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An Improved Convergence and Complexity Analysis for the Interpolatory Newton Method

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ABSTRACT

We provide an improved compared to [5]–[7] local convergence analysis and complexity for the interpolatory Newton method for solving equations in a Banach space setting. The results are obtained using more precise error bounds than before [5]–[7] and the same hypotheses/computational cost.

RESUMEN

Nosotros entregamos aquí un análisis de convergencia local y complejidad para el método de interpolación de Newton para resolver ecuaciones en espacios de Banach. Los resultados mejoran los de [5]–[7] e son obtenidos usando mas precisas cotas de error y las mismas hipotesis y costo computacional.

Key words and phrases: *Newton's method, local convergence, Banach space, interpolatory Newton method, complexity, radius of convergence.*

Math. Subj. Class.: *65G99, 65H10, 65B05, 47H17, 49M15.*

1 Introduction

In this study we are concerned with the problem of approximating a simple solution α of the equation

$$F(x) = 0, \tag{1.1}$$

where F is an operator defined on a convex subset D of a Banach space X with values in a Banach space Y over the real or complex fields of dimension N,

$$\dim(X) = \dim(Y) = N, \quad 1 \leq N \leq +\infty.$$

We consider interpolatory iteration I_n for approximating x^* defined as follows: Let x_i be an approximation to α and let w_i be the interpolatory polynomial of degree $\leq n-1$ such that

$$w_i^{(j)}(x_i) = F^{(j)}(x_i), \quad j = 0, 1, \ldots, n-1 \ \ (n \geq 2). \tag{1.2}$$

The next approximation x_{i+1}^* is a zero of w_i. For $n = 2$ we obtain Newton's method:

$$x_{i+1}^* = x_i - F'(x_i)^{-1} F(x_i) \quad (i \geq 0). \tag{1.3}$$

We approximate x_{i+1} by applying a number of Newton iterations to $w_i(x) = 0$. Let $\{x_i\}$ be the interpolatory Newton iteration IN_n given by:

$$\begin{aligned} z_0 &= x_i \\ z_{j+1} &= z_j - w_i'(z_j)^{-1} w_i(z_j), \quad j = 0, 1, \ldots, k-1 \\ x_{i+1} &= z_k, \quad k = [\log 2n]. \end{aligned} \tag{1.4}$$

A local convergence analysis and the corresponding complexity of method (1.4) was studied in the elegant paper by Traub and Wozniakowski [7]. Relevant works can be found in [1]–[7], and the references there.

Here we are motivated by paper [7] and optimization considerations. In particular using more precise estimates on the distances $\|x_i - \alpha\|$ $(i \geq 0)$ we show that under the same hypotheses and computational cost as in [5]–[7], we can provide a larger convergence radius, sharper error bounds on the distances and consequently a finer complexity for method (1.4).

Numerical examples are introduced which compare favorably with results to the corresponding ones in [5]–[7].

2 Local Convergence Analysis of Method (1.4)

Let $\Gamma \geq 0$. We introduce the closed ball $U = U(\alpha, \Gamma) = \{x \in X \mid \|x - \alpha\| \leq \Gamma\}$, and the parameters

$$A_j = A_j(\Gamma) = \sup_{x \in U} \left\| F'(\alpha)^{-1} \frac{F^{(j)}(x)}{j!} \right\|, \quad (j \geq 2) \tag{2.1}$$

provided that $F^{(j)}$ exists.

Moreover we introduce the parameter A by

$$A = A(\Gamma) = \sup_{x \in U} \frac{\|F'(\alpha)^{-1}[F'(x) - F'(\alpha)]\|}{2\|x - \alpha\|}\,. \tag{2.2}$$

The foundation of our approach and what makes it more precise than the corresponding one in [7] is the fact that we use (2.2) instead of (2.1) (for $j = 2$) to obtain upper bounds on the crucial quantity $\|w_j'(x)^{-1}F'(\alpha)\|$.

Indeed, on the one hand note that

$$A \leq A_2 \tag{2.3}$$

holds in general and $\frac{A_2}{A}$ can be arbitrarily large [1], [2]. On the other hand see (2.28), (2.46), and Remark 2.4.

Let us set

$$a = \frac{A}{A_2}\,, \quad A_2 \neq 0. \tag{2.4}$$

Note that $a \in [0, 1]$.

We showed in [3] the following improvement of Theorem 2.1 in [6] and Theorem 2.1 in [5] respectively:

Theorem 2.1. *If F is twice differentiable in U, (2.2) holds and*

$$A_2\Gamma \leq \frac{1}{2(1+a)}\,, \tag{2.5}$$

$$x_i \in U, \tag{2.6}$$

then the next approximation x_{i+1}^* generated by Newton method (1.3) is well defined, and satisfies for all $i \geq 0$:

$$\|x_{i+1}^* - \alpha\| \leq \frac{A_2}{1 - 2aA_2}\|x_i - \alpha\|^2 \leq \frac{1}{2}\|x_i - \alpha\| \tag{2.7}$$

and

$$x_{i+1}^* - \alpha = \frac{1}{2}F'(\alpha)^{-1}F'(\alpha)(x_i - \alpha)^2 + O(\|x_i - \alpha\|^2). \tag{2.8}$$

Theorem 2.2. *If F is n-times differentiable, $n \geq 3$ in U, (2.2) holds, and*

$$\frac{nA_n\Gamma^{n-1}}{1 - aA_2\Gamma} < \left(\frac{2}{3}\right)^{n-1} \tag{2.9}$$

$$x_i \in U,$$

then the polynomial w_i has a unique zero in $U^* = U^*\left(\alpha, \frac{\Gamma}{2}\right)$ and defining x_{i+1}^* as the zero of w_i in U^* the following estimates hold for all $i \geq 0$

$$\|x_{i+1}^* - \alpha\| \leq \frac{A_n(1 + \|x_{i+1}^* - \alpha\|/\|x_i - \alpha\|)^n}{1 - aA_2\|x_{i+1}^* - \alpha\|}\|x_i - \alpha\|^n \leq \frac{1}{2}\|x_i - \alpha\|, \tag{2.10}$$

and

$$x_{i+1}^* - \alpha = \frac{(-1)^n}{n!} F'(\alpha)^{-1} F^{(n)}(\alpha)(x_i - \alpha)^n + O(\|x_i - \alpha\|^n). \tag{2.11}$$

We can show the main local convergence theorem for method (1.4):

Theorem 2.3. *If F is n-times differentiable, $n \geq 3$ in U, (2.2) holds, and*

$$0 \leq \tilde{A}_2 \Gamma \leq \frac{1}{3 + 2a} \tag{2.12}$$

where,

$$\tilde{A}_2 = \frac{A_2 + \frac{n(n-1)}{2} A_n (2\Gamma)^{n-2}}{1 - aA_2\Gamma - nA_n \left(\frac{3}{2}\right)^{n-1} \Gamma^{n-1}} \tag{2.13}$$

$$x_0 \in U, \tag{2.14}$$

then sequence $\{x_i\}$ $(i \geq 0)$ generated by interpolary-Newton iteration IN_n is well defined, remains in U for all $i \geq 0$, converges to α so that the following estimates hold for all $i \geq 0$:

$$e_{i+1} = \|x_{i+1} - \alpha\| \leq \left\{ \frac{1}{2} + \frac{3}{2} \left(\frac{1}{2} \right)^k \right\} e_i, \tag{2.15}$$

$$e_{i+1} \leq c_{i,n} e_i^n \tag{2.16}$$

where,

$$\begin{aligned} c_{i,n} = \left(1 + \frac{e_{i+1}^*}{e_i}\right) \Bigg[& \frac{A_n}{1 - aA_2 e_{i+1}^*} \\ & + (\tilde{A}_2(1 + H_i))^{2^k - 1} \Bigg] \left(\left(1 + \frac{e_{i+1}^*}{e_i}\right) e_i \right)^{2^k - n}, \end{aligned} \tag{2.17}$$

for

$$e_{i+1}^* = \|x_{i+1}^* - \alpha\|, \ H_i = O(e_i), \ 0 \leq H_i \leq \frac{3 + 2a}{2}, \ k = [\log 2n], \tag{2.18}$$

$$\begin{aligned} \lim_{i \to \infty} c_{i,n} = A_n + \delta \tilde{A}_2^{n-1} \quad & \text{where } \delta = 0 \\ & \text{if } 2^k > n \text{ and } \delta = 1, \text{ if } 2^k = n, \end{aligned} \tag{2.19}$$

$$x_{i+1} - \alpha = F_n (x_i - \alpha)^n + b_{i,k} + O(\|x_i - \alpha\|^n), \tag{2.20}$$

where

$$b_{i,1} = F_2 (x_i - \alpha)^2, \tag{2.21}$$

$$b_{i,j+1} = F_2 b_{i,j}^2, \quad j = 1, 2, \ldots, k - 1, \tag{2.22}$$

and

$$F_j = \frac{(-1)^j}{j!} F'(\alpha)^{-1} F^{(j)}(\alpha) \quad \text{for } j = 2 \text{ and } n. \tag{2.23}$$

The proof is similar to Theorem 3.1 in [7], but there are differences where we use (2.2) instead of (2.1) (for $i=2$).

Proof. We shall first show using induction on $j \geq 0$ that $w_j'(z_j)$ is invertible and $z_j \in U$.

Set

$$F^{(j)}(x) - w_i^{(j)}(x) = R_n^{(j)}(x; x_i), \quad x \in U, \ j = 0,1,2, \tag{2.24}$$

where,

$$\|F'(\alpha)^{-1} R_n^{(j)}(x; x_i)\| \leq j! \binom{n}{j} A_n \|x - x_i\|^{n-1}. \tag{2.25}$$

We can write

$$\begin{aligned} w_i'(x) &= F'(x) - R_n'(x; x_i) \\ &= F'(\alpha)[I + F'(\alpha)^{-1}\{F'(x) - F'(\alpha)\} - F'(\alpha)^{-1} R_n'(x; x_i)] \end{aligned} \tag{2.26}$$

and in view of (2.2), (2.12) and (2.24) for $x \in U$ we get in turn

$$\|F'(\alpha)^{-1}[w_j'(x) - F'(\alpha)]\| \leq 2aA_2\|x - \alpha\| + nA_n\|x - x_i\|^{n-1} \tag{2.27}$$

$$\leq 2aA_2\Gamma + nA_n(2\Gamma)^{n-1} \leq \frac{2}{3+2a} < 1. \tag{2.28}$$

It follows from (2.28) and the Banach Lemma on invertible operators [4] that $w_i'(x)$ is invertible for all $x \in U$, and

$$\|w_i'(x)^{-1} F'(\alpha)\| \leq \frac{1}{1 - 2aA_2\|x - \alpha\| - nA_n\|x - x_i\|^{n-1}}\,. \tag{2.29}$$

Since the denominator in (2.13) is positive we get

$$\frac{nA_n\Gamma^{n-1}}{1 - aA_2\Gamma} < \left(\frac{2}{3}\right)^{n-1} \tag{2.30}$$

and from Theorem 2.2 w_i has a unique zero x_{i+1}^* in U^* and (2.10) holds.

Using (2.24) and (2.29) we get for $x \in U$

$$\begin{aligned} &\left\| w_j'(x_{i+1}^*)^{-1} \frac{w_i''(x)}{2} \right\| \\ &\leq \ \|w_i'(x_{i+1}^*)^{-1} F'(\alpha)\| \left\| F'(\alpha)^{-1} \frac{w_i''(x)}{2} \right\| \\ &\leq \ \frac{A_2 + \frac{n(n-1)}{2} A_n \|x - x_i\|^{n-2}}{1 - 2aA_2\|x_{i+1}^* - \alpha\| - nA_n\|x_{i+1}^* - x_i\|^{n-1}} \\ &\leq \ \frac{A_2 + \frac{n(n-1)}{2} A_n (2\Gamma)^{n-2}}{1 - aA_2\Gamma - nA_n\left(\frac{3}{2}\Gamma\right)^{n-1}} = \tilde{A}_2. \end{aligned} \tag{2.31}$$

It follows from Theorem 3.1 and (2.12) that for $z_1 = x_i - F'(x_i)^{-1} F(x_i)$

$$\|z_1 - \alpha\| \leq \frac{1}{2}\|x_i - \alpha\|. \tag{2.32}$$

Since $x_{i+1}^* \in U^*$, $\|z_1 - x_{i+1}^*\| \le \Gamma$, we shall show

$$z_{j+1} \in D_j = \left\{ x \colon \|x - x_{i+1}^*\| \le \frac{1}{2}\|z_j - x_{i+1}^*\| \right\} \cap U. \tag{2.33}$$

Set

$$w_i(x) = w_i(z_j) + w_i'(z_j)(x - z_j) + \overline{R}_2(x; z_j), \tag{2.34}$$

where,

$$\overline{R}_2(x; y) = \int_0^1 w_i''(y + t(x - y))(x - y)^2(1 - t)dt. \tag{2.35}$$

Note that z_{j+1} is the solution of equation

$$x = H(x) = x_{i+1}^* + w'(x_{i+1})^{-1}\big\{\overline{R}_2(x; z_j) - \overline{R}_2(x; x_{i+1}^*)\big\}. \tag{2.36}$$

We shall show H is contractive on D_j.

It follows from (2.12), (2.31) and (2.36):

$$\begin{aligned} \|H(x) - x_{i+1}^*\| &\le \tilde{A}_2(\|x - z_j\|^2 + \|x - x_{i+1}^*\|^2) \\ &\le \frac{2 + 3a}{2}\tilde{A}_2\|z_j - x_{i+1}^*\| \le \frac{1}{2}\|z_j - x_{i+1}^*\|. \end{aligned} \tag{2.37}$$

Moreover we have

$$\|H(x) - \alpha\| \le \|x_{i+1}^* - \alpha\| + \|H(x) - x_{i+1}^*\| \le \left(\frac{1}{2} + \frac{1}{2}\right)\Gamma = \Gamma. \tag{2.38}$$

It follows by the contraction mapping principle [4], (2.37) and (2.38) that z_{j+1} is the unique zero of H in D_j. It follows that $x_{i+1} = z_k \in U$, and

$$\begin{aligned} \|x_{i+1} - \alpha\| &\le \|x_{i+1} - x_{i+1}^*\| + \|x_{i+1}^* - \alpha\| \\ &\le \left(\frac{1}{2}\right)^k \|z_0 - x_{i+1}^*\| + \frac{1}{2}\|x_i - \alpha\| \\ &\le \left[\frac{3}{2}\left(\frac{1}{2}\right)^k + \frac{1}{2}\right]\|x_i - \alpha\| \le \frac{7}{8}\|x_i - \alpha\|, \end{aligned} \tag{2.39}$$

which shows $x_i \in U$ and (2.15) hold true.

Set $\overline{e}_j = \|z_j - x_{i+1}^*\|$ and $x = z_{j+1}$ in (2.36). Then we get

$$\overline{e}_{j+1} \le \frac{\tilde{A}_2\big(1 + \frac{\overline{e}_{j+1}}{\overline{e}_j}\big)^2}{1 - \tilde{A}_2\overline{e}_{j+1}}\overline{e}_j^2 \le \tilde{A}_2(1 + H_i)\overline{e}_j, \tag{2.40}$$

where $H_i = O(\overline{e}_j)$ and $0 \le H_i \le \frac{2+3a}{2}$ compare to (2.7). In view of $\overline{e}_j = O(e_j)$ we can set

$H_i = O(e_i)$. It follows from (2.10) and (2.40)

$$\begin{aligned} e_{i+1} &\le \|x_{i+1} - x^*_{i+1}\| + \|x^*_{i+1} - \alpha\| = \overline{e}_k + \|x^*_{i+1} - \alpha\| \\ &\le \left[\tilde{A}_2(1+H_i)\right]^{2^k-1} \|x_i - x^*_{i+1}\|^{2^k} \\ &\quad + \frac{A_n}{1 - aA_2 e^*_{i+1}} \left(1 + \frac{e^*_{i+1}}{e_i}\right)^n e_i^n \\ &\le \left(1 + \frac{e^*_{i+1}}{e_i}\right)^n \left(\frac{A_n}{1 - aA_2 e^*_{i+1}}\right. \\ &\quad \left. + \left[\tilde{A}_2(1+H_i)\right]^{2^k-1} \left[\left(1 + \frac{e^*_{j+1}}{e_i}\right) e_i\right]^{2^k-n}\right) e_i^n = c_{i,n} e_i^n. \end{aligned} \tag{2.41}$$

In view of $\frac{e^*_{i+1}}{e_i}$ and H_i tending to zero we get

$$\lim_{i\to\infty} c_{i,n} = A_n + \delta \tilde{A}_2^{n-1}, \tag{2.42}$$

where $\delta = 0$ if $2^k > n$ and $\delta = 1$ otherwise. Hence, (2.16) holds.

Furthermore, we have

$$\begin{aligned} z_{j+1} - x^*_{i+1} &= w_i'(x^*_{i+1})^{-1} \frac{w_i''(x^*_{i+1})}{2} (z_j - x^*_{i+1})^2 + O(\tilde{e}_j^3) \\ &= F'(\alpha)^{-1} \frac{F''(\alpha)}{2} (z_j - x^*_{i+1})^2 + O(e^*_{i+1}\tilde{e}_j^2 + \tilde{e}_j^3) \\ &= F_2 (z_j - x^*_{i+1})^2 + O(\tilde{e}_j^2). \end{aligned} \tag{2.43}$$

Therefore, we get

$$\begin{aligned} z_k - x^*_{i+1} &= F_2\big(F_2 \cdots (F_2(x_i - x^*_{i+1})^2)^2 \cdots\big)^2 + O(e_i^{2k}) \\ &= F_2\big(F_2 \cdots (F_2(x_i - \alpha)^2)^2 \cdots\big)^2 + O(e_i^{2k}). \end{aligned} \tag{2.44}$$

In view of (2.21), (2.22), and (2.44) we have

$$z_k - x^*_{i+1} = b_{i,k} + O(e_i^{2k}). \tag{2.45}$$

In view of (2.11) and (2.45) we deduce

$$x_{i+1} - \alpha = z_k - x^*_{i+1} + x^*_{i+1} - \alpha = b_{i,k} + F_n(x_i - \alpha)^n + O(e_i^n), \tag{2.46}$$

which shows (2.20).

That completes the proof of the theorem. ■

Remark 2.4. The less precise estimate (using (2.1) for $j = 2$ instead of sharper (2.2) that is actually needed)

$$\|F'(\alpha)^{-1}[w_j'(x) - F'(\alpha)]\| \le 2A_2\|x - \alpha\| + nA_n\|x - x_i\|^{n-1} \tag{2.47}$$

was used in [7] instead of (2.28), together with

$$0 \leq \tilde{A}_2 \Gamma \leq \frac{1}{5} \tag{2.48}$$

instead of weaker (2.12).

If $A = A_2$ our results Theorem 2.1, Theorem 2.2 and Theorem 2.3 reduce to the corresponding Theorem 2.1 in [6], Theorem 2.1 in [5] and Theorem 3.1 in [7] respectively. Otherwise our results constitute improvements with advantages already stated in the Introduction.

We now give conditions under which IN_n enjoys a "type of global convergence".

Let

$$F(x) = \sum_{i=1}^{\infty} \frac{1}{\iota!} F^{(i)} (x_i - \alpha)^i \tag{2.49}$$

be analytic in $D = U^0(\alpha, R)$, and

$$\frac{\|F'(\alpha)^{-1} F^{(i)}(\alpha)\|}{\iota!} \leq K^{i-1} \tag{2.50}$$

for $i \geq 2$ and $R \geq \frac{1}{K}$.

As in [7], one way to find K is to use Cauchy's formula

$$\frac{\|F'(\alpha)^{-1} F^{(i)}(\alpha)\|}{\iota!} \leq \frac{M}{R^i}\,, \tag{2.51}$$

where,

$$M = \sup_{x \in D} \|F'(\alpha)^{-1} F(x)\|. \tag{2.52}$$

Let $K = \max\left[\frac{1}{R}, \frac{M}{R^2}\right]$. Then

$$\frac{M}{R} \leq KR \leq (KR)^{i-1} \tag{2.53}$$

and

$$\frac{M}{R^i} \leq K^{i-1}. \tag{2.54}$$

We can show:

Theorem 2.5. *If (2.2) and (2.50) hold then the interpolary Newton method (1.4) converges provided that $x_0 \in U(\alpha, \Gamma_n)$, where*

$$\Gamma_n = \frac{x_n}{K} \tag{2.55}$$

and x_n, $0 < x_n < x_\infty$, satisfies the equation

$$(3+2a)\left[\frac{x}{(1-x)^3} + \frac{n(n-1)}{4(1-x)^2}\left(\frac{2x}{1-x}\right)^{n-1}\right] = 1 - \frac{ax}{(1-x)^3} - \frac{n}{(1-x)^2}\left[\frac{3x}{2(1-x)}\right]^{n-1} \tag{2.56}$$

and $x_n \to x_\infty$*, where*

$$x_\infty \geq .12 \tag{2.57}$$

is the positive solution of equation

$$\frac{x}{(1-x)^3} = \frac{1}{4+2a}\,. \tag{2.58}$$

Proof. In view of (2.50) we have for

$$f(x) = \frac{x}{1-Kx}\,, \tag{2.59}$$

that

$$\|F'(\alpha)^{-1}F^{(i)}(x)\| \leq f^{(i)}(\|x-\alpha\|). \tag{2.60}$$

Using

$$f^{(i)}(x) = \frac{i!K^{i-1}}{(1-Kx)^{i+1}} \quad (i \geq 2), \tag{2.61}$$

we get

$$A_i(\Gamma) \leq \frac{K^{i-1}}{(1-K\Gamma)^{i+1}} \quad (i \geq 2). \tag{2.62}$$

It follows from (2.13) and (2.62) that

$$\tilde{A}_2\Gamma \leq \frac{\left[\frac{K\Gamma}{(1-K\Gamma)^3} + \frac{n(n-1)}{4(1-K\Gamma)^2}\left(\frac{2K\Gamma}{1-K\Gamma}\right)^{n-1}\right]}{1 - \frac{aK\Gamma}{(1-K\Gamma)^3} - \frac{n}{(1-K\Gamma)^2}\left(\frac{3K\Gamma}{2(1-K\Gamma)}\right)^{n-1}} = \frac{1}{3+2a}\,. \tag{2.63}$$

Letting $K\Gamma = x$ we see that x satisfies equation (2.56). It is simple calculus to show that $x = x(n)$ is an increasing function of n and $x_\infty = \lim\limits_{n\to\infty} x(n)$ satisfies equaiton (2.58).

Remark 2.6. If $A = A_2$ (i.e. $a = 1$) our Theorem 2.5 reduces to Theorem 3.2 in [7]. Otherwise it is an improvement, since the limit of sequence $x(n)$ in [7] is .12 which is smaller than ours implying by (2.55) that we provide a larger radius of convergence.

In particular if R is related to $\frac{1}{K}$, say $R = \frac{c_1}{K}$, then

$$\Gamma_n = \frac{x_n}{K} = \frac{x_n}{c_1 R} \leq \frac{x_\infty}{c_1 R}\,. \tag{2.64}$$

The rest of the results introduced in [7] are improved. In particular with the notation introduced in [7] we have for

I: $e_i = G_i e_{i-1}^n$, $G_i \leq \overline{G}$,

$$\overline{G} = \overline{G}(n) = \begin{cases} \dfrac{A_2}{1-2aA_2\Gamma}, & n = 2 \\[2ex] (1+q)^n\left[\dfrac{A_n}{1-aA_2\frac{\Gamma}{2}} + \left(\dfrac{7}{2}\tilde{A}_2\right)^{2^k-1}\right][(1+q)\Gamma]^{2^k-n}, & n > 2 \end{cases} \tag{2.65}$$

where $\tilde{A}_2$ is given by (2.13), $q = \frac{1}{2} + \frac{3}{2}\left(\frac{1}{2}\right)^k$, and $K = [\log 2n]$.

II: If the total number of arithmetic operations necessary to solve a system of N linear equations is $O(n^\beta)$, $\beta \leq 3$, then

$$d(IN_n) = \begin{cases} O\left(N^\beta[\log 2n] + N^2\binom{N+n-2}{n-2}([\log 2n] - 1\right) & \text{for } N \geq 2, \\ \\ (3+2a)[\log 2n] + O(1), & \text{for } n = 1. \end{cases} \tag{2.66}$$

Remark 2.7. If $A = A_2$ our results reduce to the ones in [7]. Otherwise they constitute an improvement.

We complete this study with an example to show that strict inequality can hold in (2.3):

Example 2.8. Let $X = Y = \mathbf{R}$, $x^* = 0$ and define function F on $U = U(0,1)$ by

$$F(x) = e^x - 1. \tag{2.67}$$

Using (2.1), (2.2), (2.4) and (2.66) we obtain

$$A = \frac{e-1}{2} < \frac{e}{2} = A_2 \tag{2.68}$$

and

$$a = .632120588. \tag{2.69}$$

It follows from (2.5) that our radius of convergence is given by

$$\Gamma_A = .112699836. \tag{2.70}$$

The corresponding radius Γ_{TW} given in Theorem 2.1 in [6] or [7] is:

$$\Gamma_{TW} = \frac{1}{4A_2} = .09196986. \tag{2.71}$$

That is

$$\Gamma_{TW} < \Gamma_A. \tag{2.72}$$

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