CUBO A Mathematical Journal
Vol.10, N$^{\underline{o}}$03, (21–41). October 2008

Multiple Solutions for Doubly Resonant Elliptic Problems Using Critical Groups

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ABSTRACT

We consider a semilinear elliptic equation, with a right hand side nonlinearity which may grow linearly. Throughout we assume a double resonance at infinity in the spectral interval $[\lambda_1, \lambda_2]$. In this paper, we can also have resonance at zero or even double resonance in the order interval $[\lambda_m, \lambda_{m+1}]$, $m \geq 2$. Using Morse theory and in particular critical groups, we prove two multiplicity theorems.

RESUMEN

Nosotros consideramos una ecuación semilinear eliptica con una no-linealidad la cual puede crecer linealmente. Asumimos una doble resonancia en infinito en el intervalo espectral $[\lambda_1, \lambda_2]$. En este artículo, podemos también tener resonancia en cero o incluso doble resonancia en el intervalo ordenado $[\lambda_m, \lambda_{m+1}]$, $m \geq 2$. Usando teoria de Morse y en particular grupos críticos, provamos dos teoremas de mulplicidad.

Key words and phrases: *Double resonance, C-condition, critical groups, critical point of mountain pass-type, Poincare-Hopf formula.*

Math. Subj. Class.: *35J20, 35J25.*

1 Introduction

Let $Z \subseteq \mathbb{R}^{\mathbb{N}}$ be a bounded domain with a C^2-boundary ∂Z. We consider the following semilinear elliptic problem:

$$\left\{ \begin{array}{l} -\triangle x(z) = \lambda_1 x(z) + f(z, x(z)) \text{ a.e. on } Z, \\ x|_{\partial Z} = 0. \end{array} \right\} \tag{1.1}$$

Here $\lambda_1 > 0$ is the principal eigenvalue of $(-\triangle, H_0^1(Z))$. Assume that

$$\lim_{|x| \to \infty} \frac{f(z, x)}{x} = 0 \text{ uniformly for a.a. } z \in Z. \tag{1.2}$$

The problem (1.1) is resonant at infinity with respect to the principal eigenvalue $\lambda_1 > 0$. Resonant problems, were first studied by Landesman-Lazer [7], who assumed a bounded nonlinearity and introduced the well-known sufficient asymptotic solvability conditions, which carry their name (the LL-conditions for short). We can be more general and instead of (1.2), assume only that

$$\liminf_{|x| \to \infty} \frac{f(z, x)}{x} \text{ and } \limsup_{|x| \to \infty} \frac{f(z, x)}{x}$$

belong in the interval $[0, \lambda_2 - \lambda_1]$ uniformly for a.a. $z \in Z$, with λ_2 ($\lambda_2 > \lambda_1$) being the second eigenvalue of $(-\triangle, H_0^1(Z))$. In this more general setting, the nonlinearity $f(z, x)$ need not be bounded. This more general situation was examined by Berestycki-De Figueiredo [2], Landesman-Robinson-Rumbos [8], Nkashama [11], Robinson [13],[14], Rumbos [15] and Su [16]. From these works, Berestycki-De Figueiredo [2], Nkashama [11], Robinson [13] and Rumbos [15], prove existence theorems in a double resonance setting (i.e. asymptotically at $\pm\infty$, we have complete interaction of the "slope" $\frac{f(z,x)}{x}$ with both ends of the spectral interval $[0, \lambda_2 - \lambda_1]$; see Berestycki-De Figueiredo [2] who coined the term "double resonance" and Robinson [13]) or in a one-sided resonance setting (i.e. the "slope" $\frac{f(z,x)}{x}$ is not allowed to cross $\lambda_2 - \lambda_1$; see Nkashama [11] and Rumbos [15]). Multiplicity results were proved by Landesman-Robinson-Rumbos [8] (one-sided resonant problems) and by Robinson [14] and Su [16] (doubly resonant problems).

In this paper, we extend the work of Landesman-Robinson-Rumbos [8] and partially extend and complement the works of Robinson [14] and Su [16], by covering cases which are not included in their multiplicity results.

2 Mathematical background

We start by recalling some basic facts about the following weighted linear eigenvalue problem:

$$\left\{ \begin{array}{l} -\triangle u(z) = \widehat{\lambda} m(z) u(z) \text{ a.e. on } Z, \\ u|_{\partial Z} = 0, \ \ \widehat{\lambda} \in \mathbb{R}. \end{array} \right\} \tag{2.1}$$

Here $m \in L^\infty(Z)_+ = \{m \in L^\infty(Z) : m(z) \geq 0 \text{ a.e. on } Z\}$, $m \neq 0$ (the weight function). By an eigenvalue of (2.1), we mean a real number $\widehat{\lambda}$, for which problem (2.1) has a nontrivial solution $u \in H_0^1(Z)$. It is well-known (see for example Gasinski-Papageorgiou [5]), that problem (2.1) (or equivalently that $(-\triangle, H_0^1(Z), m)$), has a sequence $\{\widehat{\lambda}_k(m)\}_{k\geq 1}$ of distinct eigenvalues, $\widehat{\lambda}_1(m) > 0$ and $\widehat{\lambda}_k(m) \to +\infty$ as $k \to +\infty$. Moreover, $\widehat{\lambda}_1(m) > 0$ is simple (i.e. the corresponding eigenspace $E(\widehat{\lambda}_1)$ is one-dimensional). Also we can find an orthonormal basis $\{u_n\}_{n\geq 1} \subseteq H_0^1(Z) \cap C^\infty(Z)$ for the Hilbert space $L^2(Z)$ consisting of eigenfunctions corresponding to the eigenvalues $\{\widehat{\lambda}_k(m)\}_{k\geq 1}$. Note that $\{u_n\}_{n\geq 1}$ is also an orthogonal basis for the Hilbert space $H_0^1(Z)$. Moreover, since by hypothesis ∂Z is a C^2-manifold, then $u_n \in C^2(\overline{Z})$ for all $n \geq 1$. For every $k \geq 1$, by $E(\widehat{\lambda}_k)$ we denote the eigenspace corresponding to the eigenvalue $\widehat{\lambda}_k(m)$. This space has the so-called "unique continuation property", namely, if $u \in E(\widehat{\lambda}_k)$ is such that it vanishes on a set of positive measure, then $u(z) = 0$ for all $z \in \overline{Z}$. We set

$$\begin{aligned} \overline{H}_k &= \overset{k}{\underset{i=1}{\oplus}} E(\widehat{\lambda}_i) \\ \text{and } \ \widehat{H}_{k+1} &= \overline{\underset{i\geq k+1}{\oplus} E(\lambda_i)} = \overline{H}_k^{\perp}, \ \ k \geq 1. \end{aligned}$$

We have the orthogonal direct sum decomposition

$$H_0^1(Z) = \overline{H}_k \oplus \widehat{H}_{k+1}.$$

Using these spaces, we can have useful variational characterizations of the eigenvalues $\{\widehat{\lambda}_k(m)\}_{k\geq 1}$ using the Rayleigh quotient. Namely we have:

$$\widehat{\lambda}_1(m) = \min\left[\frac{\|Du\|_2^2}{\int_Z mu^2 dz} : u \in H_0^1(Z), u \neq 0\right]. \tag{2.2}$$

In (2.2) the minimum is attained on $E(\widehat{\lambda}_1)\backslash\{0\}$. By $u_1 \in C_0^2(\overline{Z})$, we denote the principal eigenfunction satisfying $\int_Z mu_1^2 dz = 1$. For $k \geq 2$, we have

$$\widehat{\lambda}_k(m) = \max\left[\frac{\|D\overline{u}\|_2^2}{\int_Z m\overline{u}^2 dz} : \overline{u} \in \overline{H}_k, \overline{u} \neq 0\right] \tag{2.3}$$

$$= \min\left[\frac{\|D\widehat{u}\|_2^2}{\int_Z m\widehat{u}^2 dz} : \widehat{u} \in \widehat{H}_k, \widehat{u} \neq 0\right]. \tag{2.4}$$

In (2.3) (resp.(2.4)), the maximum (resp.minimum) is attained on $E(\widehat{\lambda}_k)$. From these variational characterizations of the eigenvalues and the unique continuation property of the eigenspaces $E(\widehat{\lambda}_k)$, we see that the eigenvalues $\{\widehat{\lambda}_k(m)\}_{k\geq 1}$ have the following strict monotonicity property:

"If $m_1, m_2 \in L^\infty(Z)_+$, $m_1(z) \leq m_2(z)$ a.e. on Z and $m_1 \neq m_2$, then $\widehat{\lambda}_k(m_2) < \widehat{\lambda}_k(m_1)$ for all $k \geq 1$."

If $m \equiv 1$, then we simply write λ_k for all $k \geq 1$ and we have the full-spectrum of $(-\triangle, H_0^1(Z))$.

Let H be a Hilbert space and $\varphi \in C^1(H)$. We say that φ satisfies the "Cerami condition" (the C-condition for short), if the following is true:"every sequence $\{x_n\}_{n\geq 1} \subseteq H$ such that $|\varphi(x_n)| \leq M_1$ for some $M_1 > 0$, all $n \geq 1$ and $(1 + \|x_n\|)\varphi'(x_n) \to 0$ in H^* as $n \to \infty$, has a strongly convergent subsequence".

This condition is a weakened version of the well-known Palais-Smale condition (PS-condition for short). Bartolo-Benci-Fortunato [1], showed that the C-condition suffices to prove a deformation theorem and from this produce minimax expressions for the critical values of the functional φ.

For every $c \in \mathbb{R}$, let

$$\begin{aligned} \varphi^c &= \{x \in X : \varphi \leq c\} \text{ (the sublevel set at } c \text{ of } \varphi), \\ K &= \{x \in x : \varphi'(x) = 0\} \text{ (the set of critical points of } \varphi) \\ \text{and } K_c &= \{x \in K : \varphi(x) = c\} \text{ (the critical points of } \varphi \text{ at level } c). \end{aligned}$$

If X is a Hausdorff topological space and Y a subspace of it, for every integer $n \geq 0$, by $H_n(X, Y)$ we denote the n^{th}-relative singular homology group with integer coefficients. The critical groups of φ at an isolated critical point $x_0 \in H$ with $\varphi(x_0) = c$, are defined by

$$C_n(\varphi, x_0) = H_n(\varphi^c \cap U, (\varphi^c \cap U)\backslash\{x_0\}),$$

where U is a neighborhood of x_0 such that $K \cap \varphi^c \cap U = \{x_0\}$. By the excision property of singular homology theory, we see that the above definition of critical groups, is independent of U (see for example Mawhin-Willem [10]).

Suppose that $-\infty < \inf \varphi(K)$. Choose $c < \inf \varphi(K)$. The critical groups at infinity, are defined by

$$C_k(\varphi, \infty) = H_k(H, \varphi^c) \text{ for all } k \geq 0.$$

If K is finite, then the Morse-type numbers of φ, are defined by

$$M_k = \sum_{x \in K} \mathrm{rank} C_k(\varphi, x).$$

The Betti-type numbers of φ, are defined by

$$\beta_k = \mathrm{rank} C_k(\varphi, \infty).$$

By Morse theory (see Chang [4] and Mawhin-Willem [10]), we have

$$\sum_{k=0}^{m} (-1)^{m-k} M_k \geq \sum_{k=0}^{m} (-1)^{m-k} \beta_k$$
$$\text{and} \ \sum_{k \geq 0} (-1)^k M_k = \sum_{k \geq 0} (-1)^k \beta_k.$$

From the first relation, we deduce that $\beta_k \leq M_k$ for all $k \geq 0$. Therefore, if $\beta_k \neq 0$ for some $k \geq 0$, then φ must have a critical point $x \in H$ and the critical group $C_k(\varphi, x)$ is nontrivial. The second relation (the equality), is known as the "Poincare-Hopf formula". Finally, if $K = \{x_0\}$, then $C_k(\varphi, \infty) = C_k(\varphi, x_0)$ for all $k \geq 0$.

3 Multiplicity of solutions

The hypotheses on the nonlinearity $f(z, x)$ are the following:

$\underline{H(f)}$: $f : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(z, 0) = 0$ a.e. on Z and

- **(i)** for all $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable;
- **(ii)** for almost all $z \in Z$, $f(z, \cdot) \in C^1(\mathbb{R})$;
- **(iii)** $|f'_x(z, x)| \leq c(1 + |x|^r), \ r < \frac{4}{N-2}, c > 0.$
- **(iv)** $0 \leq \liminf_{|x| \to \infty} \frac{f(z,x)}{x} \leq \limsup_{|x| \to \infty} \frac{f(z,x)}{x} \leq \lambda_2 - \lambda_1$ uniformly for a.a. $z \in Z$;
- **(v)** suppose that $\|x_n\| \to \infty$,
  - (i) if $\frac{\|x_n^0\|}{\|x_n\|} \to 1$, $x_n = x_n^0 + \widehat{x}_n$ with $x_n^0 \in E(\lambda_1) = \overline{H}_1$, $\widehat{x}_n \in \widehat{H}_2$, then there exist $\gamma_1 > 0$ and $n_1 \geq 1$ such that

$$\int_Z f(z, x_n(z)) x_n^0(z) dz \geq \gamma_1 \ \text{ for all } \ n \geq n_1;$$

  - (ii) if $\frac{\|x_n^0\|}{\|x_n\|} \to 1$, $x_n = x_n^0 + \widehat{x}_n$ with $x_n^0 \in E(\lambda_2)$, $\widehat{x}_n \in W = E(\lambda_2)^{\perp}$, then there exist $\gamma_2 > 0$ and $n \geq 1$ such that

$$\int_Z (f(z, x_n(z)) - (\lambda_2 - \lambda_1) x_n(z)) x_n^0(z) dz \leq -\gamma_2 \ \text{ for all } \ n \geq n_2;$$

(vi) if $F(z,x) = \int_0^x f(z,s)ds$, then there exist $\eta \in L^\infty(Z)$ and $\delta > 0$, such that $\eta(z) \leq 0$ a.e. on Z with strict inequality on a set of positive measure and

$$F(z,x) \leq \frac{\eta(z)}{2} x^2 \text{ for a.a. } z \in Z \text{ and all } |x| \leq \delta.$$

Remark 3.1. *Hypothesis $H(f)(iv)$ implies that asymptotically at $\pm\infty$, we have double resonance. Hypothesis $H(f)(v)$ is a generalized LL-condition. Similar conditions can be found in the works of Landesman-Robinson-Rumbos [8], Robinson [13],[14] and Su [16]. Consider a C^2-function $x \to F(x)$ which in a neighborhood of zero equals $x^4 - sinx^2$, while for $|x|$ large (say $|x| \geq M > 0$), $F(x) = c|x|^{\frac{3}{2}}, c > 0$. If $f(x) = F'(x)$, then $f \in C^1(\mathbb{R})$ satisfies hypothesis $H(f)$ above. To verify the generalized LL-condition in hypothesis $H(f)(v)$, we use Lemma 2.1 of Su-Tang [17]. Similarly we can consider if near the origin, $F(x) = \frac{1}{2}x^2 - tan^{-1}x^2$ or $F(x) = -cosx^2$. This second case is interesting because then $f(x) = 2xsinx^2$ and $f'(x) = 2sinx^2 + 4x^2cosx^2$. So $f'(0) = 0$. This example, which is covered by hypotheses $H(f)$, illustrates that our framework of analysis incorporates also problems with resonance at zero with respect to $\lambda_1 > 0$ (double-double resonance). This is not possible in the setting of Landesman-Robinson-Rumbos [8] (see Theorem 2 in [8]). Also such a potential function is not covered by the multiplicity results of Robinson [14] (theorem 2) and Su [16] (Theorem 2).*

We consider the Euler functional for problem (1.1), $\varphi : H_0^1(Z) \to \mathbb{R}$ defined by

$$\varphi(x) = \frac{1}{2}\|Dx\|_2^2 - \frac{\lambda_1}{2}\|x\|_2^2 - \int_Z F(z,x(z))dz \text{ for all } x \in H_0^1(Z).$$

It is well-known that $\varphi \in C^2(H_0^1(Z))$ and if by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair $(H_0^1(Z), H^{-1}(Z) = H_0^1(Z)^*)$, we have

$$\langle \varphi'(x), y \rangle = \int_Z (Dx, Dy)_{\mathbb{R}^N} dz - \lambda_1 \int_Z xydz - \int_Z f(z,x(z))y(z)dz$$
$$\text{and } \varphi''(x)(u,v) = \int_Z (Du, Dv)_{\mathbb{R}^N} dz - \lambda_1 \int_Z uvdz - \int_Z f'(z,x(z))u(z)v(z)dz$$

for all $x, y, u, v \in H_0^1(Z)$.

Proposition 3.2. *If hypotheses $H(f)$ hold then φ satisfies the C-condition.*

Proof. Let $\{x_n\}_{n\geq 1} \subseteq H_0^1(Z)$ be a sequence such that

$$(1 + \|x_n\|)\varphi'(x_n) \to 0 \text{ as } n \to \infty.$$

We will show that $\{x_n\}_{n\geq 1} \subseteq H_0^1(Z)$ is bounded. We argue indirectly. Suppose that $\{x_n\}_{n\geq 1} \subseteq H_0^1(Z)$ is unbounded. We may assume that $\|x_n\| \to \infty$. Let $y_n = \frac{x_n}{\|x_n\|}$, $n \geq 1$. By passing to a suitable subsequence if necessary, we may assume that

$$y_n \xrightarrow{w} y \text{ in } H_0^1(Z),\ y_n \to y \text{ in } L^2(Z),\ y_n(z) \to y(z) \text{ a.e. on } Z$$
$$\text{and } |y_n(z)| \leq k(z) \text{ a.e. on } Z, \text{ for all } n \geq 1, \text{ with } k \in L^2(Z)_+.$$

Hypotheses $H(f)(iii)$ and (iv), imply that

$$|f(z,x)| \leq a(z) + c|x| \text{ for a.a. } z \in Z, \text{ all } x \in \mathbb{R}, \text{ with } a \in L^\infty(Z)_+, c > 0,$$

$$\Rightarrow \frac{|f(z,x_n(z))|}{\|x_n\|} \leq \frac{a(z)}{\|x_n\|} + c|y_n(z)| \text{ for a.a. } z \in Z, \text{ all } n \geq 1, \tag{3.1}$$
$$\Rightarrow \left\{ \frac{f(\cdot,x_n(\cdot))}{\|x_n\|} \right\}_{n\geq 1} \subseteq L^2(Z) \text{ is bounded.}$$

Thus we may assume that

$$\frac{f(\cdot,x_n(\cdot))}{\|x_n\|} \xrightarrow{w} h \text{ in } L^2(Z) \text{ as } n \to \infty.$$

For every $\varepsilon > 0$ and $n \geq 1$, we set

$$C^+_{\varepsilon,n} = \{z \in Z : x_n(z) > 0, \ -\varepsilon \leq \frac{f(z,x_n(z))}{x_n(z)} \leq \lambda_2 - \lambda_1 + \varepsilon\}$$
$$\text{and } C^-_{\varepsilon,n} = \{z \in Z : x_n(z) < 0, \ -\varepsilon \leq \frac{f(z,x_n(z))}{x_n(z)} \leq \lambda_2 - \lambda_1 + \varepsilon\}$$

Note that $x_n(z) \to +\infty$ a.e. on $\{y > 0\}$ and $x_n(z) \to -\infty$ a.e. on $\{y < 0\}$. Then by virtue of hypothesis $H(f)(iv)$, we have

$$\chi_{C^+_{\varepsilon,n}}(z) \to \chi_{\{y>0\}}(z) \text{ and } \chi_{C^-_{\varepsilon,n}}(z) \to \chi_{\{y<0\}}(z) \text{ a.e. on } Z.$$

Using the dominated convergent theorem, we see that

$$\|(1-\chi_{C^+_{\varepsilon,n}})\frac{f(\cdot,x_n(\cdot))}{\|x_n\|}\|_{L^2(\{y>0\})} \to 0$$
$$\text{and } \|(1-\chi_{C^-_{\varepsilon,n}})\frac{f(\cdot,x_n(\cdot))}{\|x_n\|}\|_{L^2(\{y<0\})} \to 0 \text{ as } n \to \infty.$$

It follows that

$$\chi_{C^+_{\varepsilon,n}}(\cdot)\frac{f(\cdot,x_n(\cdot))}{\|x_n\|} \xrightarrow{w} h \text{ in } L(\{y > 0\})$$
$$\text{and } \chi_{C^-_{\varepsilon,n}}(\cdot)\frac{f(\cdot,x_n(\cdot))}{\|x_n\|} \xrightarrow{w} h \text{ in } L(\{y < 0\}) \text{ as } n \to \infty.$$

From the definitions of the sets $C^+_{\varepsilon,n}$ and $C^-_{\varepsilon,n}$ we have

$$-\varepsilon y_n(z) \leq \frac{f(z,x_n(z))}{\|x_n\|} = \frac{f(z,x_n(z))}{x_n(z)} y_n(z) \leq (\lambda_2 - \lambda_1 + \varepsilon) y_n(z) \text{ a.e. on } C^+_{\varepsilon,n}$$

and

$$-\varepsilon y_n(z) \geq \frac{f(z,x_n(z))}{\|x_n\|} = \frac{f(z,x_n(z))}{x_n(z)} y_n(z) \geq (\lambda_2 - \lambda_1 + \varepsilon) y_n(z) \text{a.e. on } C^-_{\varepsilon,n}.$$

Passing to the limit as $n \to \infty$, using Mazur's lemma and recalling that $\varepsilon > 0$ is arbitrary, we obtain

$$0 \le h(z) \le (\lambda_2 - \lambda_1) y(z) \text{ a.e. on } \{y > 0\} \tag{3.2}$$
$$\text{and } 0 \ge h(z) \ge (\lambda_2 - \lambda_1) y(z) \text{ a.e. on } \{y < 0\}. \tag{3.3}$$

Moreover, from (3.1) it is clear that

$$h(z) = 0 \text{ a.e. on } \{y = 0\}. \tag{3.4}$$

From (3.2), (3.3) and (3.4), it follows that

$$h(z) = g(z)y(z) \text{ a.e. on } Z,$$

where $g \in L^\infty(Z)_+$, $0 \le g(z) \le \lambda_2 - \lambda_1$ a.e. on Z.

Recall that by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair $(H_0^1(Z), H^{-1}(Z))$.

Let $A \in \mathcal{L}(H_0^1(Z), H^{-1}(Z))$ be defined by

$$\langle A(x), y \rangle = \int_Z (Dx, Dy)_{\mathbb{R}^N} dz \text{ for all } x, y \in H_0^1(Z).$$

Also let $N : L^2(Z) \to L^2(Z)$ be the Nemitskii operator corresponding to the nonlinearity $f(z, x)$, i.e.

$$N(x)(\cdot) = f(\cdot, x(\cdot)) \text{ for all } x \in L^2(Z).$$

Because of (3.1), by Krasnoselskii's theorem, we know that N is continuous and bounded. Moreover, exploiting the compact embedding of $H_0^1(Z)$ into $L^2(Z)$, we see that N is completely continuous (hence compact too) as a map from $H_0^1(Z)$ into $L^2(Z)$ (see for example Gasinski-Papageorgiou [5], pp.267-268). We have

$$\varphi'(x_n) = A(x_n) - \lambda_1 x_n - N(x_n) \text{ for all } n \ge 1.$$

From the choice of the sequence $\{x_n\}_{n\ge1} \subseteq H_0^1(Z)$, we know that

$$|\langle \varphi'(x_n), v \rangle| \le \varepsilon_n \text{ for all } v \in H_0^1(Z) \text{ with } \varepsilon_n \downarrow 0,$$
$$\Rightarrow \left| \langle A(y_n) - \lambda_1 y_n - \frac{N(x_n)}{\|x_n\|}, v \rangle \right| \le \frac{\varepsilon_n}{\|x_n\|} \text{ for all } n \ge 1. \tag{3.5}$$

Let $v = y_n - y \in H_0^1(Z)$, $n \ge 1$. Then

$$\left| \langle A(y_n), y_n - y \rangle - \lambda_1 \int_Z y_n(y_n - y) dz - \int_Z \frac{N(x_n)}{\|x_n\|}(y_n - y) dz \right| \le \frac{\varepsilon_n}{\|x_n\|} \text{ for all } n \ge 1. \tag{3.6}$$

Evidently

$$\int_Z y_n(y_n - y)dz \to 0 \text{ and } \int_Z \frac{N(x_n)}{\|x_n\|}(y_n - y) \to 0 \text{ as } n \to \infty.$$

So from (3.6), we infer that

$$\langle A(y_n), y_n - y\rangle \to 0. \tag{3.7}$$

We have $A(y_n) \xrightarrow{w} A(y)$ in $H^{-1}(Z)$. From (3.7) it follows that

$$\begin{aligned}&\langle A(y_n), y_n\rangle \to \langle A(y), y\rangle,\\ \Rightarrow&\|Dy_n\|_2 \to \|Dy\|_2.\end{aligned}$$

Also $Dy_n \xrightarrow{w} Dy$ in $L^2(Z, \mathbb{R}^{\mathbb{N}})$. Since the Hilbert space $L^2(Z, \mathbb{R}^{\mathbb{N}})$ has the Kadec-Klee property, we deduce that

$$Dy_n \to Dy \text{ in } L^2(Z, \mathbb{R}^{\mathbb{N}}) \Rightarrow y_n \to y \text{ in } H_0^1(Z), \text{ i.e. } \|y\| = 1,\ y \neq 0.$$

We return to (3.5) and we pass to the limit as $n \to \infty$. We obtain

$$\begin{aligned}&\langle A(y) - \lambda_1 y - gy, v\rangle = 0 \text{ for all } v \in H_0^1(Z),\\ \Rightarrow&A(y) = (\lambda_1 + g)y \text{ in } H^{-1}(Z),\\ \Rightarrow& -\triangle y(z) = (\lambda_1 + g(z))y(z) \text{ a.e. on } Z,\ y|_{\partial Z} = 0.\end{aligned} \tag{3.8}$$

We distinguish three cases for problem (3.8) depending on where the function $g \in L^\infty(Z)_+$ stands in the interval $[0, \lambda_2 - \lambda_1]$.

<u>Case 1:</u> $g(z) = 0$ a.e. on Z.

Then from (3.8), we have

$$\begin{aligned}&-\triangle y(z) = \lambda_1 y(z) \text{ a.e. on } Z,\ \ y|_{\partial Z} = 0,\\ \Rightarrow&y \in E(\lambda_1),\ y \neq 0.\end{aligned}$$

We consider the orthogonal direct sum decomposition $H_0^1(Z) = E(\lambda_1) \oplus \widehat{H}_2$, $\widehat{H}_2 = E(\lambda_1)^\perp$.

Then for every $n \geq 1$, we have

$$x_n = x_n^0 + \widehat{x}_n \text{ and } x_n^0 \in E(\lambda_1),\ \widehat{x}_n \in \widehat{H}_2.$$

We have $y_n = y_n^0 + \widehat{y}_n$, with

$$y_n^0 = \frac{x_n^0}{\|x_n\|} \in E(\lambda_1) \text{ and } \widehat{y}_n = \frac{\widehat{x}_n}{\|x_n\|} \in \widehat{H}_2 \text{ for all } n \geq 1.$$

Since $y \in E(\lambda_1)$, $\|y\| = 1$, we have

$$\frac{\|x_n^0\|}{\|x_n\|} \to 1 \text{ as } n \to \infty.$$

Recall that

$$\left|\langle A(x_n), v\rangle - \lambda_1 \int_Z x_n v dz - \int_Z N(x_n) v dz\right| \le \varepsilon_n \text{ for all } v \in H_0^1(Z).$$

Let $v = x_n^0 \in H_0^1(Z)$. We have

$$\left|\|Dx_n^0\|_2^2 - \lambda_1 \|x_n^0\|_2^2 - \int_Z f(z, x_n(z)) x_n^0(z) dz\right| \le \varepsilon_n,$$
$$\Rightarrow \int_Z f(z, x_n(z)) x_n^0(z) dz \le \varepsilon_n \text{ (see (2.2)) for all } n \ge 1. \tag{3.9}$$

But by virtue of hypothesis $H(f)(v)$

$$0 < \gamma_1 \le \int_Z f(z, x(z)) x_n^0(z) dz \text{ for all } n \ge n_1. \tag{3.10}$$

Comparing (3.9) and (3.10), we reach a contradiction.

Case 2: $g(z) = \lambda_2 - \lambda_1$ a.e. on Z.

In this case, from (3.8) we have

$$\begin{aligned} &-\triangle y(z) = \lambda_2 y(z) \text{ a.e. on Z}, \; y|\partial Z = 0, \\ \Rightarrow &y \in E(\lambda_2), \; y \neq 0. \end{aligned}$$

Now we consider the orthogonal direct sum decomposition $H_0^1(Z) = E(\lambda_2) \oplus W$, with $W = E(\lambda_2)^\perp$. Then

$$x_n = x_n^0 + \widehat{x}_n \text{ with } x_n^0 \in E(\lambda_2), \; \widehat{x}_n \in W, \; n \ge 1.$$

Since $y \in E(\lambda_2)$, $\|y\| = 1$, we have

$$\frac{\|x_n^0\|}{\|x_n\|} \to 1 \text{ as } n \to \infty. \tag{3.11}$$

We have

$$|\langle A(x_n), v\rangle - \lambda_1 \int_Z x_n v dz - \int_Z f(z, x_n(z)) v(z) dz| \le \varepsilon_n$$
$$\text{for all } v \in H_0^1(Z), \text{ with } \varepsilon_n \downarrow 0.$$

Let $v = x_n^0$. Then

$$\left| \|Dx_n^0\|_2^2 - \lambda_1 \|x_n^0\|_2^2 - \int_Z f(z, x_n(z)) x_n^0(z) dz \right| \leq \varepsilon_n,$$

$$\Rightarrow \left| \|Dx_n^0\|_2^2 - \lambda_2 \|x_n^0\|_2^2 - \int_Z (f(z, x_n(z)) - (\lambda_2 - \lambda_1) x_n(z)) x_n^0(z) dz \right| \leq \varepsilon_n,$$

$$\Rightarrow \int_Z (f(z, x_n(z)) - (\lambda_2 - \lambda_1) x_n(z)) x_n^0(z) dz \geq -\varepsilon_n \ \text{ (see (2.3) and (2.4))}. \tag{3.12}$$

But again hypothesis $H(f)(v)$ implies

$$0 > -\gamma_2 \geq \int_Z (f(z, x_n(z)) - (\lambda_2 - \lambda_1) x_n(z)) x_n^0(z) dz \ \text{ for all } \ n \geq n_2. \tag{3.13}$$

Comparing (3.12) and (3.13) we reach a contradiction.

<u>Case 3:</u> $0 \leq g(z) \leq \lambda_2 - \lambda_1$ a.e. on Z with $g \neq 0$, $g \neq \lambda_2 - \lambda_1$.

Note that

$$\lambda_1 \leq \lambda_1 + g(z) \leq \lambda_2 \ \text{ a.e. on } \ Z$$

and the inequalities are strict on sets (in general different) of positive measure. Exploiting the strict monotonicity property of the eigenvalues of $(-\triangle, H_0^1(Z), m)$ on the weight function m (see Section 2), we have

$$\widehat{\lambda}_1(\lambda_1 + g) < \widehat{\lambda}_1(\lambda_1) = 1$$
$$\text{and } \ \widehat{\lambda}_2(\lambda_1 + g) > \widehat{\lambda}_2(\lambda_2) = 1.$$

Combining this with (2.2), we see that $y = 0$, a contradiction to the fact that $\|y\| = 1$.

So in all these cases we have reached a contradiction. This means that $\{x_n\}_{n \geq 1}$ is bounded and so we may assume (at least for a subsequence) that

$$x_n \xrightarrow{w} x \ \text{ in } \ H_0^1(Z), \ x_n \to x \ \text{ in } \ L^2(Z), \ x_n(z) \to x(z) \ \text{ a.e. on } \ Z$$
$$\text{and } \ |x_n(z)| \leq k(z) \ \text{ a.e. on } \ Z \ \text{ for all } \ n \geq 1, \ \text{ with } \ k \in L^2(Z)_+.$$

Recall that

$$\left| \langle A(x_n), x_n - x \rangle - \lambda_1 \int_Z x_n(x_n - x) dz - \int_Z f(z, x_n(z))(x_n - x) dz \right| \leq \varepsilon_n.$$

Since

$$\int_Z x_n(x_n - x) dz \to 0 \ \text{ and } \ \int_Z f(z, x_n(z))(x_n - x)(z) dz \to 0 \ \text{ as } \ n \to \infty,$$

we obtain

$$\langle A(x_n), x_n - x \rangle \to 0 \ \text{ as } \ n \to \infty.$$

We know that $A(x_n) \xrightarrow{w} A(x)$ in $H^{-1}(Z)$. So as before, via the Kadec-Klee property of $H_0^1(Z)$, we conclude that $x_n \to x$ in $H_0^1(Z)$. This proves that φ satisfies the C-condition. $\square$

In the sequel, we will need the following simple lemma:

Lemma 3.3. *If $\beta \in L^\infty(Z)$, $\beta(z) \le \lambda_1$ a.e. on Z and the inequality is strict on a set of positive measure, then there exists $\xi_1 > 0$ such that*

$$\psi(x) = \|Dx\|_2^2 - \int_Z \beta(z)x(z)^2 dz \ge \xi_1 \|Dx\|_2^2 \ \textit{for all} \ x \in H_0^1(Z).$$

Proof. From (2.2), we see that $\psi \ge 0$. Suppose that the lemma is not true. Exploiting the 2-homogeneity of ψ, we can find $\{x_n\}_{n\ge1} \subseteq H_0^1(Z)$ such that

$$\|Dx_n\|_2 = 1 \ \text{for all} \ n \ge 1 \ \text{and} \ \psi(x_n) \downarrow 0 \ \text{as} \ n \to \infty.$$

By Poincare's inequality $\{x_n\}_{n\ge1} \subseteq H_0^1(Z)$ is bounded. So we may assume that

$$x_n \xrightarrow{w} x \ \text{in} \ H_0^1(Z), \ x_n \to x \ \text{in} \ L^2(Z), \ x_n(z) \to x(z) \ \text{a.e. on} \ Z$$
$$\text{and} \ |x_n(z)| \le k(z) \ \text{a.e. on} \ Z \ \text{for all} \ n \ge 1, \ \text{with} \ k \in L^2(Z)_+.$$

From the weak lower semicontinuity of the norm functional, we have

$$\|Dx\|_2^2 \le \liminf_{n\to\infty} \|Dx_n\|_2^2,$$

while from the dominated convergence theorem, we have

$$\int_Z \beta(z)x_n(z)^2 dz \to \int_Z \beta(z)x(z)^2 dz \ \text{as} \ n \to \infty.$$

Hence

$$\begin{aligned} &\psi(x) \le \liminf_{n\to\infty} \psi(x_n) = 0, \qquad (3.14)\\ \Rightarrow &\|Dx\|_2^2 \le \int_Z \beta(z)x(z)^2 dz \le \lambda_1 \|x\|_2^2,\\ \Rightarrow &\|Dx\|_2^2 = \lambda_1 \|x\|_2^2 \ \text{(see (2.2))},\\ \Rightarrow &x = 0 \ \text{or} \ x = \pm u_1 \ \text{with} \ u_1 \in E(\lambda_1). \end{aligned}$$

If $x = 0$, then $\|Dx_n\|_2 \to 0$, a contradiction to the fact that $\|Dx_n\|_2 = 1$ for all $n \ge 1$.

If $x = \pm u_1$, then $|x(z)| > 0$ for all $z \in Z$ and so from the first inequality in (3.9) and the hypothesis on β, we have

$$\|Dx\|_2^2 < \lambda_1 \|x\|_2^2,$$

a contradiction to (2.2). ∎

Using this lemma, we prove the following proposition.

Proposition 3.4. *If hypotheses $H(f)$ hold, then the origin is a local minimizer of φ.*

Proof. Let $\delta > 0$ be as in hypothesis $H(f)(vi)$ and consider the closed ball

$$\overline{B}_{\delta}^{C_0^1} = \{x \in C_0^1(\overline{Z}) : \|x\|_{C_0^1(\overline{Z})} \leq \delta\}.$$

By virtue of hypothesis $H(f)(vi)$, for every $x \in \overline{B}_{\delta}^{C_0^1}$, we have

$$F(z, x(z)) \leq \frac{\eta(z)}{2} x(z)^2 \ \text{ for a.a. } z \in Z. \tag{3.15}$$

Thus, for all $x \in \overline{B}_{\delta}^{C_0^1}$, we have

$$\begin{aligned}
\varphi(x) &= \frac{1}{2}\|Dx\|_2^2 - \frac{\lambda_1}{2}\|x\|_2^2 - \int_Z F(z, x(z))dz \\
&\geq \frac{1}{2}\|Dx\|_2^2 - \frac{1}{2}\int_Z (\lambda_1 + \eta(z))x(z)^2 dz \ \ \text{(see (3.15))} \\
&\geq \frac{\xi_1}{2}\|Dx\|_2^2 \ \ \text{(apply Lemma 3.3 with } g = \lambda_1 + \eta \in L^{\infty}(Z)) \\
&\geq 0 = \varphi(0).
\end{aligned} \tag{3.16}$$

From (3.16) we see that $x = 0$ is a local $C_0^1(\overline{Z})$-minimizer of φ. But then from Brezis-Nirenberg [3], we have that $x = 0$ is a local $H_0^1(Z)$-minimizer of φ. ☐

We may assume that the origin is an isolated critical point of φ or otherwise we have a sequence of nontrivial solutions for problems (1.1). Then from the description of the critical groups at an isolated local minimizer (see Chang [4], p.33 and Mawhin-Willem [10], p.175), we have:

Corollary 3.5. *If hypotheses $H(f)$ hold, then $C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z}$ for all $k \geq 0$.*

In the next proposition, we produce the first nontrivial solution for problem (1.1).

Proposition 3.6. *If hypotheses $H(f)$ hold then problem (1.1) has a nontrivial solution $x_0 \in C_0^1(\overline{Z})$ and x_0 is a critical point of φ of mountain pass-type.*

Proof. Recall that $x = 0$ is an isolated local minimum of φ. So we can find $\rho_0 > 0$ such that

$$\varphi|_{\partial B_{\rho_0}} > 0. \tag{3.17}$$

Let $u_1 \in C_0^1(\overline{Z})$ be the $L^2(Z)$-normalized principal eigenfunction of $(-\triangle,\ H_0^1(Z))$ and let $t > 0$. For $0 < \beta_0 < t$, via the mean value theorem, we have

$$F(z, tu_1(z)) = F(z, \beta_0 u_1(z)) + \int_{\beta_0}^{t} f(z, \mu u_1(z))u_1(z)d\mu \ \text{ a.e. on } Z. \tag{3.18}$$

Integrating over Z and using Fubini's theorem, we obtain

$$\int_Z F(z, tu_1(z))dz = \int_Z F(z, \beta_0 u_1(z))dz + \int_{\beta_0}^{t} \frac{1}{\mu}\int_Z f(z, \mu u_1(z))\mu u_1(z)dzd\mu.$$

Choosing $\beta_0 > 0$ large, because of hypothesis $H(f)(v)$, we have

$$\int_Z f(z, \mu u_1(z))\mu u_1(z)dz \geq \gamma_1 > 0 \text{ for all } \mu \in [\beta_0, t]. \tag{3.19}$$

From (3.18) and (3.19), we obtain

$$\int_Z F(z, tu_1(z))dz \geq \int_Z F(z, \beta_0 u_1(z))dz + \int_{\beta_0}^t \frac{\gamma_1}{\mu} d\mu \text{ for } \beta_0 > 0 \text{ large,}$$

$$\Rightarrow \int_Z F(z, tu_1(z))dz \geq \int_Z F(z, \beta_0 u_1(z))dz + \gamma_1(\ln t - \ln \beta_0). \tag{3.20}$$

So from (3.20) it follows that

$$-\int_Z F(z, tu_1(z))dz \to -\infty \text{ as } t \to +\infty.$$

Hence

$$\varphi(tu_1) = -\int_Z F(z, tu_1(z))dz \to -\infty \text{ as } t \to +\infty \text{ (see (2.2)).}$$

Therefore for $t > 0$ large, we have

$$\varphi(tu_1) < \varphi(0) = 0 < \inf_{\partial B_{\rho_0}} \varphi = c.$$

This fact together with Proposition 3.2, permit the use of the mountain pass theorem (see Bartolo-Benci-Fortunato [1]), which gives $x_0 \in H_0^1(Z)$ such that

$$\varphi'(x_0) = 0 \text{ and } \varphi(0) = 0 < c \leq \varphi(x_0). \tag{3.21}$$

From (3.21), we deduce that $x_0 \neq 0$. From the equality in (3.21), we have

$$A(x_0) = \lambda_1 x_0 + N(x_0),$$

$$\Rightarrow -\triangle x_0(z) = \lambda_1 x_0(z) + f(z, x_0(z)) \text{ a.e. on } Z, \ x_0|_{\partial Z} = 0.$$

Thus $x_0 \in H_0^1(\overline{Z})$ is a nontrivial solution of problem (1.1) and from regularity theory (see for example Gasinski-Papageorgiou [5], pp.737-738), we have $x_0 \in C_0^1(\overline{Z})$. Let $d = \varphi(x_0)$ and assume without loss of generality that K_d is discrete (otherwise we have a whole sequence of nontrivial solutions for problem (1.1)). Then invoking Theorem 1 of Hofer [6], we can say that $x_0 \in C_0^1(\overline{Z})$ is a critical point of φ which is of mountain pass-type. $\square$

From the description of the critical groups for a critical point of a mountain pass-type (see Chang [4], p.91 and Mawhin-Willem [10], pp.195-196), we have:

Corollary 3.7. *If hypotheses $H(f)$ hold and $x_0 \in C_0^1(\overline{Z})$ is the nontrivial solution of (1.1) obtained in Proposition 3.6, then $C_k(\varphi, x_0) = \delta_{k,1}\mathbb{Z}$ for all $k \geq 0$.*

In the next proposition, we determine the critical groups of φ at infinity.

To do this, we will need the following slight generalization of Lemma 2.4 of Perera-Schechter [12].

Lemma 3.8. *If H is a Hilbert space, $\{\varphi_t\}_{t\in[0,1]}$ is a one-parameter family of $C^1(H)$-functions such that φ_t' and $\partial_t\varphi_t$ are both locally Lipschitz in $u \in H$ and there exists $R > 0$ such that*

$$\begin{aligned} &\inf[(1+\|u\|)\|\varphi_t'(u)\| : t \in [0,1], \|u\| > R] > 0 \\ \textit{and}\ \ &\inf[\varphi_t(u) : t \in [0,1], \|u\| \le R] > -\infty, \end{aligned}$$

then $C_k(\varphi_0, \infty) = C_k(\varphi_1, \infty)$ for all $k \ge 0$.

Proof. Let $\xi < \inf[\varphi_t(u) : t \in [0,1], \|u\| \le R]$. Let $h(t;u)$ $(t \in [0,1], u \in \varphi_0^\xi)$ be the flow generated by the Cauchy problem

$$\dot{h}(t) = -\frac{\partial_t\varphi_t(h(t))}{\|\varphi_t'(h(t))\|^2}\varphi_t'(h(t)) \ \text{ a.e. on } \ \mathbb{R}_+,\ h(0) = u.$$

We have

$$\begin{aligned} &\frac{d}{dt}\varphi_t(h(t)) = \langle \varphi_t'(h(t)), \dot{h}(t)\rangle + \partial_t\varphi_t(h(t)) = 0 \ \text{ for all } \ t \ge 0, \\ \Rightarrow &\varphi_t(h(t)) = \varphi_0(u) \ \text{ for all } \ t \ge 0. \end{aligned}$$

Since $u \in \varphi_0^a$, we have $\varphi_t(h(t)) \le \xi$ and so $\|h(t)\| > R$ for all $t \ge 0$. This then by virtue of the hypothesis of the lemma, implies that this flow exists for all $t \ge 0$ (see Bartolo-Benci-Fortunato [1]).

It can be reversed, if we replace φ_t with φ_{1-t}. Therefore $h(1)$ is a homeomorphism of φ_0^ξ and φ_1^ξ and so

$$C_k(\varphi_0, \infty) = H_k(H, \varphi_0^\xi) \cong H_k(H, \varphi_1^\xi) = C_k(\varphi_1, \infty).$$

□

Proposition 3.9. *If hypotheses $H(f)(i) \to (v)$ hold, then $C_k(\varphi, \infty) = \delta_{k,1}\mathbb{Z}$ for all $k \ge 0$.*

Proof. Let $0 < \sigma < \lambda_2 - \lambda_1$ and consider the following one-parameter C^2-functions on the Hilbert space $H_0^1(Z)$:

$$\varphi_t(x) = \frac{1}{2}\|Dx\|_2^2 - \frac{\lambda_1 + \sigma}{2}\|x\|_2^2 - t\int_Z (F(z, x(z)) - \sigma x(z))dz \ \text{ for all } \ x \in H_0^1(Z).$$

We claim that we can find $R > 0$ such that

$$\inf[(1+\|u\|)\|\varphi_t'(u)\| : t \in [0,1],\ \|u\| > R] > 0. \tag{3.22}$$

Suppose that this is not possible. Then we can find $t_n \to t \in [0,1]$ and $\|u_n\| \to \infty$ such that $\varphi'_{t_n}(u_n) \to 0$ in $H^{-1}(Z)$ as $n \to \infty$. Let $y_n = \frac{u_n}{\|u_n\|}$, $n \geq 1$. By passing to a suitable subsequence if necessary, we may assume that

$$\begin{aligned} &y_n \xrightarrow{w} y \text{ in } H^{-1}(Z),\ y_n \to y \text{ in } L^2(Z),\ y_n(z) \to y(z) \text{ a.e. on } Z, \\ \text{and } &|y_n(z)| \leq k(z) \text{ for a.a. } z \in Z, \text{ all } n \geq 1, \text{ with } k \in L^2(Z). \end{aligned}$$

We have

$$\begin{aligned} &\left|\langle \frac{\varphi'_{t_n}(u_n)}{\|u_n\|}, v\rangle\right| \leq \varepsilon_n \text{ for all } v \in H_0^(Z), \text{ with } \varepsilon_n \downarrow 0 \text{ (see (3.22))} \\ \Rightarrow &\left|\langle A(y_n), v\rangle - (\lambda_1 + \sigma)\int_Z y_n v dz - t_n \int_Z \frac{N(u_n)}{\|u_n\|} v dz + t_n \sigma \int_Z y_n v dz\right| \leq \varepsilon_n \end{aligned} \tag{3.23}$$

From the proof of Proposition 3.2, we know that

$$\frac{N(u_n)}{\|u_n\|} \xrightarrow{w} h = gy \text{ in } L^2(Z)$$

with $g \in L^\infty(Z)_+$, $0 \leq g(z) \leq \lambda_2 - \lambda_1$ a.e. on Z. Moreover, arguing as in that proof, we can also show that

$$y_n \to y \text{ in } H_0^1(Z), \text{ hence } \|y\| = 1, \text{ i.e. } y \neq 0.$$

So, if we pass to the limit as $n \to \infty$ in (3.23), we obtain

$$\begin{aligned} &\langle A(y), v\rangle = (\lambda_1 + \sigma)\int_Z y v dz + t\int_Z (g + \sigma) y v dz \text{ for all } v \in H_0^1(Z), \\ \Rightarrow &A(y) = (\lambda_1 + (1-t)\sigma + tg)y. \end{aligned} \tag{3.24}$$

As in the proof of Proposition 3.2, we consider three distinct possibilities for the weight function $m = \lambda_1 + (1-t)\sigma + tg \in L^\infty(Z)_+$.

<u>Case 1:</u> $t = 1$ and $g = 0$.

From (3.24), we have

$$\begin{aligned} &A(y) = \lambda_1(y), \\ \Rightarrow &-\triangle y(z) = \lambda_1 y(z) \text{ a.e. on } Z,\ y|_{\partial Z} = 0, \\ \Rightarrow &y \in E(\lambda_1),\ \ y \neq 0. \end{aligned}$$

So, if $u_n = u_n^0 + \widehat{u}_n$ with $u_n^0 \in E(\lambda_1)$, $\widehat{u}_n \in \widehat{H}_2 = E(\lambda_1)^\perp$, $n \geq 1$, then

$$\frac{\|u_n^0\|}{\|u_n\|} \to 1 \text{ as } n \to \infty. \tag{3.25}$$

We have

$$\left|\langle A(u_n), v\rangle - (\lambda_1+\sigma)\int_Z u_n v dz - t_n \int_Z N(u_n) v dz + t_n\sigma \int_Z u_n v dz\right| \le \varepsilon_n$$
for all $v \in H_0^1(Z)$.

Let $v = u_n^0 \in E(\lambda_1)$. We obtain

$$\left| \|Du_n^0\|_2^2 - (\lambda_1+\sigma)\|u_n^0\|_2^2 - t_n \int_Z f(z, u_n(z)) u_n^0(z) dz + t_n \sigma \|u_n^0\|_2^2 \right| \le \varepsilon_n. \tag{3.26}$$

Since $u_n^0 \in E(\lambda_1)$, we know that $\|Du_n^0\|_2^2 = \lambda_1 \|u_n^0\|_2^2$. Also because of (3.25) and hypothesis $H(f)(v)$, we have

$$\int_Z f(z, u_n(z)) u_n^0(z) dz \ge \gamma_1 \text{ for all } n \ge n_1.$$

Then from (3.26), we obtain

$$\begin{aligned}&(1-t_n)\sigma\|u_n^0\|_2^2 + t_n\gamma_1 \le \varepsilon_n \text{ for all } n \ge n_1,\\ \Rightarrow & t_n\gamma_1 \le \varepsilon_n \text{ for all } n \ge n_1.\end{aligned}$$

Since $t_n \to t = 1$ and $\varepsilon_n \downarrow 0$, in the limit as $n \to \infty$, we obtain

$$0 < \gamma_1 \le 0,$$

a contradiction.

Case 2: $t = 1$ and $g = \lambda_2 - \lambda_1$.

From (3.24), we have

$$\begin{aligned}&A(y) = \lambda_2 y,\\ \Rightarrow & -\triangle y(z) = \lambda_2 y(z) \text{ a.e. on } Z, \ \ y|_{\partial Z} = 0,\\ \Rightarrow & y \in E(\lambda_2), \ \ y \neq 0.\end{aligned}$$

Now we write $u_n = u_n^0 + \widehat{u}_n$ with $u_n^0 \in E(\lambda_2)$ and $\widehat{u}_n \in W = E(\lambda_2)^\perp$. We have

$$\frac{\|u_n^0\|}{\|u_n\|} \to 1 \text{ as } n \to \infty. \tag{3.27}$$

Recall that

$$\left|\langle A(u_n), v\rangle - (\lambda_1+\sigma)\int_Z u_n v dz - t_n \int_Z N(u_n) v dz + t_n\sigma \int_Z u_n v dz\right| \le \varepsilon_n$$
for all $v \in H_0^1(Z)$.

Let $v = u_n^0 \in E(\lambda_2)$. We obtain

$$\begin{aligned}\Big| &\|Du_n^0\|_2^2 - t_n\lambda_2\|u_n^0\|_2^2 - (1-t_n)(\lambda_1+\sigma)\|u_n^0\|_2^2 \\ &- t_n \int_Z (f(z,u_n(z)) - (\lambda_2-\lambda_1)u_n(z))u_n^0(z)dz\Big| \leq \varepsilon_n.\end{aligned} \tag{3.28}$$

Note that $t_n\lambda_2 + (1-t_n)(\lambda_1+\sigma) < \lambda_2$ and so

$$0 < \|Du_n^0\|_2^2 - (t_n\lambda_2 + (1-t_n)(\lambda_1+\sigma))\|u_n^0\|_2^2. \tag{3.29}$$

In addition because of (3.27) and hypothesis $H(f)(v)$, we have

$$\int_Z (f(z,u_n(z)) - (\lambda_2-\lambda_1)u_n(z))u_n^0(z)dz \leq -\gamma_2 < 0 \ \text{ for all } \ n \geq n_2. \tag{3.30}$$

Using (3.29) and (3.30) in (3.28), we obtain

$$t_n\gamma_2 \leq \varepsilon_n \ \text{ for all } \ n \geq n_2.$$

Passing to the limit as $n \to \infty$ and recalling that $t_n \to 1$ and $\varepsilon \downarrow 0$, we get

$$0 < \gamma_2 \leq 0,$$

again a contradiction.

Case 3: $t \neq 1$ or $0 \leq g(z) \leq \lambda_2 - \lambda_1$ a.e. on Z with $g \neq 0$ and $g \neq \lambda_2 - \lambda_1$.

From (3.24), we have

$$\begin{aligned} &A(y) = (\lambda_1 + \widehat{\xi})y, \ y \neq 0 \ \text{ with } \ \widehat{\xi} = (1-t)\sigma + tg \in L^\infty(Z)_+, \\ &\Rightarrow -\triangle y(z) = (\lambda_1 + \widehat{\xi}(z))y(z) \ \text{ a.e. on } \ Z, \ \ y|_{\partial Z} = 0.\end{aligned} \tag{3.31}$$

Note that since $t \neq 1$ or ($g \neq 0$ and $g \neq \lambda_2 - \lambda_1$), we have

$$\lambda_1 \leq \lambda_1 + \widehat{\xi}(z) \leq \lambda_2 \ \text{ a.e. on } \ Z, \ \ \lambda_1 \neq \lambda_1 + \widehat{\xi} \ \text{ and } \ \lambda_2 \neq \lambda_1 + \widehat{\xi}.$$

Hence from the strict monotonicity of the eigenvalues on the weight function, we infer that

$$\widehat{\lambda}_1(\lambda_1 + \widehat{\xi}) < \widehat{\lambda}_1(\lambda_1) = 1 \ \text{ and } \ \widehat{\lambda}_2(\lambda_1 + \widehat{\xi}). \tag{3.32}$$

Using (3.32) in (3.31), we infer that $y = 0$, a contradiction to the fact that $\|y\| = 1$.

So in all three cases we have reached a contradiction and this means that there exists $R > 0$ for which (3.22) is valid.

Also it is clear, that due to hypotheses $H(f)(iii), (iv)$, we have

$$\inf[\varphi_t(u) : t \in [0,1], \ \|u\| \leq R] > -\infty.$$

So we can apply Lemma 3.8 and have that

$$C_k(\varphi_0, \infty) = C_k(\varphi, \infty) \text{ for all } k \geq 0. \tag{3.33}$$

Note that

$$\varphi_0(x) = \frac{1}{2}\|Dx\|_2^2 - \frac{\lambda_1 + \sigma}{2}\|x\|_2^2 \text{ and } \varphi_1(x) = \varphi(x) \text{ for all } x \in H_0^1(Z).$$

Since $0 < \sigma < \lambda_2 - \lambda_1$, the only critical point of φ_0 is $u = 0$. Hence

$$C_k(\varphi_0, \infty) = C_k(\varphi, 0) \text{ for all } k \geq 0. \tag{3.34}$$

Moreover, from Proposition 2.3 of Su [16], we have

$$C_k(\varphi_0, 0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0. \tag{3.35}$$

From (3.33), (3.34) and (3.35), we conclude that

$$C_k(\varphi, \infty) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0.$$

□

Now we are ready for the first multiplicity theorem.

Theorem 3.10. *If hypotheses $H(f)$ hold, then problem (1.1) has at least two nontrivial solutions $x_0, v_0 \in C_0^1(\overline{Z})$.*

Proof. One nontrivial solution $x_0 \in C_0^1(\overline{Z})$, exists by virtue of Proposition 3.6.

Suppose that $\{0, x_0\}$ are the only critical points of φ. Then using Corollaries 3.5, 3.7, 3.9 and the Poincare-Hopf formula, we have

$$(-1)^0 + (-1)^1 = (-1)^1,$$

a contradiction. So there exists a third critical point $v_0 \neq x_0$, $v_0 \neq 0$. Evidently v_0 is a solution of (1.1) and by regularity theory, we have $v_0 \in C_0^1(\overline{Z})$. □

We have another multiplicity result by modifying hypothesis $H(f)(vi)$. So the new hypotheses on the nonlinearity $f(z,x)$ are the following:

$\underline{H(f)'}$: $f : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(z,0) = 0$ a.e. on Z, hypotheses $H(f)'(i) \to (v)$ are the same as hypotheses $H(f)(i) \to (v)$ respectively and

(vi) there exist $m \geq 2$ and $\delta > 0$ such that

$$\lambda_m - \lambda_1 \leq \frac{f(z,x)}{x} \leq \lambda_{m+1} - \lambda_1 \text{ for a.a. } z \in Z \text{ and all } 0 < |x| \leq \delta.$$

Remark 3.11. *Hypotheses $H(f)'(iv)$ and (vi) imply that we can have double resonance both at infinity and at zero. A double-double resonance situation.*

Theorem 3.12. *If hypotheses $H(f)'$ hold, then problem (1.1) has at least two nontrivial solutions $x_0, v_0 \in C_0^1(\overline{Z})$.*

Proof. Because of hypothesis $H(f)'(vi)$ and Proposition 1.1 of Li-Perera-Su [9], we have

$$C_k(\varphi, 0) = \delta_{k,d}\mathbb{Z}, \tag{3.36}$$

where d =sum of multiplicities of $\{\lambda_k\}_{k=1}^m = \dim \overline{H}_m \geq 2$, since $m \geq 2$.

Also from Proposition 3.9, we know that

$$C_k(\varphi, \infty) = \delta_{k,1}\mathbb{Z}. \tag{3.37}$$

So there exists a critical point x_0 of φ such that

$$C_1(\varphi, x_0) \neq 0. \tag{3.38}$$

Comparing this with (3.36), we infer that $x_0 \neq 0$. Moreover, due to (3.38) x_0 is of mountain pass type and so

$$C_1(\varphi, x_0) = \delta_{k,1}\mathbb{Z}. \tag{3.39}$$

If $\{0, x_0\}$ are the only critical points of φ, then from (3.36), (3.37) and (3.39) and the Poincare-Hopf formula, we have

$$\begin{aligned} &(-1)^d + (-1)^1 = (-1)^1, \\ \Rightarrow &(-1)^d = 0, \quad \text{a contradiction.} \end{aligned}$$

So there exists a second nontrivial critical point v_0 of φ. Evidently $x_0, v_0 \in H_0^1(Z)$ are nontrivial solutions of problem (1.1). From regularity theory, we conclude that $x_0, v_0 \in C_0^1(\overline{Z})$. ☐

Remark 3.13. *Theorem 3.12 above partially extends Theorem 3 of Robinson [14] and also Theorem 2 of Su [16].*

`Received: February 2008. Revised: April 2008.`

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