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Positive Solutions for Elliptic Boundary Value Problems with a Harnack-Like Property

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ABSTRACT

The aim of this paper is to present some existence results of positive solutions for elliptic equations and systems on bounded domains of $\mathbb{R}^N$ $(N \geq 1)$. The main tool is Krasnosel'skii's compression-expansion fixed point theorem.

RESUMEN

El objetivo de este artículo es presentar algunos resultados de existencia de soluciones positivas para ecuaciones elipticas y sistemas sobre dominios acotados de $\mathbb{R}^N$ $(N \geq 1)$. La principal herramienta es el teorema de punto fijo compresión-expansión de Krasnosel'skii.

Key words and phrases: *Positive solution, elliptic boundary value problem, elliptic systems, Harnack-like inequality, Krasnosel'skii's compression-expansion fixed point theorem.*

Math. Subj. Class.: *47H10, 35J65.*

1 Introduction

In this paper, we are concerned with the existence of positive solutions for the elliptic boundary value problem

$$\begin{cases} -\Delta u = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

and for the elliptic system

$$\begin{cases} -\Delta u = \alpha\, g(x,u,v), & \text{in } \Omega, \\ -\Delta v = \beta\, h(x,u,v), & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Here Ω is a bounded regular domain of $\mathbb{R}^N$ $(N \geq 1)$, $f : \overline{\Omega} \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and $g,\ h : \overline{\Omega} \times \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ are continuous functions, and λ, α and β are real parameters. By a *positive* solution of problem (1.1) we mean a function $u \in C^1\left(\overline{\Omega}, \mathbb{R}\right)$ which satisfies (1.1) (with Δu in the sense of distributions), and with $u(x) > 0$ for all $x \in \Omega$. A *positive* solution to problem (1.2) is a vector-valued function $(u,v) \in C^1\left(\overline{\Omega}, \mathbb{R}^2\right)$ satisfying (1.2), with $u, v \geq 0$ and $u + v > 0$ in Ω.

The main assumption will be a global weak Harnack inequality for nonnegative superharmonic functions. By a *superharmonic* function in a domain $\Omega \subset \mathbb{R}^N$ we mean a function $u \in C^1(\Omega, \mathbb{R})$ with $\Delta u \leq 0$ in the sense of distributions, i.e.,

$$\int_\Omega \nabla u \cdot \nabla v \geq 0 \quad \text{for every } v \in C_0^\infty(\Omega, \mathbb{R}) \quad \text{satisfying } v(x) \geq 0 \quad \text{on } \Omega.$$

We shall assume that the following *global weak Harnack inequality* holds:

$$\begin{cases} \text{There exists a compact set } K \subset \Omega \text{ and a number } \eta > 0 \\ \text{such that } u(x) \geq \eta \|u\|_0 \text{ for all } x \in K \\ \text{and every nonnegative superharmonic function} \\ u \in C^1(\overline{\Omega}, \mathbb{R}) \text{ with } u = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.3}$$

Here by $\|u\|_0$ we denote the sup norm in $C\left(\overline{\Omega}, \mathbb{R}\right)$, i.e., $\|u\|_0 = \sup_{x \in \overline{\Omega}} |u(x)|$.

The connection between such type of inequalities and Krasnosel'skii's compression-expansion theorem when applied to boundary value problems was first explained in [4]. Also in [4] (see also [1]), several comments on weak Harnack type inequalities can be found.

By a cone in a Banach space E we mean a closed convex subset $\mathcal{C}$ of E such that $\mathcal{C} \neq \{0\}$, $\lambda\mathcal{C} \subset \mathcal{C}$ for all $\lambda \in \mathbb{R}_+$, and $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$.

Our main tool in proving the existence of positive solutions to problems (1.1) and (1.2) is Krasnosel'skii's compression-expansion theorem [3], [2]:

Theorem 1. *Let E be a Banach space, $\mathcal{C} \subset E$ a cone in E, and assume that $T : \mathcal{C} \longrightarrow \mathcal{C}$ is a completely continuous map such that for some numbers r and R with $0 < r < R$, one of the following conditions is satisfied:*

(i) $\|Tu\| \leq \|u\|$ *for* $\|u\| = r$ *and* $\|Tu\| \geq \|u\|$ *for* $\|u\| = R$,

(ii) $\|Tu\| \geq \|u\|$ *for* $\|u\| = r$ *and* $\|Tu\| \leq \|u\|$ *for* $\|u\| = R$.

Then T has a fixed point with $r \leq \|u\| \leq R$.

2 Existence results for Problem 1.1

In this section, E is the Banach space

$$C_0(\overline{\Omega}, \mathbb{R}) = \{u \in C(\overline{\Omega}, \mathbb{R}) : \ u = 0 \quad \text{on } \partial\Omega\}$$

endowed with norm $\|.\|_0$, and $\mathcal{C}$ is the cone

$$\mathcal{C} = \{u \in C_0(\overline{\Omega}, \mathbb{R}_+) : \ u(x) \geq \eta \|u\|_0 \quad \text{for all } x \in K\}. \tag{2.1}$$

In order to state our results we introduce the notation

$$\begin{aligned} f_0 &= \limsup_{y \to 0^+} \max_{x \in \overline{\Omega}} \frac{f(x,y)}{y} \quad \text{and} \quad \underline{f}_\infty = \liminf_{y \to \infty} \min_{x \in K} \frac{f(x,y)}{y} \\ \underline{f}_0 &= \liminf_{y \to 0^+} \min_{x \in K} \frac{f(x,y)}{y} \quad \text{and} \quad f_\infty = \limsup_{y \to \infty} \max_{x \in \overline{\Omega}} \frac{f(x,y)}{y}. \end{aligned}$$

Also, for a function $h : \overline{\Omega} \to \mathbb{R}$, by $h|_K$ we mean the function $h|_K\,(x) = h(x)$ if $x \in K$ and $h|_K\,(x) = 0$ if $x \in \overline{\Omega} \setminus K$. For example, if 1 is the constant function 1 on $\overline{\Omega}$, then $1|_K\,(x) = 1$ if $x \in K$ and $1|_K\,(x) = 0$ for $x \in \overline{\Omega} \setminus K$.

Theorem 2. *Suppose* (1.3) *holds. Then for each λ satisfying*

$$\frac{1}{\underline{f}_\infty \eta \left\|(-\Delta)^{-1}\, 1|_K\right\|_0} < \lambda < \frac{1}{f_0 \|(-\Delta)^{-1} 1\|_0} \tag{2.2}$$

there exists at least one positive solution of problem (1.1).

Proof. Let λ be as in (2.2) and let $\epsilon > 0$ be such that

$$\frac{1}{(\underline{f}_\infty - \epsilon)\eta \left\|(-\Delta)^{-1}\, 1|_K\right\|_0} \leq \lambda \leq \frac{1}{(f_0 + \epsilon)\|(-\Delta)^{-1} 1\|_0}. \tag{2.3}$$

We know that u is a solution of problem (1.1) if and only if

$$u = \lambda\,(-\Delta)^{-1}Fu$$

where $F : C(\overline{\Omega},\mathbb{R}) \longrightarrow C(\overline{\Omega},\mathbb{R})$, $Fu(x) = f\,(x,u(x))$. Hence, a solution to problem (1.1) is a fixed point of the operator $T : \mathcal{C} \longrightarrow C_0(\overline{\Omega},\mathbb{R})$ given by

$$Tu = \lambda\,(-\Delta)^{-1}Fu.$$

We shall prove that the hypotheses of Theorem 1 are satisfied.

We have that the operator T satisfies

$$\begin{cases} -\Delta(Tu) = \lambda\, f\,(x,u)\,, & \text{in } \Omega, \\ Tu = 0, & \text{on } \partial\Omega. \end{cases}$$

Then by the global weak Harnack inequality (1.3), one has $T(\mathcal{C}) \subset \mathcal{C}$. Moreover, T is completely continuous by the Arzela-Ascoli Theorem.

Furthermore, by the definition of f_0, there exists an $r > 0$ such that

$$f\,(x,u) \leq (f_0+\epsilon)u \quad \text{for} \quad 0 < u \leq r \text{ and } x \in \overline{\Omega}. \tag{2.4}$$

Let $u \in \mathcal{C}$ with $\|u\|_0 = r$. Then using (2.4), the monotonicity of operator $(-\Delta)^{-1}$ and of norm $\|.\|_0$, and (2.3), we obtain

$$\begin{aligned} \|Tu\|_0 &= \lambda\left\|(-\Delta)^{-1}Fu\right\|_0 \\ &\leq \lambda(f_0+\epsilon)\|u\|_0\left\|(-\Delta)^{-1}1\right\|_0 \\ &\leq \|u\|_0. \end{aligned}$$

Hence

$$\|Tu\|_0 \leq \|u\|_0 \quad \text{for } \|u\|_0 = r. \tag{2.5}$$

By the definition of $\underline{f}_\infty$, there is $R > r$ such that

$$f\,(x,u) \geq (\underline{f}_\infty - \epsilon)u \quad \text{for} \quad u \geq \eta R \text{ and } x \in K.$$

Then, if $u \in \mathcal{C}$ with $\|u\|_0 = R$, we have

$$\begin{aligned} \|Tu\|_0 &= \lambda\left\|\,(-\Delta)^{-1}Fu\right\|_0 \\ &\geq \lambda\left\|\,(-\Delta)^{-1}\,(Fu)|_K\right\|_0 \\ &\geq \lambda(\underline{f}_\infty-\epsilon)\eta\|u\|_0\left\|(-\Delta)^{-1}\,1|_K\right\|_0 \\ &\geq \|u\|_0. \end{aligned}$$

Hence

$$\|Tu\|_0 \geq \|u\|_0 \ \text{ for } \ \|u\|_0 = R. \tag{2.6}$$

Inequalities (2.5) and (2.6) show that the expansion condition (i) in Theorem 1 is satisfied. Now Theorem 1 guarantees the existence of a fixed point u of T with $r \leq \|u\|_0 \leq R$. □

Similarly, we have the following result:

Theorem 3. *Suppose* (1.3) *holds. Then for each* λ *satisfying*

$$\frac{1}{\underline{f}_0\eta\,\|(-\Delta)^{-1}\,1|_K\|_0} < \lambda < \frac{1}{f_\infty\|(-\Delta)^{-1}1\|_0} \tag{2.7}$$

there exists at least one positive solution of problem (1.1).

Proof. Let λ be as in (2.7) and let $\epsilon > 0$ be such that

$$\frac{1}{(\underline{f}_0-\epsilon)\eta\,\|(-\Delta)^{-1}\,1|_K\|_0} \le \lambda \le \frac{1}{(f_\infty+\epsilon)\|(-\Delta)^{-1}1\|_0}. \tag{2.8}$$

By the definition of $\underline{f}_0$, there exists an $r > 0$ such that

$$f(x,u) \ge (\underline{f}_0-\epsilon)u \ \text{ for } 0 < u \le r \text{ and } x \in K.$$

If $u \in \mathcal{C}$ and $\|u\|_0 = r$, then

$$\begin{aligned}
\|Tu\|_0 &= \lambda\left\|\,(-\Delta)^{-1}Fu\right\|_0\\
&\ge \lambda\left\|\,(-\Delta)^{-1}\,(Fu)|_K\right\|_0\\
&\ge \lambda(\underline{f}_0-\epsilon)\eta\|u\|_0\,\left\|(-\Delta)^{-1}\,1|_K\right\|_0\\
&\ge \|u\|_0.
\end{aligned}$$

Hence

$$\|Tu\|_0 \ge \|u\|_0 \ \text{ for } \ \|u\|_0 = r. \tag{2.9}$$

By the definition of f_∞, there is $R_0 > 0$ such that

$$f(x,u) \le (f_\infty+\epsilon)u \text{ for } u \ge R_0 \text{ and } x \in \overline{\Omega}.$$

Let M be such that $f(x,u) \le M$ for all $u \in [0,R_0]$ and $x \in \overline{\Omega}$, and let R be such that

$$R > r \text{ and } M \le (f_\infty+\epsilon)\,R.$$

If $u \in \mathcal{C}$ with $\|u\|_0 = R$, then $0 \le u(x) \le (f_\infty+\epsilon)\,R$ for all $x \in \overline{\Omega}$. Consequently, also using (2.8), we obtain

$$\begin{aligned}
\|Tu\|_0 &= \lambda\left\|(-\Delta)^{-1}Fu\right\|_0\\
&\le \lambda(f_\infty+\epsilon)R\left\|(-\Delta)^{-1}1\right\|_0\\
&\le R\\
&= \|u\|_0.
\end{aligned}$$

Hence

$$\|Tu\|_0 \le \|u\|_0 \ \text{ for } \ \|u\|_0 = R. \tag{2.10}$$

Inequalities (2.9) and (2.10) show that the compression condition (ii) in Theorem 1 is satisfied. Now Theorem 1 guarantees the existence of a fixed point u of T with $r \le \|u\|_0 \le R$. ◻

3 Existence results for Problem 1.2

In this section, we are concerned with the existence of positive solutions to the Dirichlet problem (1.2) for elliptic systems.

Here E will be the Banach space $C_0(\overline{\Omega}, \mathbb{R}^2) := C_0(\overline{\Omega}, \mathbb{R}) \times C_0(\overline{\Omega}, \mathbb{R})$ endowed with the norm $\|(.,.)\|_0$ given by

$$\|(u,v)\|_0 = \|u\|_0 + \|v\|_0$$

and the cone in E will be $\mathcal{C} \times \mathcal{C}$, where $\mathcal{C}$ is given by (2.1).

In order to state our results in this section we introduce the notation

$$\begin{aligned} g_0 &= \limsup_{y+z\to 0^+} \max_{x\in\overline{\Omega}} \frac{g(x,y,z)}{y+z} \quad \text{and} \quad \underline{g}_\infty = \liminf_{y+z\to\infty} \min_{x\in K} \frac{g(x,y,z)}{y+z} \\ \underline{g}_0 &= \liminf_{y+z\to 0^+} \min_{x\in K} \frac{g(x,y,z)}{y+z} \quad \text{and} \quad g_\infty = \limsup_{y+z\to\infty} \max_{x\in\overline{\Omega}} \frac{g(x,y,z)}{y+z}. \end{aligned}$$

The limits $h_0, \underline{h}_0, h_\infty$ and $\underline{h}_\infty$ are defined similarly.

Theorem 4. *Suppose* (1.3) *holds. In addition assume that there are numbers* $p, q > 0$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *such that*

$$\frac{1}{\underline{g}_\infty \eta \left\|(-\Delta)^{-1}\, 1|_K\right\|_0} < \alpha < \frac{1}{p\, g_0 \|(-\Delta)^{-1} 1\|_0} \tag{3.1}$$

and

$$\frac{1}{\underline{h}_\infty \eta \left\|(-\Delta)^{-1}\, 1|_K\right\|_0} < \beta < \frac{1}{q\, h_0 \|(-\Delta)^{-1} 1\|_0}. \tag{3.2}$$

Then there exists at least one positive solution (u,v) *of problem* (1.2).

Proof. Let α, β be as in (3.1), (3.2) and let $\epsilon > 0$ be such that

$$\frac{1}{(\underline{g}_\infty - \epsilon)\eta \left\|(-\Delta)^{-1}\, 1|_K\right\|_0} \le \alpha \le \frac{1}{p\,(g_0 + \epsilon)\|(-\Delta)^{-1} 1\|_0}$$

and

$$\frac{1}{(\underline{h}_\infty - \epsilon)\eta \left\|(-\Delta)^{-1}\, 1|_K\right\|_0} \le \beta \le \frac{1}{q\,(h_0 + \epsilon)\|(-\Delta)^{-1} 1\|_0}.$$

It is easily seen that a vector-valued function (u,v) is a solution of problem (1.2) if and only if

$$\begin{aligned} u &= \alpha\, (-\Delta)^{-1} G\,(u,v) \\ v &= \beta\, (-\Delta)^{-1}\, H\,(u,v) \end{aligned}$$

where $G, H : C(\overline{\Omega}, \mathbb{R}^2) \longrightarrow C(\overline{\Omega}, \mathbb{R})$,

$$G(u,v)(x) = g\,(x, u(x), v(x))\,, \quad H\,(u,v)\,(x) = h\,(x, u\,(x)\,, v\,(x))\,.$$

Hence, (u,v) is a positive solution of (1.2) if it is a fixed point of the operator

$$T : \mathcal{C} \times \mathcal{C} \longrightarrow C_0(\overline{\Omega}, \mathbb{R}^2), \;\; T = (T_1, T_2)$$

where

$$T_1\,(u,v) = \alpha\,(-\Delta)^{-1} G\,(u,v)\,, \;\; T_2\,(u,v) = \beta\,(-\Delta)^{-1} H\,(u,v)\,.$$

We shall prove that the hypotheses of Theorem 1 are satisfied.

Clearly the operator $T = (T_1, T_2)$ satisfies

$$\begin{cases} -\Delta(T_1 u) = \alpha\, g\,(x,u,v)\,, & \text{in } \Omega, \\ -\Delta(T_2 v) = \beta\, h\,(x,u,v)\,, & \text{in } \Omega, \\ T_1 u = T_2 v = 0, & \text{on } \partial\Omega. \end{cases}$$

Then by the global weak Harnack inequality (1.3), we have $T(\mathcal{C} \times \mathcal{C}) \subset \mathcal{C} \times \mathcal{C}$. Moreover, T is completely continuous by the Arzela-Ascoli Theorem.

By the definitions of g_0 and h_0, there exists an $r > 0$ with

$$g\,(x,u,v) \leq (g_0 + \epsilon)(u+v) \quad \text{for} \quad u, v \geq 0, \; 0 < u+v \leq r \text{ and } x \in \overline{\Omega}$$

and

$$h\,(x,u,v) \leq (h_0 + \epsilon)(u+v) \quad \text{for} \quad u, v \geq 0, \; 0 < u+v \leq r \text{ and } x \in \overline{\Omega}.$$

Let $(u,v) \in \mathcal{C} \times \mathcal{C}$ with $\|(u,v)\|_0 = r$. We have

$$\begin{aligned} \|T_1\,(u,v)\|_0 &= \alpha\,\left\|(-\Delta)^{-1} G\,(u,v)\right\|_0 \\ &\leq \alpha(g_0+\epsilon)\|u+v\|_0 \left\|(-\Delta)^{-1} 1\right\|_0 \\ &\leq \frac{1}{p}\|u+v\|_0 \\ &\leq \frac{1}{p}(\|u\|_0 + \|v\|_0) \\ &= \frac{1}{p}\|(u,v)\|_0. \end{aligned}$$

Then $\|T_1\,(u,v)\,\|_0 \leq \frac{1}{p}\|(u,v)\|_0$. Similarly, we have

$$\begin{aligned} \|T_2\,(u,v)\|_0 &= \beta\,\left\|(-\Delta)^{-1} H\,(u,v)\right\|_0 \\ &\leq \beta(h_0+\epsilon)\|u+v\|_0 \left\|(-\Delta)^{-1} 1\right\|_0 \\ &\leq \frac{1}{q}\|u+v\|_0 \\ &\leq \frac{1}{q}(\|u\|_0 + \|v\|_0) \\ &= \frac{1}{q}\|(u,v)\|_0. \end{aligned}$$

Thus $\|T_2(u,v)\|_0 \le \frac{1}{q}\|(u,v)\|_0$. Combining the above two inequalities, we obtain

$$\|T(u,v)\|_0 = \|T_1(u,v)\|_0 + \|T_2(u,v)\|_0 \le (\frac{1}{p}+\frac{1}{q})\|(u,v)\|_0 = \|(u,v)\|_0.$$

Next by the definitions of $\underline{g}_\infty$ and $\underline{h}_\infty$, there is $R>0$ such that

$$g(x,u,v) \ge (\underline{g}_\infty - \epsilon)(u+v) \quad \text{for} \quad u,v\ge 0,\ u+v\ge \eta R \text{ and } x\in K$$

and

$$h(x,u,v) \ge (\underline{h}_\infty - \epsilon)(u+v) \quad \text{for} \quad u,v\ge 0,\ u+v\ge \eta R \text{ and } x\in K.$$

Let $(u,v)\in \mathcal{C}\times\mathcal{C}$ with $\|(u,v)\|_0 = R$. Then for each $x\in K$, $u(x)\ge \eta\|u\|_0$ and $v(x)\ge\eta\|v\|_0$. Hence $(u+v)(x)\ge \eta(\|u\|_0+\|v\|_0)$, that is $(u+v)(x)\ge\eta R$ for all $x\in K$. Consequently,

$$G(u,v)(x) \ge (\underline{g}_\infty - \epsilon)(u+v)(x) \quad \text{for all } x\in K.$$

Furthermore

$$\begin{aligned}
\|T_1(u,v)\|_0 &= \alpha\left\|(-\Delta)^{-1}G(u,v)\right\|_0 \\
&\ge \alpha\left\|(-\Delta)^{-1}\, G(u,v)|_K\right\|_0 \\
&\ge \alpha(\underline{g}_\infty-\epsilon)\left\|(-\Delta)^{-1}\,(u+v)|_K\right\|_0 \\
&\ge \alpha(\underline{g}_\infty-\epsilon)\left\|(-\Delta)^{-1}\,u|_K\right\|_0 \\
&\ge \alpha(\underline{g}_\infty-\epsilon)\eta\|u\|_0\left\|(-\Delta)^{-1}\,1|_K\right\|_0 \\
&\ge \|u\|_0.
\end{aligned}$$

Similarly, we have

$$\|T_2(u,v)\|_0 \ge \|v\|_0.$$

The above two inequalities give

$$\|T(u,v)\|_0 \ge \|(u,v)\|_0.$$

Thus condition (i) in Theorem 1 is satisfied. Now Theorem 1 guarantees the existence of a fixed point (u,v) of T with $r\le\|(u,v)\|_0\le R$. □

In a similar way, one can prove:

Theorem 5. *Suppose* (1.3) *holds. In addition assume that there are numbers* $p,q>0$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *such that*

$$\frac{1}{\underline{g}_0\eta\left\|(-\Delta)^{-1}\,1|_K\right\|_0} < \alpha < \frac{1}{p\, g_\infty\|(-\Delta)^{-1}1\|_0}$$

and

$$\frac{1}{\underline{h}_0\eta\left\|(-\Delta)^{-1}\,1|_K\right\|_0} < \beta < \frac{1}{q\, h_\infty\|(-\Delta)^{-1}1\|_0}.$$

Then there exists at least one positive solution (u,v) *of problem* (1.2).

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References

[1] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 1983.

[2] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer, New York, 2003.

[3] M.A. Krasnoselskii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Cambridge University Press, New York, 1964.

[4] R. Precup, *Positive solutions of semi-linear elliptic problems via Krasnoselskii type theorems in cones and Harnack's inequality*, Mathematical Analysis and Applications, AIP Conf. Proc., 835, Amer. Inst. Phys., Melville, NY, 2006, 125–132.