CUBO A Mathematical Journal
Vol.10, N$^{\underline{o}}$03, (103–114). October 2008

The Flip Crossed Products of the C^*-Algebras by Almost Commuting Isometries

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ABSTRACT

We study the flip crossed products of the C^*-algebras by almost commuting isometries and obtain some results on their structure, K-theory, and continuity.

RESUMEN

Estudiamos el produto flip crossed de una C^*-algebra mediante isometrias casi commutando y obtenemos algunos resultados sobre su estructura, K-teoria, y continuidad.

Key words and phrases: *C^*-algebra, Continuous field, K-theory, Isometry.*

Math. Subj. Class.: *46L05, 46L80.*

Introduction

Recall that the soft torus A_ε of Exel [3] (for any $\varepsilon \in [0,2]$ the closed interval) is defined to be the universal C^*-algebra generated by almost commuting two unitaries $u_{\varepsilon,1}$ and $u_{\varepsilon,2}$ in the sense that $\|u_{\varepsilon,2}u_{\varepsilon,1} - u_{\varepsilon,1}u_{\varepsilon,2}\| \leq \varepsilon$. Its K-theory is computed in [3] by showing that it can be represented as a crossed product by $\mathbb{Z}$ and applying the Pimsner-Voiculescu six-term exact sequense for the crossed product. It is shown by Exel [4] that there exists a continuous field of C^*-algebras on $[0,2]$ with fibers the soft tori varying continuously. Furthermore, K-theory and continuity of the crossed products of A_ε by the flip (a $\mathbb{Z}_2$-action) are considered by Elliott, Exel and Loring [2].

On the other hand, we [8] began to study continuous fields of C^*-algebras by almost commuting isometries and obtained some similar results (but different in some senses) on their structure, K-theory and continuity as those by Exel. In this paper we consider those properties for the flip crossed products of the C^*-algebras generated by almost commuting isometries.

Refer to [1], [5], and [9] for some basics in C^*-algebras and K-theory.

1 The flip crossed products by isometries

The Toeplitz algebra is defined to be the universal C^*-algebra generated by a (non-unitary) isometry, and it is denoted by $\mathfrak{F}$, which is also the semigroup C^*-algebra $C^*(\mathbb{N})$ of the semigroup $\mathbb{N}$ of natural numbers. The C^*-algebra $C(\mathbb{T})$ of all continuous functions on the 1-torus $\mathbb{T}$ is the universal C^*-algebra generated by a unitary, which is also the group C^*-algebra $C^*(\mathbb{Z})$ of the group $\mathbb{Z}$ of integers. There is a canonical quotient map from $\mathfrak{F}$ to $C(\mathbb{T})$ by universality, whose kernel is isomorphic to the C^*-algebra $\mathbb{K}$ of all compact operators on a separable infinite dimensional Hilbert space (cf. [5]).

Definition 1.1 For $\varepsilon \in [0,2]$, the soft Toeplitz tensor product denoted by $\mathfrak{F} \otimes_\varepsilon \mathfrak{F}$ is defined to be the universal C^*-algebra generated by two isometries $s_{\varepsilon,1}$, $s_{\varepsilon,2}$ such that $\|s_{\varepsilon,2}s_{\varepsilon,1} - s_{\varepsilon,1}s_{\varepsilon,2}\| \leq \varepsilon$ (ε-commuting). Let $\pi : \mathfrak{F} \otimes_\varepsilon \mathfrak{F} \to A_\varepsilon$ be the canonical onto $*$-homomorphism sending the isometry generators to the unitary generators.

Remark. Refer to [8], in which super-softness is further defined and assumed, but it should be unnecessary from the universality argument (as given below). Instead, in fact, another norm estimate of the form $\|s_{\varepsilon,2}s_{\varepsilon,1}^* - s_{\varepsilon,1}^*s_{\varepsilon,2}\| \leq \varepsilon$ (ε-$*$-commuting) may be required, but we omit such an estimate in what follows. If not assuming the estimate, $\mathfrak{F} \otimes_\varepsilon \mathfrak{F}$ should be replaced with $C^*(\mathbb{N}^2)_\varepsilon$, where $C^*(\mathbb{N}^2)$ is the semigroup C^*-algebra of $\mathbb{N}^2$ (in what follows).

Definition 1.2 The flip on $\mathfrak{F} \otimes_\varepsilon \mathfrak{F}$ is the (non-unital) endomorphism σ defined by $\sigma(s_{\varepsilon,j}) = s_{\varepsilon,j}^*$ for $j = 1,2$. Since σ^2 is the identity on $\mathfrak{F} \otimes_\varepsilon \mathfrak{F}$, we denote by $(\mathfrak{F} \otimes_\varepsilon \mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2$ the crossed product of $\mathfrak{F} \otimes_\varepsilon \mathfrak{F}$ by the action σ of the order 2 cyclic group $\mathbb{Z}_2$, i.e., a flip crossed product.

Definition 1.3 For $\varepsilon \in [0,2]$, we define E_ε to be the universal C^*-algebra generated by an isometry t_1 and the elements $t_{n+1} = u^n t_1 (u^*)^n$ for $n \in \mathbb{N}$, where u is an isometry, such that $\|ut_1 - t_1 u\| \le \varepsilon$. Let α_ε be the endomorphism of E_ε defined by $\alpha_\varepsilon(t_n) = t_{n+1} = ut_n u^*$ for $n \in \mathbb{N}$. Let $E_\varepsilon \rtimes_{\alpha_\varepsilon} \mathbb{N}$ be the semigroup crossed product of E_ε by the action α_ε of the additive semigroup $\mathbb{N}$ of natural numbers.

Remark. Note that $\mathfrak{F} \otimes_2 \mathfrak{F}$ (or $C^*(\mathbb{N}^2)_2$) is isomorphic to the unital full free product $\mathfrak{F} *_{\mathbb{C}} \mathfrak{F}$, which is also isomorphic to the full semigroup C^*-algebra $C^*(\mathbb{N} * \mathbb{N})$ of the free semigroup $\mathbb{N} * \mathbb{N}$. As in the above remark, another estimate $\|ut_1^* - t_1^* u\| \le \varepsilon$ may be required accordingly.

It is shown in [8] that $\mathfrak{F} \otimes_\varepsilon \mathfrak{F} \cong E_\varepsilon \rtimes_{\alpha_\varepsilon} \mathbb{N}$, where the map φ from $\mathfrak{F} \otimes_\varepsilon \mathfrak{F}$ to $E_\varepsilon \rtimes_{\alpha_\varepsilon} \mathbb{N}$ is defined by $\varphi(s_{\varepsilon,1}) = t_1$ and $\varphi(s_{\varepsilon,2}) = u$, and its inverse ψ is given by $\psi(t_{n+1}) = s_{\varepsilon,2}^n s_{\varepsilon,1} (s_{\varepsilon,2}^*)^n$ for $n \in \mathbb{N}$ and $n = 0$ and $\psi(u) = s_{\varepsilon,2}$.

Proposition 1.4 *For $\varepsilon \in [0,2]$, we have the following isomorphism:*

$$(\mathfrak{F} \otimes_\varepsilon \mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2 \cong E_\varepsilon \rtimes_{\alpha_\varepsilon * \beta} (\mathbb{N} * \mathbb{Z}_2),$$

where $\mathbb{N} * \mathbb{Z}_2$ is the free product of $\mathbb{N}$ and $\mathbb{Z}_2$, and the action β on E_ε is given by $\beta(t_n) = t_n^*$ for $n \in \mathbb{N}$.

Proof. The crossed product $(\mathfrak{F} \otimes_\varepsilon \mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2$ is the universal C^*-algebra generated by isometries $s_{\varepsilon,1}$, $s_{\varepsilon,2}$ and a unitary ρ such that $\|s_{\varepsilon,2}s_{\varepsilon,1} - s_{\varepsilon,1}s_{\varepsilon,2}\| \le \varepsilon$ and $\rho s_{\varepsilon,j}\rho^* = s_{\varepsilon,j}$ $(j = 1,2)$ with $\rho^2 = 1$, while $E_\varepsilon \rtimes_{\alpha_\varepsilon * \beta} (\mathbb{N} * \mathbb{Z}_2)$ is the C^*-algebra generated by isometries t_1, u and a unitary v such that $\|ut_1 - t_1 u\| \le \varepsilon$ and $t_{n+1} = ut_n u^* = u^n t_1 (u^*)^n$ for $n \in \mathbb{N}$, and $vt_1v^* = t_1^*$ and $vuv^* = u^*$ with $v^2 = 1$. The isomorphism between them is given by sending $s_{\varepsilon,1}, s_{\varepsilon,2}$, and ρ to t_1, u, and v respectively (cf. [2]). $\square$

Theorem 1.5 *For $0 \le \varepsilon < 2$, we obtain the K-theory isomorphisms:*

$$K_0((\mathfrak{F} \otimes_\varepsilon \mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2) \cong \mathbb{Z}^9, \quad K_1((\mathfrak{F} \otimes_\varepsilon \mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2) \cong 0.$$

Moreover, $K_j((\mathfrak{F} \otimes_\varepsilon \mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2) \cong K_j((\mathfrak{F} \otimes \mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2)$ for $j = 0, 1$.

Proof. Since $\mathfrak{F} \otimes_\varepsilon \mathfrak{F} \cong E_\varepsilon \rtimes_{\alpha_\varepsilon} \mathbb{N}$ and α_ε is a corner endomorphism on E_ε, note that $E_\varepsilon \rtimes_{\alpha_\varepsilon} \mathbb{N}$ is isomorphic to a corner of $(E_\varepsilon \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z}$, i.e., $p((E_\varepsilon \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z})p$ for a certain projection p, where $\rho_\varepsilon^\wedge$ is the dual action of the circle action on $E_\varepsilon \rtimes_{\alpha_\varepsilon} \mathbb{N}$ and id is the identity action on $\mathbb{K}$ (this is a variation of [6], and see also [7]). Hence, $(E_\varepsilon \rtimes_{\alpha_\varepsilon} \mathbb{N}) \rtimes_\sigma \mathbb{Z}_2$ is isomorphic to

$p((E_\varepsilon \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z})p \rtimes_\sigma \mathbb{Z}_2$. Therefore,

$$\begin{aligned} K_j((E_\varepsilon \rtimes_{\alpha_\varepsilon} \mathbb{N}) \rtimes_\sigma \mathbb{Z}_2) &\cong K_j(p((E_\varepsilon \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z})p \rtimes_\sigma \mathbb{Z}_2) \\ &\cong K_j^{\mathbb{Z}_2}(p((E_\varepsilon \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z})p) \\ &\cong K_j^{\mathbb{Z}_2}(p((E_\varepsilon \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z})p \otimes \mathbb{K}) \\ &\cong K_j^{\mathbb{Z}_2}(((E_\varepsilon \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z}) \otimes \mathbb{K}) \\ &\cong K_j((E_\varepsilon \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z} \rtimes \mathbb{Z}_2), \end{aligned}$$

where $K_j^{\mathbb{Z}_2}(\cdot)$ is the equivariant K-theory, and note that $p((E_\varepsilon \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z})p$ is stably isomorphic to $(E_\varepsilon \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z}$, and

$$\begin{aligned} (E_\varepsilon \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z} \rtimes \mathbb{Z}_2 &\cong (E_\varepsilon \otimes \mathbb{K}) \rtimes_{\sigma_\varepsilon' * \sigma \otimes \mathrm{id}} (\mathbb{Z}_2 * \mathbb{Z}_2) \\ &\cong (E_\varepsilon \rtimes_{\sigma_\varepsilon' * \sigma} (\mathbb{Z}_2 * \mathbb{Z}_2)) \otimes \mathbb{K} \end{aligned}$$

since $\mathbb{Z} \rtimes \mathbb{Z}_2 \cong \mathbb{Z}_2 * \mathbb{Z}_2$, where $\sigma_\varepsilon'(1) = \rho_\varepsilon^\wedge(1)\sigma(1)$ (cf. [2]). Set $F_\varepsilon = E_\varepsilon \rtimes_{\sigma_\varepsilon' * \sigma} (\mathbb{Z}_2 * \mathbb{Z}_2)$. There exists the following six-term exact sequence (A) (cf. [2]):

$$\begin{array}{ccccc} K_0(E_\varepsilon) & \longrightarrow & K_0(E_\varepsilon \rtimes_{\sigma_\varepsilon'} \mathbb{Z}_2) \oplus K_0(E_\varepsilon \rtimes_\sigma \mathbb{Z}_2) & \longrightarrow & K_0(F_\varepsilon) \\ \uparrow & & & & \downarrow \\ K_1(F_\varepsilon) & \longleftarrow & K_1(E_\varepsilon \rtimes_{\sigma_\varepsilon'} \mathbb{Z}_2) \oplus K_1(E_\varepsilon \rtimes_\sigma \mathbb{Z}_2) & \longleftarrow & K_1(E_\varepsilon). \end{array}$$

Consider the following exact sequence: $0 \to \mathfrak{I}_\varepsilon \to E_\varepsilon \to \pi(E_\varepsilon) = B_\varepsilon' \to 0$, where π is the canonical quotient map from E_ε to the quotient $\pi(E_\varepsilon) = B_\varepsilon'$, where B_ε' is the universal C^*-algebra generated by unitaries $u_{n+1} = w^n v (w^*)^n$ for $n \in \mathbb{N}$ and $n = 0$, where $\pi(t_{n+1}) = \pi(u)^n \pi(t_1) \pi(u^*)^n = u_{n+1}$ with $v = \pi(t_1)$ and $w = \pi(u)$. As shown in [8], K-theory groups of $\mathfrak{I}_\varepsilon$ are the same as those of $\mathbb{K}$. Since this quotient is invariant under the action $\beta = \sigma_\varepsilon'$ or σ, we have the following exact sequence:

$$(B): \quad 0 \to \mathfrak{I}_\varepsilon \rtimes_\beta \mathbb{Z}_2 \to E_\varepsilon \rtimes_\beta \mathbb{Z}_2 \to \pi(E_\varepsilon) \rtimes_\beta \mathbb{Z}_2 \to 0$$

and $\mathfrak{I}_\varepsilon \rtimes_\beta \mathbb{Z}_2 \cong \mathfrak{I}_\varepsilon \otimes C^*(\mathbb{Z}_2)$ and the group C^*-algebra $C^*(\mathbb{Z}_2)$ is isomorphic to $\mathbb{C}^2$ via the Fourier transform.

As shown in [2], it is deduced that $\pi(E_\varepsilon) \rtimes_\beta \mathbb{Z}_2$ is homotopy equivalent to the crossed product $C(\mathbb{T}) \rtimes_{\beta'} \mathbb{Z}_2$, where $\beta'(z) = z^{-1}$ for $z \in \mathbb{T}$. It follows that $K_j(\pi(E_\varepsilon) \rtimes_\beta \mathbb{Z}_2)$ is isomorphic to $K_j(C(\mathbb{T}) \rtimes_{\beta'} \mathbb{Z}_2)$. Since the points $\{\pm 1\}$ in $\mathbb{T}$ is fixed under the action β', we have

$$0 \to C_0(\mathbb{T} \setminus \{\pm 1\}) \rtimes_{\beta'} \mathbb{Z}_2 \to C(\mathbb{T}) \rtimes_{\beta'} \mathbb{Z}_2 \to \oplus^2 C^*(\mathbb{Z}_2) \to 0,$$

where $C_0(\mathbb{T} \setminus \{\pm 1\})$ is the C^*-algebra of all continuous functions on $\mathbb{T} \setminus \{\pm 1\}$ vanishing at infinity, and $C_0(\mathbb{T} \setminus \{\pm 1\}) \rtimes_{\beta'} \mathbb{Z}_2 \cong C_0(\mathbb{R}) \otimes (\mathbb{C}^2 \rtimes_{\beta'} \mathbb{Z}_2) \cong C_0(\mathbb{R}) \otimes M_2(\mathbb{C})$ and $C^*(\mathbb{Z}_2) \cong \mathbb{C}^2$. Hence the following six-term exact sequence is obtained:

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(C(\mathbb{T}) \rtimes_{\beta'} \mathbb{Z}_2) & \longrightarrow & \mathbb{Z}^4 \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(C(\mathbb{T}) \rtimes_{\beta'} \mathbb{Z}_2) & \longleftarrow & \mathbb{Z}, \end{array}$$

where $K_j(C_0(\mathbb{R}) \otimes M_2(\mathbb{C})) \cong K_{j+1}(\mathbb{C}) \pmod 2$ and $K_j(\oplus^2\mathbb{C}^2) \cong \oplus^4 K_j(\mathbb{C})$. It follows that $K_0(C(\mathbb{T}) \rtimes_{\beta'} \mathbb{Z}_2) \cong \mathbb{Z}^3$ and $K_1(C(\mathbb{T}) \rtimes_{\beta'} \mathbb{Z}_2) \cong 0$ (cf. [2]).

Therefore, for the above exact sequence (B), we obtain the diagram:

$$\begin{array}{ccccc} \mathbb{Z}^2 & \longrightarrow & K_0(E_\varepsilon \rtimes_\beta \mathbb{Z}_2) & \longrightarrow & \mathbb{Z}^3 \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(E_\varepsilon \rtimes_\beta \mathbb{Z}_2) & \longleftarrow & 0, \end{array}$$

where $K_j(\mathbb{K} \otimes C^*(\mathbb{Z}_2)) \cong K_j(\mathbb{C}^2)$. Hence we obtain $K_0(E_\varepsilon \rtimes_\beta \mathbb{Z}_2) \cong \mathbb{Z}^5$ and $K_1(E_\varepsilon \rtimes_\beta \mathbb{Z}_2) \cong 0$. This implies that the diagram (A) is

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & \mathbb{Z}^5 \oplus \mathbb{Z}^5 & \longrightarrow & K_0(F_\varepsilon) \\ \uparrow & & & & \downarrow \\ K_1(F_0) & \longleftarrow & 0 \oplus 0 & \longleftarrow & 0 \end{array}$$

where it is shown in [8] that $K_0(E_\varepsilon) \cong \mathbb{Z}$ and $K_1(E_\varepsilon) \cong 0$. It follows that $K_0(F_\varepsilon) \cong \mathbb{Z}^9$ and $K_1(F_\varepsilon) \cong 0$. It follows from this and the first part shown above that $K_0((\mathfrak{F} \otimes_\varepsilon \mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2) \cong \mathbb{Z}^9$ and $K_1((\mathfrak{F} \otimes_\varepsilon \mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2) \cong 0$.

The second claim follows from the case $\varepsilon = 0$ and the same argument as above. Note that $\mathfrak{F} \otimes \mathfrak{F} \cong \mathfrak{F} \rtimes_{\mathrm{id}} \mathbb{N}$, where id is the trivial action. $\square$

Corollary 1.6 *For $0 \leq \varepsilon < 2$, the natural onto $*$-homomorphism $\varphi_{\varepsilon,0}$ from $(\mathfrak{F} \otimes_\varepsilon \mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2$ to $(\mathfrak{F} \otimes \mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2$ sending $s_{\varepsilon,j}$ to $s_{0,j}$ $(j = 1, 2)$ induces the isomorphism between their K-groups.*

Proposition 1.7 *There exists a continuous field of C^*-algebras on the closed interval $[0, 2]$ such that its fibers are $(\mathfrak{F} \otimes_\varepsilon \mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2$ for $\varepsilon \in [0, 2]$, and for any $a \in (\mathfrak{F} \otimes_2 \mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2$, the sections $[0, 2] \ni \varepsilon \mapsto \varphi_\varepsilon(a) \in (\mathfrak{F} \otimes_\varepsilon \mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2$ are continuous, where $\varphi_\varepsilon : (\mathfrak{F} \otimes_2 \mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2 \to (\mathfrak{F} \otimes_\varepsilon \mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2$ is the natural onto $*$-homomorphism sending $s_{2,j}$ to $s_{\varepsilon,j}$ $(j = 0, 1)$.*

Proof. As shown before, $(\mathfrak{F} \otimes_\varepsilon \mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2 \cong (E_\varepsilon \rtimes_{\alpha_\varepsilon} \mathbb{N}) \rtimes_\sigma \mathbb{Z}_2$. Furthermore, this is isomorphic to $p((E_\varepsilon \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z})p \rtimes_\sigma \mathbb{Z}_2$. Hence it follows that

$$\begin{aligned} ((E_\varepsilon \rtimes_{\alpha_\varepsilon} \mathbb{N}) \rtimes_\sigma \mathbb{Z}_2) \otimes \mathbb{K} &\cong (p((E_\varepsilon \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z})p \rtimes_\sigma \mathbb{Z}_2) \otimes \mathbb{K} \\ &\cong (p((E_\varepsilon \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z})p \otimes \mathbb{K}) \rtimes_{\sigma \otimes \mathrm{id}} \mathbb{Z}_2 \\ &\cong (((E_\varepsilon \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z}) \otimes \mathbb{K}) \rtimes_{\sigma \otimes \mathrm{id}} \mathbb{Z}_2 \\ &\cong ((E_\varepsilon \otimes \mathbb{K} \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id} \otimes \mathrm{id}} \mathbb{Z}) \rtimes_{\sigma \otimes \mathrm{id}} \mathbb{Z}_2 \\ &\cong ((E_\varepsilon \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z}) \rtimes_\sigma \mathbb{Z}_2 \\ &\cong (E_\varepsilon \otimes \mathbb{K}) \rtimes_{\sigma'_\varepsilon * \sigma \otimes \mathrm{id}} (\mathbb{Z}_2 * \mathbb{Z}_2). \end{aligned}$$

It is deduced from [2] that there exists a continuous field of C^*-algebras on $[0, 2]$ such that its fibers are $(E_\varepsilon \otimes \mathbb{K}) \rtimes_{\sigma'_\varepsilon * \sigma \otimes \mathrm{id}} (\mathbb{Z}_2 * \mathbb{Z}_2)$ for $\varepsilon \in [0, 2]$, and for any $b \in (E_2 \otimes \mathbb{K}) \rtimes_{\sigma'_2 * \sigma \otimes \mathrm{id}} (\mathbb{Z}_2 * \mathbb{Z}_2)$, the sections $[0,2] \ni \varepsilon \mapsto \psi_\varepsilon(b) \in (E_\varepsilon \otimes \mathbb{K}) \rtimes_{\sigma'_\varepsilon * \sigma \otimes \mathrm{id}} (\mathbb{Z}_2 * \mathbb{Z}_2)$ are continuous, where ψ_ε is the unique onto $*$-homomorphism from $(E_2 \otimes \mathbb{K}) \rtimes_{\sigma'_2 * \sigma \otimes \mathrm{id}} (\mathbb{Z}_2 * \mathbb{Z}_2)$ to $(E_\varepsilon \otimes \mathbb{K}) \rtimes_{\sigma'_\varepsilon * \sigma \otimes \mathrm{id}} (\mathbb{Z}_2 * \mathbb{Z}_2)$. Cutting down this continuous field by cutting down the fibers from $((E_\varepsilon \rtimes_{\alpha_\varepsilon} \mathbb{N}) \rtimes_\sigma \mathbb{Z}_2) \otimes \mathbb{K}$ to $(E_\varepsilon \rtimes_{\alpha_\varepsilon} \mathbb{N}) \rtimes_\sigma \mathbb{Z}_2$ by minimal projections, we obtain the desired continuous field. □

2 The flip crossed products by n isometries

The n-fold tensor product $\otimes^n \mathfrak{F}$ of $\mathfrak{F}$ is the universal C^*-algebra generated by mutually commuting and $*$-commuting n isometries, while the universal C^*-algebra generated by mutually commuting n isometries is just the semigroup C^*-algebra $C^*(\mathbb{N}^n)$ of the semigroup $\mathbb{N}^n$. The C^*-algebra $C(\mathbb{T}^n)$ of all continuous functions on the n-torus $\mathbb{T}^n$ is the universal C^*-algebra generated by mutually commuting n unitaries, which is also the group C^*-algebra $C^*(\mathbb{Z}^n)$ of the group $\mathbb{Z}^n$. There is a canonical quotient map from $\otimes^n \mathfrak{F}$ to $C(\mathbb{T}^n) \cong \otimes^n C(\mathbb{T})$ by universality,

Definition 2.1 For $\varepsilon \in [0,2]$, the soft Toeplitz n-tensor product denoted by $\otimes^n_\varepsilon \mathfrak{F}$ is defined to be the universal C^*-algebra generated by n isometries $s_{\varepsilon,j}$ $(1 \le j \le n)$ such that $\|s_{\varepsilon,k}s_{\varepsilon,j} - s_{\varepsilon,j}s_{\varepsilon,k}\| \le \varepsilon$ $(1 \le j,k \le n)$.

Remark. Note that, in fact, the norm estimates of the form $\|s_{\varepsilon,k}s^*_{\varepsilon,j} - s^*_{\varepsilon,j}s_{\varepsilon,k}\| \le \varepsilon$ may be further required (and in what follows). If not assuming these estimates, $\otimes^n_\varepsilon \mathfrak{F}$ should be replaced with $C^*(\mathbb{N}^n)_\varepsilon$ in the same sense (and in what follows).

Definition 2.2 The flip on $\otimes^n_\varepsilon \mathfrak{F}$ is the (non-unital) endomorphism σ defined by $\sigma(s_{\varepsilon,j}) = s^*_{\varepsilon,j}$ for $1 \le j \le n$. Since σ^2 is the identity on $\otimes^n_\varepsilon \mathfrak{F}$, we denote by $(\otimes^n_\varepsilon \mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2$ the crossed product of $\otimes^n_\varepsilon \mathfrak{F}$ by the action σ of $\mathbb{Z}_2$.

Definition 2.3 For $\varepsilon \in [0,2]$, we define E^m_ε to be the universal C^*-algebra generated by n isometries $t^{(j)}_1$ $(1 \le j \le m)$ and the partial isometries $t^{(j)}_{n+1} = u^n t^{(j)}_1 (u^*)^n$ for $n \in \mathbb{N}$, where u is an isometry such that $\|ut^{(j)}_1 - t^{(j)}_1 u\| \le \varepsilon$ and $\|t^{(k)}_1 t^{(j)}_1 - t^{(j)}_1 t^{(k)}_1\| \le \varepsilon$ $(1 \le j,k \le m)$. Let α_ε be the endomorphism of E^m_ε defined by $\alpha_\varepsilon(t^{(j)}_n) = t^{(j)}_{n+1} = ut^{(j)}_n u^*$ for $n \in \mathbb{N}$. Let $E^m_\varepsilon \rtimes_{\alpha_\varepsilon} \mathbb{N}$ be the semigroup crossed product of E^m_ε by the action α_ε of $\mathbb{N}$.

Remark. Note that $\otimes^n_2 \mathfrak{F}$ (or $C^*(\mathbb{N}^n)_2$) is isomorphic to the unital full free product $*^n_{\mathbb{C}} \mathfrak{F}$, which is also isomorphic to the full semigroup C^*-algebra $C^*(*^n \mathbb{N})$ of the free semigroup $*^n \mathbb{N}$. As in the above remark, the additional estimates $\|u(t^{(j)}_1)^* - (t^{(j)}_1)^* u\| \le \varepsilon$ and $\|t^{(k)}_1 (t^{(j)}_1)^* - (t^{(j)}_1)^* t^{(k)}_1\| \le \varepsilon$ may be required accordingly.

It is shown as in [8] that $\otimes^{m+1}_\varepsilon \mathfrak{F} \cong E^m_\varepsilon \rtimes_{\alpha_\varepsilon} \mathbb{N}$ as in the case in Section 1.

Proposition 2.4 *For $\varepsilon \in [0,2]$, we have*

$$(\otimes^{m+1}_\varepsilon \mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2 \cong E^m_\varepsilon \rtimes_{\alpha_\varepsilon * \beta} (\mathbb{N} * \mathbb{Z}_2),$$

where the action β on E_ε^m is given by $\beta(t_n^{(j)}) = (t_n^{(j)})^*$ for $n \in \mathbb{N}$ and $1 \le j \le m$.

Proof. This is shown as in the proof of Proposition 1.4 similarly. $\square$

Theorem 2.5 *For $0 \le \varepsilon < 2$, we obtain (inductively)*

$$K_0((\otimes_\varepsilon^{m+1}\mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2) \cong \mathbb{Z}^{2^{m+2}+3}, \quad K_1((\otimes_\varepsilon^{m+1}\mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2) \cong 0.$$

Moreover, $K_j((\otimes_\varepsilon^{m+1}\mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2) \cong K_j((\otimes^{m+1}\mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2)$ for $j = 0, 1$.

Proof. Since $\otimes_\varepsilon^{m+1}\mathfrak{F} \cong E_\varepsilon^m \rtimes_{\alpha_\varepsilon} \mathbb{N}$, note that $E_\varepsilon^m \rtimes_{\alpha_\varepsilon} \mathbb{N}$ is isomorphic to a corner of $(E_\varepsilon^m \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z}$, i.e., $p((E_\varepsilon^m \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z})p$ for a certain projection p, where $\rho_\varepsilon^\wedge$ is the dual action of the circle action on $E_\varepsilon^m \rtimes_{\alpha_\varepsilon} \mathbb{N}$ and id is the identity action on $\mathbb{K}$ (this is a variation of [6], and see also [7]). Hence, $(E_\varepsilon^m \rtimes_{\alpha_\varepsilon} \mathbb{N}) \rtimes_\sigma \mathbb{Z}_2$ is isomorphic to $p((E_\varepsilon^m \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z})p \rtimes_\sigma \mathbb{Z}_2$. Therefore,

$$\begin{aligned} K_j((E_\varepsilon^m \rtimes_{\alpha_\varepsilon} \mathbb{N}) \rtimes_\sigma \mathbb{Z}_2) &\cong K_j(p((E_\varepsilon^m \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z})p \rtimes_\sigma \mathbb{Z}_2) \\ &\cong K_j^{\mathbb{Z}_2}(p((E_\varepsilon^m \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z})p) \\ &\cong K_j^{\mathbb{Z}_2}(p((E_\varepsilon^m \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z})p \otimes \mathbb{K}) \\ &\cong K_j^{\mathbb{Z}_2}(((E_\varepsilon^m \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z}) \otimes \mathbb{K}) \\ &\cong K_j((E_\varepsilon^m \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z} \rtimes \mathbb{Z}_2), \end{aligned}$$

where $p((E_\varepsilon^m \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z})p$ is stably isomorphic to $(E_\varepsilon^m \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z}$, and

$$\begin{aligned} (E_\varepsilon^m \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z} \rtimes \mathbb{Z}_2 &\cong (E_\varepsilon^m \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge * \sigma \otimes \mathrm{id}} (\mathbb{Z}_2 * \mathbb{Z}_2) \\ &\cong (E_\varepsilon^m \rtimes_{\rho_\varepsilon^\wedge * \sigma} (\mathbb{Z}_2 * \mathbb{Z}_2)) \otimes \mathbb{K} \end{aligned}$$

since $\mathbb{Z} \rtimes \mathbb{Z}_2 \cong \mathbb{Z}_2 * \mathbb{Z}_2$ (cf. [2]). Set $F_\varepsilon^m = E_\varepsilon^m \rtimes_{\rho_\varepsilon^\wedge * \sigma} (\mathbb{Z}_2 * \mathbb{Z}_2)$. There exists the following six-term exact sequence $(A)_m$ (cf. [2]):

$$\begin{array}{ccccc} K_0(E_\varepsilon^m) & \longrightarrow & K_0(E_\varepsilon^m \rtimes_{\rho_\varepsilon^\wedge} \mathbb{Z}_2) \oplus K_0(E_\varepsilon^m \rtimes_\sigma \mathbb{Z}_2) & \longrightarrow & K_0(F_\varepsilon^m) \\ \uparrow & & & & \downarrow \\ K_1(F_\varepsilon^m) & \longleftarrow & K_1(E_\varepsilon^m \rtimes_{\rho_\varepsilon^\wedge} \mathbb{Z}_2) \oplus K_1(E_\varepsilon^m \rtimes_\sigma \mathbb{Z}_2) & \longleftarrow & K_1(E_\varepsilon^m). \end{array}$$

We now have the following exact sequence:

$$0 \to \mathfrak{I}_\varepsilon^m \rtimes \mathbb{Z}_2 \to E_\varepsilon^m \rtimes \mathbb{Z}_2 \to \pi(E_\varepsilon^m) \rtimes \mathbb{Z}_2 \to 0,$$

where the map π is sending isometries of E_ε^m to unitaries with the same norm estimates by universality, and $\mathfrak{I}_\varepsilon^m$ is the kernel of π, and the action of $\mathbb{Z}_2$ is given by $\rho_\varepsilon^\wedge$ or σ. Furthermore, it follows that $\mathfrak{I}_\varepsilon^m \rtimes \mathbb{Z}_2 \cong \mathfrak{I}_\varepsilon^m \otimes C^*(\mathbb{Z}_2)$ and the K-theory of $\mathfrak{I}_\varepsilon^m$ is the same as that of $\mathbb{K}$.

It is deduced that $\pi(E_\varepsilon^m) \rtimes \mathbb{Z}_2$ is homotopy equivalent to $C(\mathbb{T}^m) \rtimes_\sigma \mathbb{Z}_2$, where $\beta(z_j) = (z_j^{-1})$ for $(z_j) \in \mathbb{T}^m$. Since the points $(\pm 1, \cdots, \pm 1) \in \mathbb{T}^m$ are fixed under α, we have

$$0 \to C_0(\mathbb{T}^m \setminus (\pm 1, \cdots, \pm 1)) \rtimes \mathbb{Z}_2 \to C(\mathbb{T}^m) \rtimes \mathbb{Z}_2 \to \oplus^{2^m} C^*(\mathbb{Z}_2) \to 0,$$

where $C_0(X)$ is the C^*-algebra of all continuous functions on a locally compact Hausdorff space X vanishing at infinity (in what follows). Set $X_{m+1} = \mathbb{T}^m \setminus (\pm 1, \cdots, \pm 1)$. By considering invariant subspaces in X_{m+1} under β, we obtain a finite composition series $\{\mathfrak{L}_j\}_{j=1}^m$ of $C_0(X_{m+1}) \rtimes \mathbb{Z}_2$ such that $\mathfrak{L}_0 = \{0\}$, $\mathfrak{L}_j = C_0(X_j) \times \mathbb{Z}_2$, and

$$\mathfrak{L}_j/\mathfrak{L}_{j-1} \cong \oplus^{{}_mC_{m-j+1}} C_0((\mathbb{T} \setminus \{\pm 1\})^{m-j+1}) \rtimes \mathbb{Z}_2,$$

where ${}_mC_{m-j+1}$ mean the combinations. Furthermore,

$$C_0((\mathbb{T} \setminus \{\pm 1\})^{m-j+1}) \rtimes \mathbb{Z}_2 \cong C_0(\mathbb{R}^{m-j+1}) \otimes (C(\Pi^{m-j+1}\{\pm i\}) \rtimes \mathbb{Z}_2)$$

and $C(\Pi^{m-j+1}\{\pm i\}) \rtimes \mathbb{Z}_2 \cong \oplus^{m-j+1}(\mathbb{C}^2 \rtimes \mathbb{Z}_2) \cong \oplus^{m-j+1} M_2(\mathbb{C})$, where $\mathbb{T} \setminus \{\pm 1\}$ is homeomorphic to $i\mathbb{R} \cup (-i)\mathbb{R}$ so that the above isomorphisms are deduced from considering orbits under β in this identification. Set $C(m, j) = {}_mC_{m-j+1}(m - j + 1)$. Thus, the following six-term exact sequences are obtained:

$$\begin{array}{ccccc}
K_0(\mathfrak{L}_{j-1}) & \longrightarrow & K_0(\mathfrak{L}_j) & \longrightarrow & K_{m-j+1}(\oplus^{C(m,j)}\mathbb{C}) \\
\uparrow & & & & \downarrow \\
K_{m-j+2}(\oplus^{C(m,j)}\mathbb{C}) & \longleftarrow & K_1(\mathfrak{L}_j) & \longleftarrow & K_1(\mathfrak{L}_{j-1}).
\end{array}$$

Now consider the case $m = 2$. Then

$$0 \to C_0(\mathbb{T}^2 \setminus (\pm 1, \pm 1)) \rtimes \mathbb{Z}_2 \to C(\mathbb{T}^2) \rtimes \mathbb{Z}_2 \to \oplus^{2^2} C^*(\mathbb{Z}_2) \to 0.$$

Furthermore, $0 \to C_0(X_1) \rtimes \mathbb{Z}_2 \to C_0(X_2) \rtimes \mathbb{Z}_2 \to C_0(X_2 \setminus X_1) \rtimes \mathbb{Z}_2 \to 0$, where $X_2 = \mathbb{T}^2 \setminus (\pm 1, \pm 1)$, $X_1 = (\mathbb{T} \setminus \{\pm 1\})^2$, and $C_0(X_2 \setminus X_1) \rtimes \mathbb{Z}_2$ is isomorphic to $\oplus^2 C_0(\mathbb{T} \setminus \{\pm 1\}) \rtimes \mathbb{Z}_2$. We have the following six-term exact sequence:

$$\begin{array}{ccccc}
\mathbb{Z}^2 & \longrightarrow & K_0(C_0(X_2) \rtimes \mathbb{Z}_2) & \longrightarrow & 0 \\
\uparrow & & & & \downarrow \\
\mathbb{Z}^2 & \longleftarrow & K_1(C_0(X_2) \rtimes \mathbb{Z}_2) & \longleftarrow & 0,
\end{array}$$

which implies $K_0(C_0(X_2) \rtimes \mathbb{Z}_2) \cong 0$ and $K_1(C_0(X_2) \rtimes \mathbb{Z}_2) \cong 0$. Thus,

$$\begin{array}{ccccc}
0 & \longrightarrow & K_0(C(\mathbb{T}^2) \rtimes \mathbb{Z}_2) & \longrightarrow & \mathbb{Z}^{2^3} \\
\uparrow & & & & \downarrow \\
0 & \longleftarrow & K_1(C(\mathbb{T}^2) \rtimes \mathbb{Z}_2) & \longleftarrow & 0,
\end{array}$$

which implies $K_0(C(\mathbb{T}^2) \rtimes \mathbb{Z}_2) \cong \mathbb{Z}^{2^3}$ and $K_1(C(\mathbb{T}^2) \rtimes \mathbb{Z}_2) \cong 0$. Therefore,

$$\begin{array}{ccccc}
\mathbb{Z}^2 & \longrightarrow & K_0(E_\varepsilon^2 \rtimes \mathbb{Z}_2) & \longrightarrow & \mathbb{Z}^{2^3} \\
\uparrow & & & & \downarrow \\
0 & \longleftarrow & K_1(E_\varepsilon^2 \rtimes \mathbb{Z}_2) & \longleftarrow & 0.
\end{array}$$

It follows that $K_0(E_\varepsilon^2 \rtimes \mathbb{Z}_2) \cong \mathbb{Z}^{2^3+2}$ and $K_1(E_\varepsilon^2 \rtimes \mathbb{Z}_2) \cong 0$. Therefore,

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & \mathbb{Z}^{2^3+2} \oplus \mathbb{Z}^{2^3+2} & \longrightarrow & K_0(F_\varepsilon^2) \\ \uparrow & & & & \downarrow \\ K_1(F_\varepsilon^2) & \longleftarrow & 0 \oplus 0 & \longleftarrow & 0. \end{array}$$

Hence, it follows that $K_0(F_\varepsilon^2) \cong \mathbb{Z}^{2^4+3}$ and $K_1(F_\varepsilon^2) \cong 0$.

Next consider the case $m = 3$. Then

$$0 \to C_0(\mathbb{T}^3 \setminus (\pm1, \pm1, \pm1)) \rtimes \mathbb{Z}_2 \to C(\mathbb{T}^3) \rtimes \mathbb{Z}_2 \to \oplus^{2^3} C^*(\mathbb{Z}_2) \to 0.$$

Furthermore, $0 \to C_0(X_2) \rtimes \mathbb{Z}_2 \to C_0(X_3) \rtimes \mathbb{Z}_2 \to C_0(X_3 \setminus X_2) \rtimes \mathbb{Z}_2 \to 0$, where $X_3 = \mathbb{T}^3 \setminus (\pm1, \pm1, \pm1)$, and

$$0 \to C_0(X_1) \rtimes \mathbb{Z}_2 \to C_0(X_2) \rtimes \mathbb{Z}_2 \to C_0(X_2 \setminus X_1) \rtimes \mathbb{Z}_2 \to 0,$$

where $X_1 = (\mathbb{T} \setminus \{\pm1\})^3$. We have the following six-term exact sequence:

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(C_0(X_2) \rtimes \mathbb{Z}_2) & \longrightarrow & \mathbb{Z}^6 \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(C_0(X_2) \rtimes \mathbb{Z}_2) & \longleftarrow & \mathbb{Z}^3, \end{array}$$

which implies $K_0(C_0(X_2) \rtimes \mathbb{Z}_2) \cong \mathbb{Z}^3$ and $K_1(C_0(X_2) \rtimes \mathbb{Z}_2) \cong 0$. Furthermore,

$$\begin{array}{ccccc} \mathbb{Z}^3 & \longrightarrow & K_0(C_0(X_3) \rtimes \mathbb{Z}_2) & \longrightarrow & 0 \\ \uparrow & & & & \downarrow \\ \mathbb{Z}^3 & \longleftarrow & K_1(C_0(X_3) \rtimes \mathbb{Z}_2) & \longleftarrow & 0, \end{array}$$

which implies $K_0(C_0(X_3) \rtimes \mathbb{Z}_2) \cong 0$ and $K_1(C_0(X_3) \rtimes \mathbb{Z}_2) \cong 0$. Thus,

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(C(\mathbb{T}^3) \rtimes \mathbb{Z}_2) & \longrightarrow & \mathbb{Z}^{2^4} \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(C(\mathbb{T}^3) \rtimes \mathbb{Z}_2) & \longleftarrow & 0, \end{array}$$

which implies $K_0(C(\mathbb{T}^3) \rtimes \mathbb{Z}_2) \cong \mathbb{Z}^{2^4}$ and $K_1(C(\mathbb{T}^2) \rtimes \mathbb{Z}_2) \cong 0$. Therefore,

$$\begin{array}{ccccc} \mathbb{Z}^2 & \longrightarrow & K_0(E_\varepsilon^3 \rtimes \mathbb{Z}_2) & \longrightarrow & \mathbb{Z}^{2^4} \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(E_\varepsilon^3 \rtimes \mathbb{Z}_2) & \longleftarrow & 0. \end{array}$$

It follows that $K_0(E_\varepsilon^3 \rtimes \mathbb{Z}_2) \cong \mathbb{Z}^{2^4+2}$ and $K_1(E_\varepsilon^3 \rtimes \mathbb{Z}_2) \cong 0$. Therefore,

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & \mathbb{Z}^{2^4+2} \oplus \mathbb{Z}^{2^4+2} & \longrightarrow & K_0(F_\varepsilon^3) \\ \uparrow & & & & \downarrow \\ K_1(F_\varepsilon^3) & \longleftarrow & 0 \oplus 0 & \longleftarrow & 0. \end{array}$$

Hence, it follows that $K_0(F_\varepsilon^3) \cong \mathbb{Z}^{2^5+3}$ and $K_1(F_\varepsilon^3) \cong 0$.

Next consider the case $m = 4$. Then

$$0 \to C_0(\mathbb{T}^4 \setminus (\pm 1, \pm 1, \pm 1, \pm 1)) \rtimes \mathbb{Z}_2 \to C(\mathbb{T}^4) \rtimes \mathbb{Z}_2 \to \oplus^{2^4} C^*(\mathbb{Z}_2) \to 0.$$

Furthermore, $0 \to C_0(X_3) \rtimes \mathbb{Z}_2 \to C_0(X_4) \rtimes \mathbb{Z}_2 \to C_0(X_4 \setminus X_3) \rtimes \mathbb{Z}_2 \to 0$, where $X_4 = \mathbb{T}^4 \setminus (\pm 1, \pm 1, \pm 1, \pm 1)$, and

$$0 \to C_0(X_1) \rtimes \mathbb{Z}_2 \to C_0(X_2) \rtimes \mathbb{Z}_2 \to C_0(X_2 \setminus X_1) \rtimes \mathbb{Z}_2 \to 0,$$

where $X_1 = (\mathbb{T} \setminus \{\pm 1\})^4$. We have the following six-term exact sequence:

$$\begin{array}{ccccc} \mathbb{Z}^4 & \longrightarrow & K_0(C_0(X_2) \rtimes \mathbb{Z}_2) & \longrightarrow & 0 \\ \uparrow & & & & \downarrow \\ \mathbb{Z}^{12} & \longleftarrow & K_1(C_0(X_2) \rtimes \mathbb{Z}_2) & \longleftarrow & 0, \end{array}$$

which implies $K_0(C_0(X_2) \rtimes \mathbb{Z}_2) \cong 0$ and $K_1(C_0(X_2) \rtimes \mathbb{Z}_2) \cong \mathbb{Z}^8$. Furthermore,

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(C_0(X_3) \rtimes \mathbb{Z}_2) & \longrightarrow & \mathbb{Z}^{12} \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(C_0(X_3) \rtimes \mathbb{Z}_2) & \longleftarrow & \mathbb{Z}^8, \end{array}$$

which implies $K_0(C_0(X_3) \rtimes \mathbb{Z}_2) \cong \mathbb{Z}^4$ and $K_1(C_0(X_3) \rtimes \mathbb{Z}_2) \cong 0$. Furthermore,

$$\begin{array}{ccccc} \mathbb{Z}^4 & \longrightarrow & K_0(C_0(X_4) \rtimes \mathbb{Z}_2) & \longrightarrow & 0 \\ \uparrow & & & & \downarrow \\ \mathbb{Z}^4 & \longleftarrow & K_1(C_0(X_4) \rtimes \mathbb{Z}_2) & \longleftarrow & 0, \end{array}$$

which implies $K_0(C_0(X_4) \rtimes \mathbb{Z}_2) \cong 0$ and $K_1(C_0(X_4) \rtimes \mathbb{Z}_2) \cong 0$. Thus,

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(C(\mathbb{T}^4) \rtimes \mathbb{Z}_2) & \longrightarrow & \mathbb{Z}^{2^5} \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(C(\mathbb{T}^4) \rtimes \mathbb{Z}_2) & \longleftarrow & 0, \end{array}$$

which implies $K_0(C(\mathbb{T}^4) \rtimes \mathbb{Z}_2) \cong \mathbb{Z}^{2^5}$ and $K_1(C(\mathbb{T}^4) \rtimes \mathbb{Z}_2) \cong 0$. Therefore,

$$\begin{array}{ccccc} \mathbb{Z}^2 & \longrightarrow & K_0(E_\varepsilon^4 \rtimes \mathbb{Z}_2) & \longrightarrow & \mathbb{Z}^{2^5} \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(E_\varepsilon^4 \rtimes \mathbb{Z}_2) & \longleftarrow & 0. \end{array}$$

It follows that $K_0(E_\varepsilon^4 \rtimes \mathbb{Z}_2) \cong \mathbb{Z}^{2^5+2}$ and $K_1(E_\varepsilon^4 \rtimes \mathbb{Z}_2) \cong 0$. Therefore,

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & \mathbb{Z}^{2^5+2} \oplus \mathbb{Z}^{2^5+2} & \longrightarrow & K_0(F_\varepsilon^4) \\ \uparrow & & & & \downarrow \\ K_1(F_\varepsilon^4) & \longleftarrow & 0 \oplus 0 & \longleftarrow & 0. \end{array}$$

Hence, it follows that $K_0(F_\varepsilon^4) \cong \mathbb{Z}^{2^6+3}$ and $K_1(F_\varepsilon^4) \cong 0$.

The case for m general can be treated by the step by step argument as shown above. The argument for K-theory is inductive in a sense that it involves essentially suspensions and direct sums inductively. The second claim follows from considering the case $\varepsilon = 0$ and the same argument as above. $\square$

Corollary 2.6 *For $0 \le \varepsilon < 2$, the natural onto $*$-homomorphism $\varphi_{\varepsilon,0}$ from $(\otimes_\varepsilon^{m+1}\mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2$ to $(\otimes^{m+1}\mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2$ sending $s_{\varepsilon,j}$ to $s_{0,j}$ $(1 \le j \le m+1)$ induces the isomorphism between their K-groups.*

Proposition 2.7 *There exists a continuous field of C^*-algebras on the closed interval $[0,2]$ such that fibers are $(\otimes_\varepsilon^{m+1}\mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2$ for $\varepsilon \in [0,2]$, and for any $a \in (\otimes_2^{m+1}\mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2$, the sections $[0,2] \ni \varepsilon \mapsto \varphi_\varepsilon(a) \in (\otimes_\varepsilon^{m+1}\mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2$ are continuous, where $\varphi_\varepsilon : (\otimes_2^{m+1}\mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2 \to (\otimes_\varepsilon^{m+1}\mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2$ is the natural onto $*$-homomorphism sending $s_{2,j}$ to $s_{\varepsilon,j}$ $(1 \le j \le m+1)$.*

Proof. As shown before, $(\otimes_\varepsilon^{m+1}\mathfrak{F}) \rtimes_\sigma \mathbb{Z}_2 \cong (E_\varepsilon^m \rtimes_{\alpha_\varepsilon} \mathbb{N}) \rtimes_\sigma \mathbb{Z}_2$. Furthermore, this is isomorphic to $p((E_\varepsilon^m \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z})p \rtimes_\sigma \mathbb{Z}_2$. Hence it follows that

$$\begin{aligned}
((E_\varepsilon^m \rtimes_{\alpha_\varepsilon} \mathbb{N}) \rtimes_\sigma \mathbb{Z}_2) \otimes \mathbb{K} &\cong (p((E_\varepsilon^m \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z})p \rtimes_\sigma \mathbb{Z}_2) \otimes \mathbb{K} \\
&\cong (p((E_\varepsilon^m \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z})p \otimes \mathbb{K}) \rtimes_{\sigma \otimes \mathrm{id}} \mathbb{Z}_2 \\
&\cong (((E_\varepsilon^m \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z}) \otimes \mathbb{K}) \rtimes_{\sigma \otimes \mathrm{id}} \mathbb{Z}_2 \\
&\cong ((E_\varepsilon^m \otimes \mathbb{K} \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id} \otimes \mathrm{id}} \mathbb{Z}) \rtimes_{\sigma \otimes \mathrm{id}} \mathbb{Z}_2 \\
&\cong ((E_\varepsilon^m \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge \otimes \mathrm{id}} \mathbb{Z}) \rtimes_\sigma \mathbb{Z}_2 \\
&\cong (E_\varepsilon^m \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge * \sigma \otimes \mathrm{id}} (\mathbb{Z}_2 * \mathbb{Z}_2).
\end{aligned}$$

It is deduced from [2] that there exists a continuous field of C^*-algebras on $[0,2]$ such that fibers are $(E_\varepsilon^m \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge * \sigma \otimes \mathrm{id}} (\mathbb{Z}_2 * \mathbb{Z}_2)$ for $\varepsilon \in [0,2]$, and for any $b \in (E_2^m \otimes \mathbb{K}) \rtimes_{\rho_2^\wedge * \sigma \otimes \mathrm{id}} (\mathbb{Z}_2 * \mathbb{Z}_2)$, the sections $[0,2] \ni \varepsilon \mapsto \psi_\varepsilon(b) \in (E_\varepsilon^m \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge * \sigma \otimes \mathrm{id}} (\mathbb{Z}_2 * \mathbb{Z}_2)$ are continuous, where ψ_ε is the unique onto $*$-homomorphism from $(E_2^m \otimes \mathbb{K}) \rtimes_{\rho_2^\wedge * \sigma \otimes \mathrm{id}} (\mathbb{Z}_2 * \mathbb{Z}_2)$ to $(E_\varepsilon^m \otimes \mathbb{K}) \rtimes_{\rho_\varepsilon^\wedge * \sigma \otimes \mathrm{id}} (\mathbb{Z}_2 * \mathbb{Z}_2)$. Cutting down this continuous field by cutting down the fibers from $((E_\varepsilon^m \rtimes_{\alpha_\varepsilon} \mathbb{N}) \rtimes_\sigma \mathbb{Z}_2) \otimes \mathbb{K}$ to $(E_\varepsilon^m \rtimes_{\alpha_\varepsilon} \mathbb{N}) \rtimes_\sigma \mathbb{Z}_2$ by minimal projections, we obtain the desired continuous field. $\square$

Received: May 2008. Revised: June 2008.

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