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A Fixed Point Theorem for Certain Operators

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ABSTRACT

We obtain a fixed point theorem for a class of operators. This result is an extension of a similar theorem of Constantin (1994).

RESUMEN

Obtenemos un teorema de punto fijo para una clase de operadores. Este resultado es una extensión de un teorema similar devido a Constantin (1994).

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Most fixed point theorems are proved either by examining the successive iterates of the operator, or by constructing an iteration scheme, such as that of Mann or Ishikawa. In this paper we consider the situation in which the operator is used on the successive iterates of a sequence.

In a recent paper, Constantin [3] obtained a fixed point theorem for a class of operators which are selfmaps of a Banach space X, and which satisfy the condition

$$\|Tx - Ty\| \leq g(\|x-y\|, \|x-Tx\|, \|y-Ty\|) \tag{1}$$

for all $x, y \in X$, where $g : \mathbb{R}_+^3 \to \mathbb{R}_+, g$ is continuous, nondecreasing in each variable, and is such that, if $h(r) := g(r,r,r)$, then $r - h(r)$ is nonnegative and strictly increasing on $\mathbb{R}_+$.

A natural extension of (1) would be: Let Λ denote the set of all continuous functions $g : \mathbb{R}_+^5 \to \mathbb{R}_+$, nondecreasing in each variable and such that, if $h(r) := g(r,r,r,r,r)$, then $h(r) < r$ for each $r > 0$. Define T from X to X satisfying

$$\begin{aligned} \|Tx - Ty\| \leq g(&\|x-y\|, \|x-Tx\|, \|y-Ty\|, \\ &\|x-Ty\|, \|y-Tx\|) \end{aligned} \tag{2}$$

for all $x, y \in X$, some $g \in \Lambda$.

However, a slightly more general extension of (1) is the following. Let X be a Banach space, $g : \mathbb{R}_+^3 \to \mathbb{R}_+, g$ continuous, nondecreasng, and satisfying $g(t) < t$ for each $t > 0$. Let T be a selfmap of X satisfying

$$\|Tx - Ty\| \leq g(M(x,y)), \quad \text{for all} \quad x, y \in X, \tag{3}$$

where

$$\begin{aligned} M(x,y) := \max\{&\|x-y\|, \|x-Tx\|, \|y-Ty\|, \\ &\|x-Ty\|, \|y-Tx\|). \end{aligned}$$

Theorem 1. *Let A satisfy* (3) *and $\{x_n\} \subset X$. Then the following are equivalent:*

(i) $Tx_n - x_n \to 0 \quad$ *as* $\quad n \to \infty$,

(ii) $\{Tx_n - x_n\}$ *is bounded, and* $\{x_n\}$ *converges to a point* p *which is the unique fixed point of* T.

Proof. (i) $\Rightarrow$ (ii). Define $y_n = Tx_n - x_n, \alpha_n = \sup_n\{\|x_m - x_n\| : m \geq n\}$, and $\beta_n = \sup_n\{\|y_m\| : m \geq n\}$. Then $\{\alpha_n\}$ and $\{\beta_n\}$ are nonincreasing nonnegative sequences. Hence $\lim \alpha_n = \alpha \geq 0$ and, from the hypotheses, $\lim y_n = 0$ and $\{y_n\}$ is bounded.

Assume that $\alpha > 0$. From (3), with $m \geq n$,

$$\begin{aligned} \|x_m - x_n\| \leq \|Tx_m - y_m - (Tx_n - y_n)\| &\leq \|Tx_m - Tx_n\| \\ &\quad + \|y_m - y_n\| \\ &\leq g(\max\{\|x_m - x_n\|, \|y_m\|, \|y_n\|, \|x_m - Tx_n\|, \|x_n - Tx_m\|\}) \\ &\quad + 2\beta_n \\ &\leq g(\max\{\alpha_n, \beta_n, \beta_n, \alpha_n + \beta_n, \alpha_n + \beta_n\}) + 2\beta_n. \end{aligned}$$

Thus, $\alpha_n \leq g(\alpha_n+\beta_n)+2\beta_n$. Taking the limit as $n \to \infty$ yields $\alpha \leq g(\alpha) < \alpha$, a contradiction. Therefore $\alpha = 0$ and $\{x_n\}$ is Cauchy, hence convergent to some point p in X.

Since $Tx_n - x_n \to 0$ and $x_n \to p$, it follows that $Tx_n \to p$. Again using (3),

$$\|Tp - Tx_n\| \leq g(\max\{\|p - x_n\|, \|p - Tp\|, \|y_n\|, \|p - Tx_n\|, \|x_n - Tp\|\}).$$

Taking the limit as $n \to \infty$ yields

$$\|Tp - p\| \leq g(\max\{0, \|p - Tp\|, 0, 0, \|p - Tp\|\}) = g(\|p - Tp\|),$$

which implies that $\|p - Tp\| = 0$, or $Tp = p$.

To prove uniqueness, suppose that q is also a fixed point of T. Then, from (3),

$$\begin{aligned} \|p - q\| &= \|Tp - Tq\| \\ &\leq g(\max\{\|p - q\|, 0, 0, \|p - q\|, \|q - p\|\}) \\ &= g(\|p - q\|), \end{aligned}$$

which implies that $p = q$.

(ii) $\Rightarrow$ (i) Using (3),

$$\begin{aligned} \|Tx_n - x_n\| &= \|Tx_n - Tp + Tp - x_n\| \leq \|Tx_n - Tp\| + \|p - x_n\| \\ &\leq g(\max\{\|x_n - p\|, \|x_n - Tx_n\|, \|p - Tp\|, \\ &\qquad \|x_n - Tp\|, \|p - Tx_n\|\}) + \|p - x_n\|. \end{aligned}$$

Taking the lim sup of both sides, since $\{y_n\}$ is bounded, one obtains, with the identification $\gamma = \limsup \|y_n\|$,

$$\gamma \leq g(\max\{0, \gamma, 0, 0, \limsup \|p - Tx_n\|\}) + 0.$$

But $\limsup \|p - Tx_n\| \leq \limsup(\|p - x_n\| + \|x_n - Tx_n\|) = \gamma$. Therefore we have $\gamma \leq g(\gamma)$, which implies that $\gamma = 0$. $\square$

The special case of (3) with $g(t) := kt$ for some $0 \leq k < 1$, and X a metric space is that of Ćirić [2], which was shown in [7] to be one of the most general contractive definitions for which a unique fixed point exists.

In order to prove that a map satisfying (3) has a fixed point, it would be necessary to show that the orbit of some $x \in X$ is bounded, which cannot be implied from (3). However, the following is true.

Theorem 2. *Let X be a complete metric space, T a selfmap of X satisfying*

$$d(Tx, Ty) \leq g(M(x, y)), \qquad \text{for each} \quad x, y \in X, \tag{4}$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}.$$

If there exists a point $x_0 \in X$ with bounded orbit, then T has a unique fixed point in X.

Proof. For any $n \in \mathbb{N}$, $O(x,n) := \{x, Tx, T^2x, \ldots, T^nx\}$, and $\delta(A)$ denotes the diameter of a set A. Let $m,n \in \mathbb{N}, n < m$. Then, from (4), with $x = x_0$,

$$\begin{aligned} d(T^nx, T^mx) &= d(T(T^{n-1}x), T(T^{m-1}x)) \\ &\le g(\max\{d(T^{n-1}x, T^{m-1}x), d(T^{n-1}x, T^nx), d(T^{m-1}x, T^mx), \\ &\qquad d(T^{n-1}x, T^mx), d(T^{m-1}x, T^nx)\} \\ &\le g(\delta[O(T^{n-1}x, n-m+1)]) \\ &\le g(g(\delta[O(T^{n-2}x, n-m+2)]) \\ &\quad \ldots \\ &\le g^n(\delta[O(x,m)]). \end{aligned} \tag{5}$$

It is well known that the hypotheses on g imply that $\lim g^n(t) = 0$ for each $t \ge 0$. Since the orbit of $x = x_0$ is bounded, (5) implies that $\{T^nx\}$ is Cauchy, hence convergent to a point $p \in X$.

Suppose that $p \neq Tp$. Then, from (4),

$$\begin{aligned} d(p,Tp) &\le d(p, T^{n+1}x) + d(T^{n+1}x, Tp) \\ &\le d(p, T^{n+1}x) + g(\max\{d(T^nx, p), d(T^nx, T^{n+1}x), d(p,Tp), \\ &\qquad d(T^nx, p), d(p, T^{n+1}x)\}. \end{aligned}$$

Taking the limit of both sides of the above inequality as $n \to \infty$ yields

$$d(p,Tp) \le g(d(p,Tp)) < d(p,Tp),$$

a contradiction, and $p = Tp$.

Suppose that p and q are fixed points of T, with $p \neq q$. Then, using (4),

$$\begin{aligned} d(p,q) &= d(Tp,Tq) \\ &\le g(\max(\{d(p,q), 0, 0, d(p,q), d(q,p)\} \\ &= g(d(p,q)) < d(p,q), \end{aligned}$$

a contadiction. Therefore $p = q$. □

If T is continuous, then, even with X unbounded, Theorem 2 is a special case of Theorem 3.3 of [4]

If one replaces $M(x,y)$ with

$$m(x,y) := \max\{d(x,y), d(x,Tx), d(y,Ty), [d(x,Ty)+d(y,Tx)]/2\},$$

in Theorem 2, then Theorem 2 is true without the boundedness assumption. See, e.g., Theorem 2.2 of [1].

Most of the recent papers on fixed ont theory, which do not involve fixed point iterations, deal with four maps. For a survey of these results the reader may wish to consult [6] and the references therein.

Let $F(T)$ denote the fixed point set of a mapping T. In [5] it was conjectured that $F(T^n) = F(T)$ for every map T which satisfies a contractive condition that does not include nonexpansive maps. That conjecture was verified in [5] for many such maps. We shall now show that the same is true for maps satisfying (4).

Theorem 3. *Let X be a metric space, T a selfmap of X satisfying* (4) *with $F(T) \neq \emptyset$. Then $F(T^n) = F(T)$ for every integer $n \geq 1$.*

Proof. Since $F(T) \neq \emptyset$. $F(T^n) \neq \emptyset$. Clearly $F(T) \subseteq F(T^n)$. Suppose that $p \in F(T^n)$, for some positive integer n. We shall assume that $n > 1$, since the case for $n = 1$ is trivial. Let i, j be integers, $0 \leq i < j \leq n$. Then, using (4),

$$d(T^i p, T^j p) \leq g(M(T^{i-1}p, T^{j-1}p)) \leq g(\delta[(O(p,n)]).$$

Suppose that $\delta[(O(p,n)] > 0$. Then the above inequality implies that

$$\delta[(O(p,n)] \leq g(\delta[(O(p,n)]) < \delta[(O(p,n)],$$

a contradiction. Therefore $\delta[(O(p,n)] = 0$, and $p \in F(T)$. ☐

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