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Fixed Points for Operators on Generalized Metric Spaces

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ABSTRACT

The purpose of this paper is to present the fixed point theory for operators (singlevalued and multivalued) on generalized metric spaces in the sense of Luxemburg.

RESUMEN

El proposito de este artículo es presentar la teoria de punto fijo para operadores (univariados y multivaluados) sobre espacios métricos generalizados en el sentido de Luxemburg.

Key words and phrases: *Generalized metric in the sense of Luxemburg, Pompeiu-Hausdorff generalized functional, weakly Picard operator, fixed point, strict fixed point, generalized contraction, fibre generalized contraction, data dependence, pseudo-contractive multivalued operator.*

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1. Introduction

Let X be a nonempty set. A functional $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ is said to be a generalized metric in the sense of Luxemburg on X ([9], [13]) if:

i) $d(x, y) = 0 \Leftrightarrow x = y$;

ii) $d(x, y) = d(y, x)$;

iii) $x, y, z \in X$ with $d(x, z), d(z, y) < +\infty \Rightarrow d(x, y) \leq d(x, z) + d(z, y)$.

The pair (X, d) is called a generalized metric space. In a generalized metric space, the concepts of open and closed ball, Cauchy sequence, convergent sequence, etc. are defined in a similar way to the case of a metric space.

There are some contributions to fixed point theory for singlevalued operators (W.A.J. Luxemburg [13], J.B. Diaz and B. Margolis [7], C.F.G. Jung [9], S. Kasahara [10], G. Dezso [6],...) and multivalued operators (H. Covitz and S.B. Nadler [5], P.Q. Khanh [11],...) on a generalized metric space in the sense of Luxemburg.

The aim of this paper is to establish some new fixed point theorems for operators on a generalized metric space and, in this framework, to study the basic problems of the metrical fixed point theory.

2. Generalized metric spaces in the sense of Luxemburg

We start our considerations by presenting some examples of generalized metric spaces.

Example 2.1 *Let X be a nonempty set and $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$, given by*

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ +\infty, & \text{otherwise.} \end{cases}$$

Example 2.2 *Let $X := C(\mathbb{R})$ and $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ given by $d(x, y) := \sup\limits_{t \in \mathbb{R}} |x(t) - y(t)|$.*

Example 2.3 *Let $X := C(\mathbb{R})$ (the space of all continuous functions on $\mathbb{R}$) and $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ given by $d(x, y) := \sup\limits_{t \in \mathbb{R}}(|x(t) - y(t)| \cdot e^{-\tau|t|})$, where $\tau > 0$.*

Example 2.4 (Generic example) *Let (X_i, d_i), $i \in I$ be a family of metric spaces such that each two elements of the family are disjoint. Denote $X := \bigcup\limits_{i \in I} X_i$. If we define*

$$d(x, y) := \begin{cases} d_i(x, y), & \text{if } x, y \in X_i \\ +\infty, & \text{if } x \in X_i, y \in X_j, i \neq j \end{cases},$$

then the pair (X, d) is a generalized metric space.

The following characterization theorem of a generalized metric space was given by Jung.

Theorem 2.5 (Jung [9]) *Let (X, d) be a generalized metric space. Then there exists a partition $X := \bigcup_{i \in I} X_i$ of X such that $d_i := d_{|X_i \times X_i}$ is a metric, for each $i \in I$. Moreover, (X, d) is complete if and only if (X_i, d_i) is complete, for each $i \in I$.*

Notice that the above partition is induced by the following equivalence relation: $x \sim y \Leftrightarrow d(x, y) < +\infty$.

Let (X, d) be a generalized metric space. Then, the partition $X := \bigcup_{i \in I} X_i$ given by Jung's theorem is called the canonical decomposition of X into metric spaces. Moreover, if $x \in X$, then there exists $i(x) \in I$ such that $x \in X_{i(x)}$.

We will denote $B_d(x_0; r) := \{x \in X | d(x_0, x) < r\}$ and $\widetilde{B}_d(x_0; r) := \{x \in X | d(x_0, x) \leq r\}$. If $x \in X_i$, then $\widetilde{B}_d(x_0; r) = \widetilde{B}_{d_i}(x_0; r)$ and $B_d(x_0; r) = B_{d_i}(x_0; r)$.

If (X, d) is a generalized metric space, then the metric topology induced on X is given by:

$$\tau_d := \{Y \subseteq X | y \in Y \Rightarrow \exists r > 0 \ : \ B_d(y, r) \subset Y\}.$$

By this definition, it follows that:

$$(x_n)_{n \in \mathbb{N}} \subset X, x^* \in X, x_n \stackrel{\tau_d}{\to} x^* \ \Leftrightarrow \ d(x_n, x^*) \to 0.$$

A subset Y of X is said to be d-closed (closed with respect to the topology induced by d) if and only if $(y_n)_{n \in \mathbb{N}} \subset Y$ with $d(y_n, y) \to 0$, as $n \to +\infty$ implies $y \in Y$. Also, Y is d-open if for each $y \in Y$ there exists a ball $B(x_0, r) := \{x \in Y | d(x_0, x) < r\} \subset Y$.
Let us remark that if $X := \bigcup_{i \in I} X_i$ is the canonical decomposition of X, then X_i is d-closed and d-open, for each $i \in I$.

Definition 2.6 Two generalized metrics d_1 and d_2 on X are said to be:

(a) topological equivalent if $\tau_{d_1} = \tau_{d_2}$;

(b) metric equivalent if there exist $c_1, c_2 > 0$ such that:

i) $d_1(x, y) < +\infty$ implies $d_2(x, y) \leq c_1 d_1(x, y)$;

ii) $d_2(x, y) < +\infty$ implies $d_1(x, y) \leq c_2 d_2(x, y)$.

Remark 2.7 *If d_1 is a generalized metric on X, then there exists a bounded metric d_2 on X, topological equivalent to d_1 (for example take $d_2(x, y) := \min\{d_1(x, y), 1\}$).*

3. Functionals on generalized metric spaces

Throughout this section (X, d) will be a generalized metric space in the sense of Luxemburg.

Let us consider now the following families of subsets of the space (X,d):

$$P(X) := \{Y \subseteq X |\ Y \neq \emptyset\}\ ;\ P_b(X) := \{Y \in P(X) |\ Y \text{ is bounded }\};$$

$$P_{cl}(X) := \{Y \in P(X) |\ Y \text{ is closed }\};\ P_{b,cl}(X) := \{Y \in P(X) |\ Y \text{ is bounded and closed }\}.$$

Consider now some functionals on $P(X) \times P(X)$ (see also [3], [16]).

(i) the gap functional D_d defined by:

$$D_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$$

$$D_d(A,B) := \inf\{d(a,b) |\ a \in A,\ b \in B\}.$$

(ii) the excess generalized functional ρ_d defined by:

$$\rho_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},$$

$$\rho_d(A,B) := \sup\{D(a,B) |\ a \in A\}.$$

(iii) the Pompeiu-Hausdorff generalized functional H_d defined by:

$$H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},$$

$$H_d(A,B) := \max\{\rho(A,B), \rho(B,A)\}.$$

(iv) the delta functional δ_d defined by:

$$\delta_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$$

$$\delta_d(A,B) := \sup\{d(a,b) |\ a \in A,\ b \in B\}.$$

Let $A,B \in P(X)$. For the rest of the paper, we denote

$$A_i := A \cap X_i \text{ and } B_i := B \cap X_i,$$

where X_i are the sets from the characterization Theorem 2.1.

From (i), Theorem 2.5 and Example 2.4 we have:

Lemma 3.1 *Let (X,d) be a generalized metric space and $A,B \in P(X)$. Then:*

(i) $D(A,B) = \inf_{i \in I} D(A_i, B_i)$;

(ii) $D(A,B) < +\infty$ if and only if there exists $i \in I$ such that $A_i \neq \emptyset$ and $B_i \neq \emptyset$.

A useful result is:

Lemma 3.2 *Let (X,d) be a generalized metric space $x \in X$ and $A \in P(X)$. Then $D(x,A) = 0$ if and only if $X_{i(x)} \cap A \neq \emptyset$ and $x \in \overline{A}$ (where $X_{i(x)}$ denotes the unique element of the canonical decomposition of X where x belongs).*

From (iv) and Theorem 2.5 we obtain:

Lemma 3.3 *Let (X, d) be a generalized metric space and $A, B \in P(X)$. Then $\delta(A, B) < +\infty$ if and only if there exists $i \in I$ such that $A, B \in P_b(X_i)$. In particular, $A \in P_b(X)$ if and only if there exists $i \in I$ such that $A \in P_b(X_i)$.*

From (ii), Theorem 2.5 and Example 2.4 we have:

Lemma 3.4 *Let (X, d) be a generalized metric space and $Y, Z \in P(X)$. Then $\rho(Y, Z) < +\infty$ if and only if there exists $\eta > 0$ such that for each $y \in Y$ there is $z \in Z$ such that $d(y, z) < \eta$.*

Proof. If $\rho(Y, Z) < +\infty$, then there is $\eta > 0$ such that $\rho(Y, Z) < \eta$. Thus $D(y, Z) < \eta$ for each $y \in Y$. Hence there exists $z \in Z$ such that $d(y, z) < \eta$.

Suppose now there is $\eta > 0$ such that for each $y \in Y$ there exists $z \in Z$ with $d(y, z) < \eta$. Then, $y, z \in X_i$, where X_i is an element of the partition of the generalized metric space X. Hence $D(y, Z) \leq \eta$, for each $y \in Y$. Thus, $\rho(Y, Z) \leq \eta$. □

Let (X, d) be a generalized metric space, $Y \in P(X)$ and $\varepsilon > 0$. An open neighborhood of radius ε for the set Y is the set denoted $V_\varepsilon(Y)$ and defined by:

$$V_\varepsilon(Y) := \{x \in X |\ D(x, Y) < \varepsilon\}.$$

Let us remark that $V_\varepsilon(Y) = \bigcup\limits_{i \in I, Y_i \neq \emptyset} V_\varepsilon(Y_i)$.

In the usual case of a metric space (X, d) the following equivalent definitions of the Pompeiu-Hausdorff functional are well-known.

$$(iii)'\ H_d(A, B) := \inf\{\varepsilon > 0 | A \subset V_\varepsilon(B), B \subset V_\varepsilon(A)\},$$

and

$$(iii)''\ H_d(A, B) := \sup_{x \in X} |D(x, A) - D(x, B)|.$$

We have:

Lemma 3.5 *Let (X, d) be a generalized metric space. Then, the definitions (iii), $(iii)'$ and $(iii)''$ are equivalent.*

We can also prove the following result.

Lemma 3.6 *Let (X, d) be a generalized metric space and $A, B \in P(X)$. Then the following assertions are equivalent:*

(a) $H(A, B) < +\infty$;

(c) there exists $\eta > 0$ such that [for each $a \in A$ there exists $b \in B$ such that $d(a, b) < \eta$] and [for each $b \in B$ there exists $a \in A$ such that $d(a, b) < \eta$].

Lemma 3.7 *Let (X, d) be a generalized metric space. Then the following assertions hold:*

i) Let $\varepsilon > 0$ and $Y, Z \in P(X)$ such that $H(Y, Z) < +\infty$. Then for each $y \in Y$ there exists $z \in Z$ such that $d(y, z) \leq H(Y, Z) + \varepsilon$.

ii) Let $q > 1$ and $Y, Z \in P(X)$ such that $H(Y, Z) < +\infty$. Then, for each $y \in Y$ there exists $z \in Z$ such that $d(y, z) \leq qH(Y, Z)$.

Proof. i) Let $Y, Z \in P(X)$ and $\varepsilon > 0$. Suppose that $H(Y, Z) < +\infty$. Then, supposing, by contradiction, there is $y \in Y$ such that for every $z \in Z$ we have $d(y, z) > H(Y, Z) + \varepsilon$. If $d(y, z) < +\infty$ then since $H(Y, Z) \geq D(y, Z) \geq H(Y, Z) + \varepsilon$ we get a contradiction. If $d(y, z) = +\infty$ then, we get a contradiction to the supposition $H(Y, Z) < +\infty$, since, by Lemma 3.6, there is $\eta > 0$ such that for each $y \in Y$ there is $z \in Z$ with $d(y, z) < \eta$. □

Lemma 3.8 *Let (X, d) be a generalized metric space and $A, B \in P(X)$.*
Then:

a) $H(A, B) = \sup\limits_{i \in I} H(A \cap X_i, B \cap X_i)$;

b) $A \in P_{cp}(X) \;\Leftrightarrow\; card\{i \in I | A \cap X_i \neq \emptyset\} < +\infty$ and $A_i \in P_{cp}(X_i)$.

Remark 3.9 *Let (X, d) be a generalized metric space. Then $P_{cp}(X) \nsubseteq P_b(X)$. Consider, for example, $x, y \in X$ with $d(x, y) = +\infty$, then $\{x, y\}$ is compact but it is not bounded.*

4. Singlevalued operators on generalized metric spaces

4.1 General considerations

Let X be a nonempty set, $s(X) := \{(x_n)_{n\in\mathbb{N}} | x_n \in X,\ n \in \mathbb{N}\}$, $c(X) \subset s(X)$ and $Lim : c(X) \to X$ an operator. By definition the triple $(X, c(X), Lim)$ is called an L-space if the following conditions are satisfied:

(i) If $x_n = x,$ for all $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$.

(ii) If $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$, then for all subsequences, $(x_{n_i})_{i\in\mathbb{N}}$, of $(x_n)_{n\in\mathbb{N}}$ we have that $(x_{n_i})_{i\in\mathbb{N}} \in c(X)$ and $Lim(x_{n_i})_{i\in\mathbb{N}} = x$.

By definition an element of $c(X)$ is convergent sequence and $x := Lim(x_n)_{n\in\mathbb{N}}$ is the limit of this sequence and we write $x_n \to x$ as $n \to \infty$.

In what follows we will denote an L-space by $(X, \to)$.

Actually, an L-space is any set endowed with a structure implying a notion of convergence for sequences. For example, Hausdorff topological spaces, metric spaces, generalized metric spaces in Perov' sense (i.e., $d(x, y) \in \mathbb{R}^m_+$), generalized metric spaces in Luxemburg' sense (i.e., $d(x, y) \in \mathbb{R}_+ \cup \{+\infty\}$), K-metric spaces (i.e., $d(x, y) \in K$, where K is a cone in an ordered Banach space), gauge spaces, 2-metric spaces, D-R-spaces, probabilistic metric spaces, syntopogenous spaces, are such L-spaces. For more details see Fréchet [8], Blumenthal [4] and I.A. Rus [22].

Let (X,d) and (Y,ρ) be two generalized metric spaces and $f : X \to Y$.

Definition 4.1 The operator $f : (X,d) \to (Y,\rho)$ is said to be:

a) continuous, if $x_n \to x^*$ implies $f(x_n) \to f(x^*)$;

b) closed, if $x_n \to x^*$ and $f(x_n) \to y^*$ imply $f(x^*) = y^*$;

c) α-Lipschitz if $\alpha > 0$ and

$$d(x,y) < +\infty \Longrightarrow \rho(f(x), f(y)) \leq \alpha \cdot d(x,y).$$

d) α-contraction if f is α-Lipschitz with $\alpha < 1$.

4.2 Weakly Picard operators on L-spaces

Let $(X,\to)$ be an L-space and $f : X \to X$. We denote by $f^0 := 1_X$, $f^1 := f$, $f^{n+1} := f \circ f^n$, $n \in \mathbb{N}$ the iterate operators of f. Also:

$$F_f := \{x \in X \mid f(x) = x\},$$

$$I(f) := \{Y \in P(X) \mid f(Y) \subseteq Y\}.$$

Definition 4.2 (I.A. Rus [22]) Let $(X,\to)$ be an L-space. Then $f : X \to X$ is said to be

1) a Picard operator if:

i) $F_f = \{x^*\}$;

ii) $(f^n(x))_{n\in\mathbb{N}} \to x^*$ as $n \to +\infty$, for all $x \in X$.

2) a weakly Picard (briefly WP) operator if the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of f.

If $f : X \to X$ is a weakly Picard operator, then we define the operator $f^\infty : X \to X$ by:

$$f^\infty(x) := \lim_{n\to\infty} f^n(x).$$

Notice that $f^\infty(X) = F_f$. Moreover, if f is a Picard operator and we denote by x^* its unique fixed point, then $f^\infty(x) = x^*$, for each $x \in X$.

Definition 4.3 Let $(X,\to)$ be an L-space, $c > 0$ and $d : X \times X \to \mathbb{R}_+$. By definition, the operator f is called c-weakly Picard with respect to d, if f is a weakly Picard operator and

$$d(x, f^\infty(x)) \leq c \cdot d(x, f(x)), \quad \text{for all } x \in X.$$

If f is Picard operator and the above condition holds, then f is said to be c-Picard.

Theorem 4.4 *(Characterization Theorem) (I.A. Rus [25], [22]) Let $(X,\to)$ be an L-space and $f : X \to X$ be an operator. Then, f is a weakly Picard operator if and only if there exists a partition of X, $X = \bigcup_{\lambda\in\Lambda} X_\lambda$, such that:*

a) $X_\lambda \in I(f)$, *for all* $\lambda \in \Lambda$;

b) $f|_{X_\lambda} : X_\lambda \to X_\lambda$ *is a Picard operator, for all* $\lambda \in \Lambda$.

4.3 Contractions on generalized metric spaces

We present first some important auxiliary results.

Lemma 4.5 *Let* (X, d) *be a complete generalized metric space and* $f : X \to X$ *be an* α*-contraction. The following statements are equivalent:*

i) $F_f \neq \emptyset$;

ii) there exists $x \in X$ *such that* $d(x, f(x)) < +\infty$;

iii) there exist $x \in X$ *and* $n(x) \in \mathbb{N}$ *such that* $d\left(f^{n(x)}(x), f^{n(x)+1}(x)\right) < +\infty$;

iv) there exists $i \in I$ *such that* $X_i \in I(f)$.

Proof. $i) \Longrightarrow ii)$ Let $x^* \in F_f$. We have

$$d(x^*, f(x^*)) = d(x^*, x^*) = 0 < +\infty.$$

$ii) \Longrightarrow iii)$ We choose $n(x) = 0$;

$iii) \Longrightarrow i)$ Since f is an α-contraction we have that $(f^n(x))$ is a Cauchy sequence. This implies $f^n(x) \to x^*$, as $n \to +\infty$. From the continuity of f it follows that $x^* \in F_f$.

$ii) \Longrightarrow iv)$ Since $d(x, f(x)) < +\infty$, there exists $i \in I$ such that $x \in X_i$. Let $y \in X_i$ then $d(x, y) < +\infty$. We have:

$$d(x, f(y)) \leq d(x, f(x)) + d(f(x), f(y)) \leq d(x, f(x)) + \alpha \cdot d(x, y) < +\infty$$

which implies $f(y) \in X_i$.

$iv) \Longrightarrow ii)$ Let $x \in X_i$. Since $X_i \in I(f)$, we get that $f(x) \in X_i$. Therefore $d(x, f(x)) < +\infty$.□

Lemma 4.6 *Let* (X, d) *be a complete generalized metric space and* $f : X \to X$ *be an* α*-contraction. We suppose that:*

i) there exists $x \in X$ *such that* $d(x, f(x)) < +\infty$;

ii) if $u, v \in F_f$ *then* $d(u, v) < +\infty$;

Then:

a) $F_f = \{x^*\}$;

b) $f\big|_{X_{i(x)}} : X_{i(x)} \to X_{i(x)}$ *is a Picard operator.*

Proof. From $i)$ and Lemma 4.5 we have that there exists $i \in I$ such that $X_i \in I(f)$, $f^n(x) \in X_i$ for every $n \in \mathbb{N}$, $F_f \neq \emptyset$, $f^n(x) \to x^* \in F_f \cap X_i$. Let $u, v \in F_f$. Then $d(u, v) < +\infty$ and

$$d(u, v) = d(f(u), f(v)) \leq \alpha \cdot d(u, v).$$

Therefore $d(u,v)=0$, which implies $u=v$. Hence $F_f=\{x^*\}$.

Since $X_i \in I(f)$ then $d(y,f(y))<+\infty$ for every $y \in X_i$ and applying again Lemma 4.5 we get that $f\big|_{X_{i(x)}} : X_{i(x)} \to X_{i(x)}$ is a Picard operator. □

Theorem 4.7 *Let (X,d) be a complete generalized metric space and $f : X \to X$. We suppose that:*

i) f is an α-contraction;

ii) for every $x \in X$ there exists $n(x) \in \mathbb{N}$ such that $d\left(f^{n(x)}(x), f^{n(x)+1}(x)\right) < +\infty$.

Then:

a) f is a weakly Picard operator. If in addition, for every $x \in X$ we have $d(x,f(x))<+\infty$, then f is $\frac{1}{1-\alpha}$-weakly Picard;

b) If, in addition:

b_1) for every $x \in X$ we have $d(x,f(x))<+\infty$;

b_2) $u,v \in F_f$ implies $d(u,v)<+\infty$,

then f is $\frac{1}{1-\alpha}$-Picard.

Proof. a) The first part follows from Lemma 4.5 and Lemma 4.6. For the second conclusion, notice that for every $x \in X$ such that $d(x,f(x))<+\infty$ and each $n \in \mathbb{N}$ we have:

$$d\left(f^n(x), f^\infty(x)\right) \le \frac{\alpha^n}{1-\alpha} \cdot d(x,f(x))$$

which implies

$$d\left(x, f^\infty(x)\right) \le \frac{1}{1-\alpha} \cdot d(x,f(x)).$$

b) From b_2) we obtain $F_f=\{x^*\}$ and from a) we obtain that f is $\frac{1}{1-\alpha}$-Picard operator. □

Theorem 4.8 *Let (X,d) be a complete generalized metric space and $f,g : X \to X$ two operators. We suppose that:*

i) f and g are α-contractions;

ii) $d(x,f(x))<+\infty$ and $d(x,g(x))<+\infty$, for every $x \in X$;

iii) there exists $\eta>0$ such that

$$d(f(x),g(x)) \le \eta, \quad \text{for all } x \in X.$$

Then:

$$H(F_f,F_g) \le \frac{\eta}{1-\alpha}.$$

Proof. Let $x \in F_f$ and $y \in F_g$. From ii) and Theorem 4.7 we have:

$$d\left(x, g^\infty(x)\right) \le \frac{1}{1-\alpha} \cdot d(x,g(x)) = \frac{1}{1-\alpha} \cdot d(f(x),g(x)) \le \frac{\eta}{1-\alpha}.$$

Since $g^{\infty}(x) \in F_g$ then

$$D(x, F_g) \leq d(x, g^{\infty}(x)) \leq \frac{\eta}{1-\alpha}.$$

By taking the supremum over $x \in F_f$ we get

$$\rho(F_f, F_g) \leq \frac{\eta}{1-\alpha}.$$

Using the same technique we have:

$$\rho(F_g, F_f) \leq \frac{\eta}{1-\alpha}$$

which implies the conclusion. □

Theorem 4.9 *(Fibre contraction principle) Let $(X_0, \to)$ be an L-space and (X_k, d_k), $k \in \{0, 1, \cdots, p\}$ (where $p \geq 1$) be complete generalized metric spaces. We consider the operators:*

$$f_k : X_0 \times ... \times X_k \to X_k, \quad k \in \{0, 1, \cdots, p\}.$$

We suppose that:

i) $f_0 : X_0 \to X_0$ *is a weakly Picard operator;*

ii) $f_k(x_0, ..., x_{k-1}, \cdot)$ *is an α_k-contraction, $k \in \{1, 2, \cdots, p\}$;*

iii) f_k *is continuous, $k \in \{1, 2, \cdots, p\}$;*

iv) for every $(x_0, x_1, ..., x_k) \in X_0 \times ... \times X_k$ we have

$$d_k(x_k, f_k(x_0, x_1, ..., x_k)) < +\infty, \quad k \in \{1, 2, \cdots, p\}.$$

Then the operator

$$g_p : X_0 \times ... \times X_p \to X_0 \times ... \times X_p$$
$$g_p(x_0, x_1, ..., x_p) = (f_0(x_0), f_1(x_0, x_1), ..., f_p(x_0, x_1, ..., x_p))$$

is weakly Picard.

Proof. We will prove by induction. For $p = 1$ the conclusion follows by Theorem 3.1 in M.A. Şerban [31]. We suppose that conclusion holds for $k \leq p$ and we prove the conclusion for $k+1$. We know that $g_{k+1} = (g_k, f_{k+1})$, g_k are weakly Picard and from *ii)* $f_{k+1}(x_0, ..., x_k, \cdot)$ is an α_{k+1}-contraction, so we apply again Theorem 3.1 from M.A. Şerban [31] and we get that g_{k+1} is weakly Picard. □

Theorem 4.10 *Let X be a nonempty set, $\alpha \in]0; 1[$ and $f : X \to X$ an operator. The following statements are equivalent:*

i) $F_f = F_{f^n} \neq \emptyset$ *for every $n \in \mathbb{N}$;*

ii) there exists a complete generalized metric d on X such that:

a) $f : (X, d) \to (X, d)$ *is an α-contraction;*

b) $d(x, f(x)) < +\infty$ *for every* $x \in X$.

Proof. $i) \Longrightarrow ii)$ $F_f = F_{f^n} \neq \emptyset$ for every $n \in \mathbb{N}$ implies that there exists a partition of X, $X = \bigcup_{i \in I} X_i$ such that $X_i \in I(f)$, $card(F_f \cap X_i) = 1$ and $f|_{X_i}$ is a Bessaga operator (see I.A. Rus [24]). From Bessaga's theorem [2] there exists a complete metric d_i on X_i such that $f|_{X_i} : X_i \to X_i$ is an α-contraction for all $i \in I$. So, $d : X \times X \to R_+ \cup \{+\infty\}$

$$d(x, y) = \begin{cases} d_i(x, y) & if \;\; x, y \in X_i \\ +\infty & if \;\; x \in X_i, \; y \in X_i, \; i \neq j \end{cases}$$

is the complete generalized metric on X that we are looking for.

$ii) \Longrightarrow i)$ is Theorem 4.7. □

4.4 Graphic contractions

Let (X, d) be a generalized metric space and $f : X \to X$.

Definition 4.11 $f : X \to X$ is a graphic contraction if there exists $\alpha \in [0; 1[$ such that:

$$d\left(f^2(x), f(x)\right) \leq \alpha \cdot d(x, f(x)) \text{ for all } x \in X \text{ with } d(x, f(x)) < +\infty.$$

Theorem 4.12 *Let* (X, d) *be a complete generalized metric space and* $f : X \to X$. *We suppose that:*

i) f *is a closed graphic contraction;*

ii) for every $x \in X$ *there exists* $n(x) \in \mathbb{N}$ *such that* $d\left(f^{n(x)}(x), f^{n(x)+1}(x)\right) < +\infty$.

Then:

a) f *is a weakly Picard operator. If, in addition, for every* $x \in X$ *we have that* $d(x, f(x)) < +\infty$, *then* f *is* $\frac{1}{1-\alpha}$*-weakly Picard;*

b) If, in addition:

b_1) *for every* $x \in X$ *we have* $d(x, f(x)) < +\infty$;

b_2) *if* $u, v \in F_f$ *implies* $d(u, v) < +\infty$,

then f *is* $\frac{1}{1-\alpha}$*-Picard.*

Proof. $a)$ From $i)$ and $ii)$ we have that for each $x \in X$, the sequence $(f^n(x))$ is Cauchy. Therefore there exists $x^* \in X$ such that $f^n(x) \to x^*$, as $n \to +\infty$ and

$$d(f^n(x), x^*) \leq \frac{\alpha^{n-n(x)}}{1-\alpha} \cdot d\left(f^{n(x)}(x), f^{n(x)+1}(x)\right), \quad n \geq n(x).$$

Since f is closed we get that $x^* \in F_f$ and $f^\infty(x) = x^*$. This means that f is a weakly Picard operator.

If for every $x \in X$ we have $d(x, f(x)) < +\infty$, then $n(x) = 0$ and letting $n = 0$ in the above relation, we conclude that f is $\frac{1}{1-\alpha}$-weakly Picard operator.

b) If for $u, v \in F_f$ we have $d(u, v) < +\infty$ then $F_f = \{x^*\}$, which means that f is a $\frac{1}{1-\alpha}$-Picard operator. □

4.5 Meir-Keeler operators

Let us consider now the case of Meir-Keeler operators on generalized metric spaces.

Definition 4.13 *Let (X, d) be a generalized metric space. Then, $f : X \to X$ is called a Meir-Keeler type operator if for each $\epsilon > 0$ there exists $\eta = \eta(\epsilon) > 0$ such that for $x, y \in X$ with $\epsilon \le d(x, y) < \epsilon + \eta$ we have $d(f(x), f(y)) < \epsilon$.*

By using an argument similar to the one in the Meir-Keeler fixed point theorem [14] we have:

Theorem 4.14 *Let (X, d) be a generalized complete metric space and $f : X \to X$ be a Meir-Keeler type operator. Suppose there exists $x_0 \in X$ such that $d(x_0, f(x_0)) < +\infty$.*
Then $F_f \neq \emptyset$. Moreover, if additionally $x, y \in F_f$ implies $d(x, y) < +\infty$, then $F_f = \{x^*\}$.

Proof. Denote $x_n := f^n(x_0)$, $n \in \mathbb{N}$.

The proof of the theorem can be organized in five steps.
Step 1. We prove that

$$d(f(x), f(y)) < d(x, y), \text{ for each } x, y \in X \text{ with } x \neq y \text{ and } d(x, y) < +\infty.$$

Let $x, y \in X$ be such that $x \neq y$ and $d(x, y) < +\infty$. Then by letting $\epsilon := d(x, y)$ in the definition of Meir-Keeler operators we get $d(f(x), f(y)) < d(x, y)$.
Step 2. We can prove, by induction, that $d(x_n, x_{n+1}) < +\infty$, for all $n \in \mathbb{N}$.
Step 3. We prove that the sequence $a_n := d(x_n, x_{n+1}) \searrow 0$ as $n \to +\infty$.

If there is $n_0 \in \mathbb{N}$ such that $a_{n_0} = 0$ then $x_{n_0} \in F_f$.

If $a_n \neq 0$, for each $n \in \mathbb{N}$, then $a_n = d(f(x_{n-1}), f(x_n)) < d(x_{n-1}, x_n) = a_{n-1}$. Hence the sequence $(a_n)_{n\in\mathbb{N}}$ converges to a certain $a \ge 0$. Suppose that $a > 0$. Then, for each $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that $\epsilon \le a_n < \epsilon + \eta$, for all $n \ge n_\epsilon$. Then, by the Meir-Keeler condition we obtain $a_{n+1} < \epsilon$, which is a contradiction with the above relation.
Step 4. We will prove that the sequence (x_n) is Cauchy.

Suppose, by contradiction, that (x_n) is not a Cauchy sequence. Then, there exists $\epsilon > 0$ such that $\limsup d(x_m, x_n) > 2\epsilon$. For this ϵ there exists $\eta := \eta(\epsilon) > 0$ such that for $x, y \in X$ with $\epsilon \le d(x, y) < \epsilon + \eta$ we have $d(f(x), f(y)) < \epsilon$. Choose $\delta := \min\{\epsilon, \eta\}$. Since $a_n \searrow 0$ as $n \to +\infty$ it follows that there is $p \in \mathbb{N}$ such that $a_p < \frac{\delta}{3}$. Let $m, n \in \mathbb{N}^*$ with $n > m > p$ such that $d(x_n, x_m) > 2\epsilon$. For $j \in [m, n]$ we have $|d(x_m, x_j) - d(x_m, x_{j+1}| \le a_j < \frac{\delta}{3}$. Also, $d(x_m, x_{m+1} < \epsilon$ and $d(x_m, x_n) > \epsilon + \delta$ we obtain that there exists $k \in [m, n]$ such that $\epsilon < \epsilon + \frac{2\delta}{3} < d(x_m, x_k) < \epsilon + \delta$. On the other hand, for any $m, l \in \mathbb{N}$ we have: $d(x_m, x_l) \le d(x_m, x_{m+1}) + d(x_{m+1}, x_{l+1}) +$

$d(x_{l+1}, x_l) = a_m + d(f(x_m), f(x_l)) + a_l < \frac{\delta}{3} + \epsilon + \frac{\delta}{3}$. The contradiction proves that (x_n) is Cauchy.

Step 5. We prove that $x^* := \lim\limits_{n \to +\infty} x_n$ is a fixed point of f.

Since f is continuous and $x_{n+1} = f(x_n)$, we get by passing to the limit that $x^* = f(x^*)$.

If $x^*, y \in F_f$ are two distinct fixed points of f then, by the contractive condition, we get the following contradiction: $d(x^*, y) = d(f(x^*), f(y)) < d(x^*, y)$. This completes the proof. □

4.6 Caristi operators

Let (X, d) be a generalized metric space.

Definition 4.15 *A space X is said to be sequentially complete in Weierstrass' sense (see [33]) if each sequence $(x_n)_{n \in \mathbb{N}}$ in X such that $\sum\limits_{n=0}^{+\infty} d(x_n, x_{n+1}) < +\infty$ is convergent in X.*

Definition 4.16 *Let (X, d) be a generalized metric space. Then, $f : X \to X$ is called a Caristi operator if there exists a functional $\varphi : X \to \mathbb{R}_+$ such that*

$$d\left(x, f\left(x\right)\right) \leq \varphi\left(x\right) - \varphi\left(f\left(x\right)\right), \text{ for every } x \in X \ .$$

Theorem 4.17 *Let (X, d) be a sequentially complete (in Weierstrass' sense) generalized metric space and $f : X \to X$ be a closed Caristi operator. Then f is a weakly Picard operator.*

Proof. We remark that if f is a Caristi operator, then $d\left(x, f\left(x\right)\right) < +\infty$ for every $x \in X$. Denote by $x_n := f^n(x)$, for $n \in \mathbb{N}$. Then:

$$\sum_{n=0}^{+\infty} d(x_n, x_{n+1}) = \sum_{n=0}^{+\infty} d(f^n(x), f^{n+1}(x)).$$

We will prove that the series $\sum\limits_{n=0}^{+\infty} d(f^n(x), f^{n+1}(x))$ is convergent. For this purpose we need to show that the sequence of its partial sums is convergent in $\mathbb{R}_+$. Denote by $s_n := \sum\limits_{k=0}^{n} d(f^k(x), f^{k+1}(x))$. Then $s_{n+1} - s_n = d(f^{n+1}(x), f^{n+2}(x)) \geq 0$, for each $n \in \mathbb{N}$. Moreover $s_n = \sum\limits_{k=0}^{n} d(f^k(x), f^{k+1}(x)) \leq \varphi(x)$. Hence $(s_n)_{n \in \mathbb{N}}$ is upper bounded and increasing in $\mathbb{R}_+$. Then the sequence $(s_n)_{n \in \mathbb{N}}$ is convergent.

It follows that the sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy and, from the sequentially completeness of the space, convergent to a certain element $x^* \in X$. The conclusion follows from the fact that f is closed. □

4.7 Fixed point theorems in a set with two generalized metrics

Let X be a nonempty set and $d, \rho : X \times X \to R_+ \cup \{+\infty\}$ be two generalized metrics on X. In this subsection we will present Maia's fixed point theorem for the case of a set with two generalized metrics.

Theorem 4.18 *Let X be a nonempty set, $d, \rho : X \times X \to R_+ \cup \{+\infty\}$ two generalized metrics on X and $f : X \to X$. We suppose that:*

i) (X, d) is a complete generalized metric space;

ii) there exists $c > 0$ such that $d(x, y) \le c \cdot \rho(x, y)$ for all $x, y \in X$ with $\rho(x, y) < +\infty$;

iii) for every $x \in X$ there exists $n(x) \in \mathbb{N}$ such that $\rho\left(f^{n(x)}(x), f^{n(x)+1}(x)\right) < +\infty$;

iv) $f : (X, \rho) \to (X, \rho)$ is an α-contraction.

Then f is weakly Picard.

Proof. For each $x \in X$ there exists $n(x) \in \mathbb{N}$ such that $\rho\left(f^{n(x)}(x), f^{n(x)+1}(x)\right) < +\infty$. Also, there exists $i \in I$ such that $X_i \in I(f)$ and $f^n(x) \in X_i$ for all $n \ge n(x)$. Since $f : (X, \rho) \to (X, \rho)$ is an α-contraction, the sequence $(f^n(x))_{n\in\mathbb{N}}$ is Cauchy in (X, ρ). Using conditions $ii)$, $iii)$ and $iv)$ we get

$$d\left(f^n(x), f^{n+p}(x)\right) \le c \cdot \rho\left(f^n(x), f^{n+p}(x)\right) \le c \cdot \frac{\alpha^{n-n(x)}}{1-\alpha}\rho\left(f^{n(x)}(x), f^{n(x)+1}(x)\right), \quad n \ge n(x),$$

so $d(f^n(x), f^{n+p}(x)) \to 0$ as $n \to +\infty$. Thus $(f^n(x))_{n\in\mathbb{N}}$ is Cauchy sequence in (X, d), which implies that $f^n(x) \to x^* \in X_i$. By condition $iv)$ we have that $x^* \in F_f$. Hence f is weakly Picard. □

An improved version of Maia's theorem can be obtained by replacing the assumption $ii)$ with a more useful condition (from an application point of view), see I.A. Rus [20].

Theorem 4.19 *Let X be a nonempty set, $d, \rho : X \times X \to R_+ \cup \{+\infty\}$ two generalized metrics on X and $f : X \to X$. We suppose that:*

i) (X, d) is a complete generalized metric space;

ii) there exists $c > 0$ such that $d(f(x), f(y)) \le c \cdot \rho(x, y)$, for all $x, y \in X$ with $\rho(x, y) < +\infty$;

iii) for every $x \in X$ there exists $n(x) \in \mathbb{N}$ such that $\rho\left(f^{n(x)}(x), f^{n(x)+1}(x)\right) < +\infty$;

iv) $f : (X, \rho) \to (X, \rho)$ is an α-contraction.

Then f is a weakly Picard operator.

Proof. The proof follows the method in Theorem 4.18. □

5. Multivalued operators in generalized metric spaces

5.1 General considerations

Let (X, d) be a generalized metric space. Let Y, Z be two nonempty subsets of X and $T : Y \to P(Z)$ be a multivalued operator. By definition, $t : Y \to Z$ is a selection of T if $t(x) \in T(x)$, for each $x \in Y$. If $T : X \to P(X)$ is a multivalued operator, then $x^* \in X$ is a fixed point for T if and only if $x^* \in T(x^*)$. Denote by F_T the set of all fixed points for T. Also, $x^* \in X$ is called a strict fixed point for T if and only if $\{x^*\} = T(x^*)$. We will denote by $(SF)_T$ the set of all strict fixed points of T. By $Graph(T) := \{(x, y) \in X \times X | y \in T(x)\}$ we denote the graph of the multivalued operator T and by $T(Y) := \bigcup_{x \in Y} T(x)$ the image through T of the set $Y \in P(X)$.

Recall that if $Y \subseteq X$, then $T(Y) := \bigcup_{x \in Y} T(x)$. We also denote by $T^n := T \circ T \cdots \circ T$ (the n times composition).

Recall that, if (X, d) is a metric space, then $T : X \to P_{cl}(X)$ is said to be a multivalued a-contraction if

$$a \in [0, 1[\text{ and } H_d(T(x), T(y)) \leq ad(x, y), \text{ for each } x, y \in X.$$

The following result is known as Covitz-Nadler fixed point principle.

Theorem 5.1 (Covitz-Nadler [5]) *Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multivalued a-contraction. Then, for each $x_0 \in X$ there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in X with $x_{n+1} \in T(x_n)$ for all $n \in \mathbb{N}$, which converges to a fixed point of T.*

Remark 5.2 *From the proof of the above result it follows that for each $x \in X$ and each $y \in T(x)$ there exists in X a sequence $(x_n)_{n \in \mathbb{N}}$ with the properties:*
a) $x_0 = x, x_1 = y$;
b) $x_{n+1} \in T(x_n)$ for all $n \in \mathbb{N}^*$;
c) $(x_n)_{n \in \mathbb{N}}$ converges to a fixed point of T.

This principle gave rise to the following concept.

Definition 5.3 (Rus-Petruşel-Sîntămărian [28], [29]) Let $(X, \to)$ be an L-space. Then $T : X \to P(X)$ is a multivalued weakly Picard operator (briefly MWP operator) if for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in X such that:

i) $x_0 = x,\ x_1 = y$

ii) $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$

iii) the sequence $(x_n)_{n \in \mathbb{N}}$ is convergent and its limit is a fixed point of T.

A sequence $(x_n)_{n \in \mathbb{N}}$ in X satisfying the conditions (i) and (ii) in Definition 5.3 is called a sequence of successive approximations for T starting from (x, y).

The aim of this section is to establish some fixed point results for multivalued operators of contractive type on generalized metric space.

5.2 Multivalued contractions on generalized metric spaces

Let us recall first some contractive-type conditions for multivalued operators.

Definition 5.4 Let (X, d) be a generalized metric space. Then $T : X \to P_{cl}(X)$ is called a multivalued a-contraction if $a \in [0, 1[$ and

$$H_d(T(x), T(y)) \leq ad(x, y), \text{ for each } x, y \in X, \text{ with } d(x, y) < +\infty.$$

Let (X, d) be a generalized metric space. We denote by $\mathcal{P}(X)$ the set of all subsets of a nonempty set X.

Definition 5.5 Let (X, d) be a generalized metric space. If $T : X \to P(X)$ is a multivalued operator, then we consider the following multivalued operators generated by T:

$$\widehat{T} : X \to \mathcal{P}(X),\ \widehat{T}(x) := T(x) \cap X_{i(x)}$$

(where $X_{i(x)}$ denotes the unique element of the canonical decomposition of X where x belongs),

$$\tilde{T}^i : X \to \mathcal{P}(X),\ \tilde{T}^i(x) := T(x) \cap X_i$$

(where X_i denotes an arbitrary element of the canonical decomposition of X).

Then we have:

Lemma 5.6 $F_T = F_{\widehat{T}}$.

Lemma 5.7 $F_T \neq \emptyset \ \Leftrightarrow$ *if there exists* $i \in I$ *such that* $F_{\tilde{T}^i} \neq \emptyset$.

The following result is a straightforward version of Covitz and Nadler alternative theorem in [5].

Theorem 5.8 *Let* (X, d) *be a generalized complete metric space and* $T : X \to P_{cl}(X)$ *be a multivalued a-contraction. Suppose that for each* $x \in X$ *there is* $y \in T(x)$ *such that* $d(x, y) < +\infty$. *Then there exists a sequence of successive approximations of* T *starting from any arbitrary* $x \in X$ *which converges to a fixed point of* T.

The previous result gives rise to the following open question.

Open question. Let $T : X \to P_{cl}(X)$ be a multivalued a-contraction as in the above Covitz-Nadler fixed point result. Is T a MWP operator ?

Theorem 5.9 *Let* (X, d) *be a generalized complete metric space and* $T : X \to P_{cl}(X)$ *be a multivalued a-contraction. Suppose there exists* $x_0 \in X$ *and* $x_1 \in T(x_0)$ *such that* $d(x_0, x_1) < +\infty$.

Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for T starting from x_0 which converges to a fixed point of T.

Proof. Let $X := \bigcup_{i\in I} X_i$ be the canonical decomposition of X into metric spaces. Recall that X is complete if and only if X_i is complete for each $i \in I$. Let $j \in I$ such that $x_0 \in X_j$.

For $x \in X$ we successively have:

$$D(x, T(x)) < +\infty \Leftrightarrow \text{ there exists } y \in T(x) \text{ such that } d(x,y) < +\infty \Leftrightarrow y \in T(x) \cap X_{i(x)}.$$

Hence

$$D(x, T(x)) < +\infty \Leftrightarrow T(x) \cap X_{i(x)} \neq \emptyset.$$

Consider now the multivalued operator

$$\tilde{T}^j : X \to \mathcal{P}(X),\ \tilde{T}^j(x) := T(x) \cap X_j.$$

We will prove that $\tilde{T}^j_{|X_j} : X_j \to P_{cl}(X_j)$. For this purpose, it is enough to show that

$$D(x, T(x)) < +\infty, \text{ for each } x \in X_j.$$

For $x \in X_j$ we have:
$D(x, T(x)) \leq D(x, T(x_0)) + H(T(x_0), T(x)) \leq d(x, x_0) + D(x_0, T(x_0)) + ad(x_0, x) < +\infty.$

Hence $\tilde{T}^j_{|X_j} : X_j \to P_{cl}(X_j)$ is a multivalued a-contraction on the complete metric space $(X_j, d_{|X_j\times X_j})$. The conclusion follows from Lemma 5.7 and Theorem 5.1. □

An answer to the above problem is the following result.

Theorem 5.10 *Let (X, d) be a generalized complete metric space and $T : X \to P_{cl}(X)$ be a multivalued a-contraction. Suppose that for each $x \in X$ and $y \in T(x)$ we have $d(x, y) < +\infty$ (or equivalently, for each $x \in X$ we have $T(x) \subset X_{i(x)}$). Then T is a MWP operator.*

Proof. From the hypothesis we have that $D(x, T(x)) < +\infty$, for each $x \in X$. Hence, for each $x \in X$ we have that $T : X_{i(x)} \to P_{cl}(X_{i(x)})$. Since $(X_{i(x)}, d_{|X_{i(x)}\times X_{i(x)}})$ is a complete metric space, by Theorem 5.1 and Remark 5.2, we conclude that T is a MWP operator. □

We introduce now the following concepts.

Definition 5.11 (Rus-Petruşel-Sîntămărian [29]) Let $(X, \to)$ be an L-space and $T : X \to P(X)$ be a MWP operator. Define the multivalued operator $T^\infty : Graph(T) \to P(F_T)$ by the formula $T^\infty(x, y) = \{\ z \in F_T \mid$ there exists a sequence of successive approximations of T starting from (x, y) that converges to $z\ \}$.

Definition 5.12 (see also Rus-Petruşel-Sîntămărian [29]) Let (X, d) be a generalized metric space and $T : X \to P(X)$ be a MWP operator such that for each $x \in X$ and $y \in T(x)$ we have that $d(x, y) < +\infty$. Then, T is called a c-multivalued weakly Picard operator (briefly c-MWP operator) if there exists a selection t^∞ of T^∞ such that $d(x, t^\infty(x, y)) \leq c\ d(x, y)$, for all $(x, y) \in Graph(T)$.

As an example, we have:

Theorem 5.13 *Let (X, d) be a generalized complete metric space and $T : X \to P_{cl}(X)$ be a multivalued a-contraction, such that for each $x \in X$ and $y \in T(x)$ we have $d(x, y) < +\infty$.*

Then T is a $\frac{1}{1-a}$-MWP operator.

We present now an abstract data dependence theorem for the fixed point set of c-MWP operators on generalized metric spaces.

Theorem 5.14 *Let (X, d) be a generalized metric space and $T_1, T_2 : X \to P(X)$ be two multivalued operators. We suppose that:*

i) T_i is a c_i-MWP operator, for $i \in \{1, 2\}$

ii) there exists $\eta > 0$ such that $H(T_1(x), T_2(x)) \leq \eta$, for all $x \in X$.

Then $H(F_{T_1}, F_{T_2}) \leq \eta \ \max \{ \ c_1, \ c_2 \ \}$.

Proof. The proof follows in a similar way to Rus-Petruşel-Sîntămărian [29]. For the sake of completeness we present it here.
Let $t_i : X \to X$ be a selection of T_i for $i \in \{1, 2\}$. Let us remark that

$$H(F_{T_1}, F_{T_2}) \leq \max \left\{ \sup_{x \in F_{T_2}} d(x, t_1^{\infty}(x, t_1(x))), \sup_{x \in F_{T_1}} d(x, t_2^{\infty}(x, t_2(x))) \right\}.$$

Let $q > 1$. Then we can choose t_i $(i \in \{1, 2\})$ such that

$$d(x, t_1^{\infty}(x, t_1(x))) \leq c_1 q H(T_2(x), T_1(x)), \text{ for all } x \in F_{T_2}$$

and

$$d(x, t_2^{\infty}(x, t_2(x)) \leq c_2 q H(T_1(x), T_2(x)), \text{ for all } x \in F_{T_1}.$$

Thus we have $H(F_{T_1}, F_{T_2}) \leq q\eta \max\{c_1, c_2\}$. Letting $q \searrow 1$, the proof is complete. □

Notice that the above conclusions means that the data dependence phenomenon of the fixed point set for c-MWP operators holds.

We also have:

Theorem 5.15 *Let (X, d) be a generalized complete metric space and $T : X \to P_{cl}(X)$ be a multivalued a-contraction. Suppose:*

(i) $(SF)_T \neq \emptyset$;

(ii) If $x, y \in F_T$ then $d(x, y) < +\infty$.

Then $F_T = (SF)_T = \{x^*\}$.

Proof. We will prove first that $(SF)_T = \{x^*\}$. Indeed, if $z \in (SF)_T$ with $z \neq x^*$, then $d(z, x^*) < +\infty$ and $d(z, x^*) = H(T(z), T(x^*)) \leq ad(z, x^*)$, a contradiction. Next we will prove that $F_T \subseteq$

$(SF)_T$. Let $y \in F_T$. Then $d(y, x^*) < +\infty$. Thus $d(y, x^*) = D(y, T(x^*)) \leq H(T(y), T(x^*)) \leq ad(y, x^*)$, which implies $y = x^*$. This completes the proof. $\square$

5.3 Pseudo-contractive multivalued operators on generalized metric spaces

In D. Azé and J.-P. Penot [1] the following concept is introduced.

Definition 5.16 (Azé-Penot [1]) Let (X, d) be a metric space. A multivalued operator $T : X \to P(X)$ is said to be pseudo-a-Lipschitzian with respect to the subset $U \subset X$ whenever, for all $x, y \in U$, we have

$$\rho_d(T(x) \cap U, T(y)) \leq ad(x, y).$$

Also, the multivalued opeator T is called pseudo-a-contractive with respect to U if it is pseudo-a-Lipschitzian with respect to U for some $a \in [0, 1[$.

In Azé-Penot [1], the fixed point theory for multivalued pseudo-a-contractive operators with respect to the open ball $B_d(x_0, r)$ of a complete metric space (X, d) is studied. The aim of this section is to give some fixed point results for multivalued pseudo-a-contractive operators in the setting of a generalized metric space.

Theorem 5.17 *Let (X, d) be a generalized complete metric space and $T : X \to P_{cl}(X)$ be a multivalued operator. Let $X := \bigcup_{i \in I} X_i$ be the canonical decomposition of X. Suppose that there exists $x_0 \in X$ such that $D(x_0, T(x_0)) < +\infty$ and T is pseudo a-contractive with respect to $X_{i(x_0)}$. Then $F_T \neq \emptyset$.*

Proof. Since $D(x_0, T(x_0)) < +\infty$ there exists $b > 0$ and $x_1 \in T(x_0)$ such that $d(x_0, x_1) < b < +\infty$. Then $x_1 \in X_{i(x_0)}$ and thus $x_1 \in T(x_0) \cap X_{i(x_0)}$. Hence we have $D(x_1, T(x_1)) \leq \rho(T(x_0) \cap X_{i(x_0)}, T(x_1)) \leq ad(x_0, x_1) < ab$. Thus there exists $x_2 \in T(x_1)$ such that $d(x_1, x_2) < ab < +\infty$. Thus $x_2 \in T(x_1) \cap X_{i(x_0)}$. In a similar way, we have $D(x_2, T(x_2)) \leq \rho(T(x_1) \cap X_{i(x_0)}, T(x_2)) \leq ad(x_1, x_2) < a^2 b < +\infty$.

By induction, we obtain a sequence $(x_n)_{n \in \mathbb{N}}$ with the following properties:

(a) $x_{n+1} \in T(x_n) \cap X_{i(x_0)}$, for all $n \in \mathbb{N}$;

(b) $d(x_n, x_{n+1}) < a^n b$, for all $n \in \mathbb{N}$.

From (b) we get that $(x_n)_{n \in \mathbb{N}}$ is Cauchy and hence convergent in $X_{i(x_0)}$. Thus there exists $x^* \in X_{i(x_0)}$ (since $X_{i(x_0)}$ is d-closed), such that $x_n \to x^*$ as $n \to +\infty$. Let us show now that $x^* \in F_T$. We have $D(x^*, T(x^*)) \leq d(x^*, x_{n+1}) + D(x_{n+1}, T(x^*)) \leq d(x^*, x_{n+1}) + \rho(T(x_n) \cap X_{i(x_0)}, T(x^*)) \leq d(x^*, x_{n+1}) + ad(x^*, x_n)) \to 0$ as $n \to +\infty$. Hence $x^* \in T(x^*)$. $\square$

A second answer to the open problem mentioned in Section 3 is the following:

Theorem 5.18 *Let (X, d) be a generalized complete metric space and $T : X \to P_{cl}(X)$ be a* *multivalued operator such that for each $x \in X$ and $y \in T(x)$ we have $d(x, y) < +\infty$. Let $X := \bigcup_{i \in I} X_i$ be the canonical decomposition of X. Suppose that T is pseudo a-contractive with respect to $X_{i(x)}$, for each $x \in X$. Then T is a MWP operator.*

Proof. Let $x_0 \in X$ and $x_1 \in T(x)$ such that $d(x_0, x_1) < b < +\infty$, for some $b > 0$. Thus $x_1 \in T(x_0) \cap X_{i(x_0)}$. Hence we have $D(x_1, T(x_1)) \leq \rho(T(x_0) \cap X_{i(x_0)}, T(x_1)) \leq ad(x_0, x_1) < ab$. We obtain that there exists $x_2 \in T(x_1)$ such that $d(x_1, x_2) < ab < +\infty$. Thus $x_2 \in T(x_1) \cap X_{i(x_0)}$. In a similar way, we have $D(x_2, T(x_2)) \leq \rho(T(x_1) \cap X_{i(x_0)}, T(x_2)) \leq ad(x_1, x_2) < a^2 b < +\infty$.

By induction, we obtain a sequence $(x_n)_{n\in\mathbb{N}}$ with the following properties:

(a) $x_{n+1} \in T(x_n) \cap X_{i(x_0)}$, for all $n \in \mathbb{N}$;

(b) $d(x_n, x_{n+1}) < a^n b$, for all $n \in \mathbb{N}$.

From (b) we get that $(x_n)_{n\in\mathbb{N}}$ is Cauchy and hence convergent in $X_{i(x_0)}$ to a certain x^*. As before, we obtain $x^* \in T(x^*)$. Since $x_0 \in X$ and $x_1 \in T(x_0)$ were arbitrarily chosen, we get that T is a MWP operator. □

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