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A Strong Convergence Theorem by a New Hybrid Method for an Equilibrium Problem with Nonlinear Mappings in a Hilbert Space

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ABSTRACT

In this paper, we prove a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem, the set of solutions of the variational inequality for a monotone mapping and the set of fixed points of a nonexpansive mapping in a Hilbert space by using a new hybrid method. Using this theorem, we obtain three new results for finding a solution of an equilibrium problem, a solution of the variational inequality for a monotone mapping and a fixed point of a nonexpansive mapping in a Hilbert space.

RESUMEN

En este artículo, probamos un teorema de convergencia fuerte para encontrar un elemento común del conjunto de soluciones de un problema de equilibrio; del conjunto de soluciones de una desigualdad variacional para una aplicación monótona y del conjunto de punto fijos de una aplicación no expansiva en un espacio de Hilbert mediante el uso de un nuevo método híbrido. Usando nuestro teorema obtenemos tres nuevos resultados para encontrar una solución de un problema de equilíbrio; una solución de la desigualdad variacional para una aplicación monótona y un punto fijo para una aplicación no expansiva en un espacio de Hilbert.

Key words and phrases: *Hilbert space, equilibrium problem, nonexpansive mapping, inverse-strongly monotone mapping, iteration, strong convergence theorem.*

Math. Subj. Class.: *47H05, 47H09, 47J25.*

1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let C be a nonempty closed convex subset of H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $f : C \times C \to \mathbb{R}$ is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0 \tag{1.1}$$

for all $y \in C$. The set of such solutions $\hat{x}$ is denoted by $EP(f)$. The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncoopetative games and others; see, for instance, [1] and [6]. A mapping S of C into H is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|$$

for all $x, y \in C$. We denote by $F(S)$ the set of fixed points of S. A mapping $A : C \to H$ is called inverse-strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2$$

for all $x, y \in C$. The variational inequality problem is to find a $u \in C$ such that

$$\langle v - u, Au \rangle \geq 0 \tag{1.2}$$

for all $v \in C$. The set of such solutions u is denoted by $VI(C, A)$. Setting $A = I - S$, where $S : C \to H$ is nonexpansive, we have from [14] that $A : C \to H$ is a $\frac{1}{2}$-inverse-strongly monotone mapping. Recently, Tada and Takahashi [9, 10] and Takahashi and Takahashi [11] obtained weak and strong convergence theorems for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. In particular, Tada and Takahashi [10] established a strong convergence theorem for finding a common element of such two sets by using the hybrid method introduced in Nakajo and Takahashi [7]. On the other hand, Takahashi and Toyoda [16] introduced an iterative method for finding a common element of the set of solutions of the variational inequality for an inverse-strongly monotone mapping and the set of fixed points of a nonexpansive mapping. Very recently, Takahashi, Takeuchi and Kubota [15] proved the following theorem by a new hybrid method which is different from Nakajo and Takahashi's hybrid method. We call such a method the shrinking projection method.

Theorem 1.1 (Takahashi, Takeuchi and Kubota [15]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into H such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases}$$

where $0 \leq \alpha_n \leq a < 1$. Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$, where $P_{F(T)}$ is the metric projection of H onto $F(T)$.

In this paper, motivated by Tada and Takahashi [10], Takahashi and Toyoda [16], and Takahashi, Takeuchi and Kubota [15], we prove a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem, the set of solutions of the variational inequality for an inverse-strongly monotone mapping and the set of fixed points of a nonexpansive mapping in a Hilbert space by using the shrinking projection method. Using this theorem, we obtain three new results for finding a solution of an equilibrium problem, a solution of the variational inequality for an inverse-strongly monotone mapping and a fixed point of a nonexpansive mapping in a Hilbert space.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. We denote by "$\to$" strong convergence and by "$\rightharpoonup$" weak convergence. We know from [14] that, for all $x, y \in H$ and $\lambda \in [0,1]$, there holds

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2.$$

Let C be a nonempty closed convex subset of H. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|$$

for all $y \in C$. P_C is called the metric projection of H onto C. We know that P_C satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle \tag{2.1}$$

for all $x, y \in H$. Further, we have that

$$\langle x - P_C x, P_C x - y\rangle \geq 0 \tag{2.2}$$

for all $x \in H$ and $y \in C$. A mapping $A : C \to H$ is called inverse-strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha \|Ax - Ay\|^2$$

for all $x, y \in C$. The set of solutions of the variational inequality for A is denoted by $VI(C, A)$. We know that, for all $\lambda > 0$,

$$u \in VI(C, A) \iff u = P_C(u - \lambda Au).$$

We also know that, for any λ with $0 < \lambda \leq 2\alpha$, a mapping $I - \lambda A : C \to H$ is nonexpansive; see [16, 14] for more details. It is also known that H satisfies Opial's condition, i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. A Hilbert space H also has the Kadec-Klee property, i.e., if $\{x_n\}$ is a sequence of H with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then there holds $x_n \to x$.

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be an inverse-strongly monotone mapping of C into H and let $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \ \forall u \in C\}$, and define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$; see [8].

For solving an equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$, let us assume that f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e. $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for all $x, y, z \in C$,

$$\limsup_{t\downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y);$$

(A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

The following lemma appears implicitly in Blum and Oettlli [1].

Lemma 2.1 (Blum and Oettli)**.** *Let C be a nonempty closed convex subset of H and let f be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$. Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \ \textit{for all} \ y \in C.$$

The following lemma was also given in [2].

Lemma 2.2. *Assume that $f : C \times C \to \mathbb{R}$ satisfies $(A1) - (A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \text{ for all } y \in C \right\}$$

for all $x \in H$. Then, the following hold:

(1) T_r is single-valued;

(2) T_r is a firmly nonexpansive mapping, i.e., for all $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) $F(T_r) = EP(f)$;

(4) $EP(f)$ is closed and convex.

3 Strong convergence theorem

In this section, using the shrinking projection method, we prove a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem, the set of solutions of the variational inequality for an inverse-strongly monotone mapping and the set of fixed points of a nonexpansive mapping in a Hilbert space.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$ and let S be a nonexpansive mapping from C into H and let A be an α-inverse-strongly monotone mapping of C into H such that $F(S) \cap VI(C, A) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by $x_0 = x \in C$, $C_0 = C$ and*

$$\begin{cases} u_n = T_{r_n}(x_n), \\ y_n = \alpha_n x_n + (1 - \alpha_n) S P_C(u_n - \lambda_n A u_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x, \quad n \in \mathbb{N} \cup \{0\}, \end{cases}$$

where $0 \leq \alpha_n \leq c < 1$, $\quad 0 < d \leq r_n < \infty$ and $0 < a \leq \lambda_n \leq b < 2\alpha$. Then, $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(C,A)\cap EP(f)} x$.

Proof. From [7], we know that

$$\begin{aligned} &\|y_n - z\| \leq \|x_n - z\| \\ \iff &\|y_n - x_n\|^2 + 2\langle y_n - x_n, x_n - z\rangle \leq 0. \end{aligned}$$

So, C_n is a closed convex subset of H for all $n \in \mathbb{N} \cup \{0\}$. Next we show by mathematical induction that $F(S) \cap VI(C, A) \cap EP(f) \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$. Put $z_n = P_C(u_n - \lambda_n A u_n)$ for all $n \in \mathbb{N} \cup \{0\}$. From $C_0 = C$, we have

$$F(S) \cap VI(C, A) \cap EP(f) \subset C_0.$$

Suppose that $F(S) \cap VI(C, A) \cap EP(f) \subset C_k$ for some $k \in \mathbb{N} \cup \{0\}$. Let $u \in F(S) \cap VI(C, A) \cap EP(f)$. Since $I - \lambda_k A$ and T_{r_k} are nonexpansive and $u = P_C(u - \lambda_k A u)$, we have

$$\begin{aligned}
\|z_k - u\| &= \|P_C(u_k - \lambda_k A u_k) - P_C(u - \lambda_k A u)\| \\
&\leq \|(I - \lambda_k A)u_k - (I - \lambda_k A)u\| \\
&\leq \|u_k - u\| \\
&= \|T_{r_k} x_k - T_{r_k} u\| \\
&\leq \|x_k - u\|.
\end{aligned}$$

So, we have

$$\begin{aligned}
\|y_k - u\| &= \|\alpha_k x_k + (1 - \alpha_k) S z_k - u\| \\
&\leq \alpha_k \|x_k - u\| + (1 - \alpha_k)\|S z_k - u\| \\
&\leq \alpha_k \|x_k - u\| + (1 - \alpha_k)\|z_k - u\| \\
&\leq \alpha_k \|x_k - u\| + (1 - \alpha_k)\|x_k - u\| \\
&= \|x_k - u\|.
\end{aligned}$$

Since $u \in C_k$, we have $u \in C_{k+1}$. This implies that

$$F(S) \cap VI(C, A) \cap EP(f) \subset C_n$$

for all $n \in \mathbb{N} \cup \{0\}$. So, $\{x_n\}$ is well-defined.

From the definition of x_{n+1}, we have

$$\|x_{n+1} - x\| \leq \|u - x\|$$

for all $u \in F(S) \cap VI(C, A) \cap EP(f) \subset C_{n+1}$. Then, $\{x_n\}$ is bounded. Therefore, $\{y_n\}$, $\{z_n\}$, $\{u_n\}$ and $\{S z_n\}$ are also bounded.

Let us show that $\|x_{n+1} - x_n\| \to 0$. From $x_{n+1} \in C_{n+1} \subset C_n$ and $x_n = P_{C_n} x$, we have

$$\|x_n - x\| \leq \|x_{n+1} - x\|$$

for all $n \in \mathbb{N} \cup \{0\}$. Thus $\{\|x_n - x\|\}$ is nondecreasing. Thus $\lim_{n\to\infty} \|x_n - x\|$ exists. Since

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|x_{n+1} - x\|^2 + \|x_n - x\|^2 + 2\langle x_{n+1} - x, x - x_n \rangle \\
&= \|x_{n+1} - x\|^2 - \|x_n - x\|^2 - 2\langle x_n - x_{n+1}, x - x_n \rangle \\
&\leq \|x_{n+1} - x\|^2 - \|x_n - x\|^2
\end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$, we have $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Since $x_{n+1} \in C_{n+1}$, we have

$$\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \le 2\|x_n - x_{n+1}\|.$$

This together with $\|x_{n+1} - x_n\| \to 0$ implies that

$$\|x_n - y_n\| \to 0.$$

We also show that $\|Au_n - Au\| \to 0$. For all $u \in F(S) \cap VI(C, A) \cap EP(f)$, we have

$$\begin{aligned}
\|z_n - u\|^2 &= \|P_C(u_n - \lambda_n Au_n) - P_C(u - \lambda_n Au)\|^2 \\
&\le \|(u_n - \lambda_n Au_n) - (u - \lambda_n Au)\|^2 \\
&= \|u_n - u - \lambda_n(Au_n - Au)\|^2 \\
&= \|u_n - u\|^2 - 2\lambda_n\langle u_n - u, Au_n - Au\rangle + \lambda_n^2\|Au_n - Au\|^2 \\
&\le \|u_n - u\|^2 - 2\lambda_n\alpha\|Au_n - Au\|^2 + \lambda_n^2\|Au_n - Au\|^2 \\
&= \|u_n - u\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Au_n - Au\|^2 \\
&\le \|u_n - u\|^2 + a(b - 2\alpha)\|Au_n - Au\|^2.
\end{aligned}$$

Since $\|\cdot\|^2$ is convex and $\|u_n - u\| \le \|x_n - u\|$, we have

$$\begin{aligned}
\|y_n - u\|^2 &\le \alpha_n\|x_n - u\|^2 + (1 - \alpha_n)\|Sz_n - u\|^2 \\
&\le \alpha_n\|x_n - u\|^2 + (1 - \alpha_n)\{\|u_n - u\|^2 + a(b - 2\alpha)\|Au_n - Au\|^2\} \\
&\le \|x_n - u\|^2 + a(b - 2\alpha)\|Au_n - Au\|^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
-a(b - 2\alpha)\|Au_n - Au\|^2 &\le \|x_n - u\|^2 - \|y_n - u\|^2 \\
&= (\|x_n - u\| + \|y_n - u\|)(\|x_n - u\| - \|y_n - u\|) \\
&\le (\|x_n - u\| + \|y_n - u\|)\|x_n - y_n\|.
\end{aligned}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded and $\|x_n - y_n\| \to 0$, we obtain $\|Au_n - Au\| \to 0$. Further we show that $\|z_n - u_n\| \to 0$. For all $u \in F(S) \cap VI(C, A) \cap EP(f)$, we have from (2.1) that

$$\begin{aligned}
\|z_n - u\|^2 &= \|P_C(u_n - \lambda_n Au_n) - P_C(u - \lambda_n Au)\|^2 \\
&\le \langle (u_n - \lambda_n Au_n) - (u - \lambda_n Au), z_n - u\rangle \\
&= \frac{1}{2}\{\|(u_n - \lambda_n Au_n) - (u - \lambda_n Au)\|^2 + \|z_n - u\|^2 \\
&\quad - \|(u_n - \lambda_n Au_n) - (u - \lambda_n Au) - (z_n - u)\|^2\} \\
&\le \frac{1}{2}\{\|u_n - u\|^2 + \|z_n - u\|^2 - \|(u_n - z_n) - \lambda_n(Au_n - Au)\|^2\} \\
&= \frac{1}{2}\{\|u_n - u\|^2 + \|z_n - u\|^2 - \|u_n - z_n\|^2 \\
&\quad + 2\lambda_n\langle u_n - z_n, Au_n - Au\rangle - \lambda_n^2\|Au_n - Au\|^2\},
\end{aligned}$$

and hence

$$\|z_n - u\|^2 \leq \|u_n - u\|^2 - \|u_n - z_n\|^2 + 2\lambda_n \langle u_n - z_n, Au_n - Au\rangle.$$

From this inequality and $\|u_n - u\| \leq \|x_n - u\|$, we have

$$\begin{aligned}\|y_n - u\|^2 &\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|z_n - u\|^2 \\ &\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\{\|u_n - u\|^2 - \|u_n - z_n\|^2 \\ &\qquad + 2\lambda_n\langle u_n - z_n, Au_n - Au\rangle\} \\ &\leq \|x_n - u\|^2 - (1-\alpha_n)\|u_n - z_n\|^2 \\ &\qquad + 2\lambda_n(1-\alpha_n)\langle u_n - z_n, Au_n - Au\rangle,\end{aligned}$$

and hence

$$\begin{aligned}(1-\alpha_n)\|u_n - z_n\|^2 &\leq \|x_n - u\|^2 - \|y_n - u\|^2 \\ &\qquad + 2\lambda_n(1-\alpha_n)\langle u_n - z_n, Au_n - Au\rangle \\ &\leq (\|x_n - u\| + \|y_n - u\|)\|x_n - y_n\| \\ &\qquad + 2\lambda_n(1-\alpha_n)\langle u_n - z_n, Au_n - Au\rangle.\end{aligned}$$

Since $0 \leq \alpha_n \leq c < 1$, $\|x_n - y_n\| \to 0$ and $\|Au_n - Au\| \to 0$, we have that

$$\|u_n - z_n\| \to 0.$$

Let us show $\|x_n - u_n\| \to 0$. For all $u \in F(S) \cap VI(C, A) \cap EP(f)$, we have from Lemma 2.2 and $F(T_{r_n}) = EP(f)$ that

$$\begin{aligned}\|u_n - u\|^2 &= \|T_{r_n}x_n - T_{r_n}u\|^2 \leq \langle T_{r_n}x_n - T_{r_n}u, x_n - u\rangle \\ &= \langle u_n - u, x_n - u\rangle \\ &= \frac{1}{2}\{\|u_n - u\|^2 + \|x_n - u\|^2 - \|u_n - x_n\|^2\},\end{aligned}$$

and hence

$$\|u_n - u\|^2 \leq \|x_n - u\|^2 - \|u_n - x_n\|^2.$$

From this inequality and $\|z_n - u\| \leq \|u_n - u\|$, we have

$$\begin{aligned}\|y_n - u\|^2 &\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|z_n - u\|^2 \\ &\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\{\|x_n - u\|^2 - \|u_n - x_n\|^2\},\end{aligned}$$

and hence

$$(1-\alpha_n)\|u_n - x_n\|^2 \leq \|x_n - u\|^2 - \|y_n - u\|^2 \leq (\|x_n - u\| + \|y_n - u\|)\|x_n - y_n\|.$$

Therefore, we obtain

$$\|u_n - x_n\| \to 0.$$

Since $(1-\alpha_n)(Sz_n - z_n) = \alpha_n(z_n - x_n) + (y_n - z_n)$, we have

$$\begin{aligned}(1-\alpha_n)\|Sz_n - z_n\| &\le \|z_n - x_n\| + \|y_n - z_n\| \\ &\le \|z_n - x_n\| + \|y_n - x_n\| + \|x_n - z_n\| = 2\|z_n - x_n\| + \|y_n - x_n\| \\ &\le 2(\|z_n - u_n\| + \|u_n - x_n\|) + \|y_n - x_n\|.\end{aligned}$$

Therefore, we also obtain $\|Sz_n - z_n\| \to 0$.

Since $\{z_n\}$ is bounded, there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightharpoonup z_0$. Then, we can obtain that $z_0 \in F(S) \cap VI(C, A) \cap EP(f)$. In fact, let us first show $z_0 \in F(S)$. Assume that $z_0 \notin F(S)$. By Opial's condition,

$$\begin{aligned}\liminf_{i\to\infty} \|z_{n_i} - z_0\| < \liminf_{i\to\infty} \|z_{n_i} - Sz_0\| &= \liminf_{i\to\infty} \|z_{n_i} - Sz_{n_i} + Sz_{n_i} - Sz_0\| \\ &= \liminf_{i\to\infty} \|Sz_{n_i} - Sz_0\| \\ &\le \liminf_{i\to\infty} \|z_{n_i} - z_0\|.\end{aligned}$$

This is a contradiction. Therefore, we have $z_0 \in F(S)$. Let us show $z_0 \in VI(C, A)$. Define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $T^{-1}0 = VI(C, A)$; see [8]. Let $(v, u) \in G(T)$. Since $u - Av \in N_C v$ and $z_n = P_C(u_n - \lambda_n Au_n) \in C$, we have $\langle v - z_n, u - Av\rangle \ge 0$. By the definition of z_n, we also have

$$\langle v - z_n, z_n - (u_n - \lambda_n Au_n)\rangle \ge 0,$$

and hence

$$\langle v - z_n, \frac{z_n - u_n}{\lambda_n} + Au_n\rangle \ge 0.$$

Therefore,

$$\begin{aligned}\langle v - z_{n_i}, u\rangle &\ge \langle v - z_{n_i}, Av\rangle \\ &\ge \langle v - z_{n_i}, Av - \{\frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}} + Au_{n_i}\}\rangle \\ &= \langle v - z_{n_i}, Av - Az_{n_i}\rangle + \langle v - z_{n_i}, Az_{n_i} - Au_{n_i}\rangle - \langle v - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle \\ &\ge -\|v - z_{n_i}\|\|Az_{n_i} - Au_{n_i}\| - \|v - z_{n_i}\|\|\frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}}\|.\end{aligned}$$

Since $\|z_n - u_n\| \to 0$ and A is Lipschits continuous, we have $\langle v - z_0, u\rangle \ge 0$. Since T is maximal monotone, we have $z_0 \in T^{-1}0$ and hence $z_0 \in VI(C, A)$.

Finally, we show that $z_0 \in EP(f)$. By $u_n = T_{r_n}x_n$, we have

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0$$

for all $y \in C$. From $(A2)$ we also have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq f(y, u_n)$$

and hence

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq f(y, u_{n_i}).$$

Since $\|u_n - z_n\| \to 0$ and $z_{n_i} \rightharpoonup z_0$, we have $u_{n_i} \rightharpoonup z_0$. Since $0 < d \leq r_n < \infty$ and $\|u_n - x_n\| \to 0$, we have from $(A4)$ that $0 \geq f(y, z_0)$ for all $y \in C$. For $t \in (0, 1]$ and $y \in C$, let $y_t = ty + (1-t)z_0$. Since $y \in C$ and $z_0 \in C$, we have $y_t \in C$ and hence $f(y_t, z_0) \leq 0$. So, from $(A1)$ and $(A4)$ we have

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, z_0) \leq tf(y_t, y)$$

and hence $0 \leq f(y_t, y)$. From $(A3)$, we have $0 \leq f(z_0, y)$ for all $y \in C$ and hence $z_0 \in EP(f)$. Therefore $z_0 \in F(S) \cap VI(C, A) \cap EP(f)$.

From $z' = P_{F(S)\cap VI(C,A)\cap EP(f)}x$, $z_0 \in F(S) \cap VI(C, A) \cap EP(f)$ and $\|x_n - x\| \leq \|z' - x\|$, we have

$$\begin{aligned}
\|z' - x\| \leq \|z_0 - x\| &\leq \liminf_{i\to\infty} \|z_{n_i} - x\| \\
&\leq \limsup_{i\to\infty} \|z_{n_i} - x\| \\
&\leq \limsup_{i\to\infty} \{\|z_{n_i} - u_{n_i}\| + \|u_{n_i} - x_{n_i}\| + \|x_{n_i} - x\|\} \\
&\leq \|z' - x\|.
\end{aligned}$$

Thus, we have

$$\lim_{i\to\infty} \|z_{n_i} - x\| = \|z_0 - x\| = \|z' - x\|.$$

This implies $z_0 = z'$. Further, since a Hilbert space has the Kadec-Klee property, we have that $z_{n_i} \to z'$. From $\|z_n - x_n\| \to 0$, we also have $x_{n_i} \to z'$. Therefore, $x_n \to z'$. This completes the proof. $\square$

4 Applications

In this section, using Theorem 3.1, we prove three new results for finding a solution of an equilibrium problem, a solution of the variational inequality for an inverse-strongly monotone mapping and a fixed point of a nonexpansive mapping in a Hilbert space. First, we obtain a result for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space.

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$ and let S be a nonexpansive mapping from C*

into H such that $F(S)\cap EP(f)\neq\emptyset$. Let $\{x_n\}$ be a sequence in C generated by $x_0 = x\in C$, $C_0 = C$ and

$$\begin{cases} u_n = T_{r_n}(x_n), \\ y_n = \alpha_n x_n + (1-\alpha_n)S(u_n), \\ C_{n+1} = \{z\in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x, \quad n\in\mathbb{N}\cup\{0\}, \end{cases}$$

where $0\le\alpha_n\le c<1$ and $0<d\le r_n<\infty$. Then, $\{x_n\}$ converges strongly to $P_{F(S)\cap EP(f)}x$.

Proof. Putting $A = 0$ in Theorem 3.1, we obtain the desired result. □

Next, we obtain a result for finding a common element of the set of solutions of an equilibrium problem and the set of solutions of the variational inequality for an inverse-strongly monotone mapping in a Hilbert space.

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$ and let A be an α-inverse-strongly monotone mapping of C into H such that $VI(C,A)\cap EP(f)\neq\emptyset$. Let $\{x_n\}$ be a sequence in C generated by $x_0 = x\in C$, $C_0 = C$ and*

$$\begin{cases} u_n = T_{r_n}(x_n), \\ y_n = \alpha_n x_n + (1-\alpha_n)P_C(u_n - \lambda_n A u_n), \\ C_{n+1} = \{z\in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x, \quad n\in\mathbb{N}\cup\{0\}, \end{cases}$$

where $0\le\alpha_n\le c<1$, $\quad 0<d\le r_n<\infty$ and $0<a\le\lambda_n\le b<2\alpha$. Then, $\{x_n\}$ converges strongly to $P_{VI(C,A)\cap EP(f)}x$.

Proof. Putting $S = I$ in Theorem 3.1, we obtain the desired result. □

Finally, we obtain a result for finding a common element of the set of solutions of the variational inequality for an inverse-strongly monotone mapping and the set of fixed points of a nonexpansive mapping in a Hilbert space.

Theorem 4.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let S be a nonexpansive mapping from C into H and let A be an α-inverse-strongly monotone mapping of C into H such that $F(S)\cap VI(C,A)\neq\emptyset$. Let $\{x_n\}$ be a sequence in C generated by $x_0 = x\in C$, $C_0 = C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)SP_C(x_n - \lambda_n A x_n), \\ C_{n+1} = \{z\in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x, \quad n\in\mathbb{N}\cup\{0\}, \end{cases}$$

where $0\le\alpha_n\le c<1$ and $0<a\le\lambda_n\le b<2\alpha$. Then, $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(C,A)}x$.

Proof. Putting $f = 0$ in Theorem 3.1, we obtain the desired result. □

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