_n) \leq G(x_n, x_{n+1}, x_{n+1}) \leq \frac{\alpha}{1 - \beta} G(x_{n-1}, x_n, x_n)$$

which is a contradiction since $\frac{\alpha}{1 - \beta} < 1$.

Then, it must be the case (1) is true, which says that

$$G(x_n, x_{n+1}, x_{n+1}) \leq (\alpha + \beta)G(x_{n-1}, x_n, x_n).$$

Let $q = \alpha + \beta$, then $0 \leq q < 1$ since $0 \leq \alpha + \beta < 1$, therefor

$$G(x_n, x_{n+1}, x_{n+1}) \leq q G(x_{n-1}, x_n, x_n).$$

Continuing in the same argument, we will find

$$G(x_n, x_{n+1}, x_{n+1}) \leq q^n G(x_0, x_1, x_1).$$

For all $n, m \in \mathbf{N}$; $n < m$ we have by repeated use the rectangle inequality that $G(x_n, x_m, x_m) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+3}, x_{n+3}) + \dots + G(x_{m-1}, x_m, x_m) \leq (q^n + q^{n+1} + \dots + q^{m-1})G(x_0, x_1, x_1) \leq \frac{q^n}{1-q}G(x_0, x_1, x_1)$.

This proved that, $\lim G(x_n, x_m, x_m) = 0$, as $n, m \rightarrow \infty$, thus (x_n) is G -Cauchy sequence. Due to the completeness of (X, G) , there exists $u \in X$ such that (x_n) is G -convergent to u in (X, G) .

Suppose that $T(u) \neq u$. Then by condition (2.14), we have

$$G(x_n, T(u), T(u)) \leq \alpha G(x_{n-1}, u, u) + \beta \max \{G(x_{n-1}, x_n, x_n), G(u, T(u), T(u))\}.$$

Taking the limit as $n \rightarrow \infty$, and using the fact that the function G is continuous, we get $G(u, T(u), T(u)) \leq \beta G(u, T(u), T(u))$, this contradiction implies that $u = T(u)$.

To prove uniqueness, suppose that u and v are two fixed points for T . Then by (2.14) we have

$G(u, v, v) \leq \alpha G(u, v, v) + \beta \max \{G(u, T(u), T(u)), G(v, T(v), T(v))\} = \alpha G(u, v, v)$, since $\alpha < 1$ this implies that $u = v$.

To show that T is G -continuous at u , let $(y_n) \subseteq X$ be a sequence converging to u in (X, G) , then

$$G(u, T(y_n), T(y_n)) \leq \alpha G(u, y_n, y_n) + \beta \max \{G(u, T(u), T(u)), G(y_n, T(y_n), T(y_n))\}, \text{ hence} \\ G(u, T(y_n), T(y_n)) \leq \alpha G(u, y_n, y_n) + \beta G(y_n, T(y_n), T(y_n)) \quad (2.19)$$

But, by G -metric axioms we have

$$G(y_n, T(y_n), T(y_n)) \leq G(y_n, u, u) + G(u, T(y_n), T(y_n)).$$

Thus equation (2.19) implies that,

$$G(u, T(y_n), T(y_n)) \leq \frac{\alpha}{1-\beta} G(u, y_n, y_n) + \frac{\beta}{1-\beta} G(y_n, u, u).$$

Taking the limit as $n \rightarrow \infty$, from which we see that

$G(u, T(y_n), T(y_n)) \rightarrow 0$ and so, by Proposition (3), we have $T(y_n) \rightarrow u = Tu$ which implies that T is G -continuous at u . This completes the proof of Theorem (2.3). \square

Corollary 3. Let (X, G) be a complete G -metric spaces, and let $T : X \rightarrow X$ be a mapping satisfying the following condition for some $m \in \mathbf{N}$

$$G(T^m(x), T^m(y), T^m(z)) \leq \alpha G(x, y, z) + \beta \max \left\{ \begin{array}{l} G(x, T^m(x), T^m(x)), \\ G(y, T^m(y), T^m(y)), \\ G(z, T^m(z), T^m(z)) \end{array} \right\} \quad (2.20)$$

for all $x, y, z \in X$, where $0 \leq \alpha + \beta < 1$. Then T has unique fixed point (say u), and T^m is G -continuous at u .

Proof. We use the same argument in Corollary 1. □

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