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Limit Cycles of Liénard-Type Dynamical Systems

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ABSTRACT

In this paper, using geometric properties of the field rotation parameters, we present a solution of *Smale's Thirteenth Problem* on the maximum number of limit cycles for Liénard's polynomial system, generalize the obtained results for special classes of polynomial systems, and complete the global qualitative analysis of a piecewise linear dynamical system approximating a Liénard-type polynomial system with an arbitrary number of finite singularities.

RESUMEN

En este artículo, usando propriedades geometricas del campo de rotación de parametros, nosotros presentamos una solución del problema trece de Smale sobre el número máximo de ciclos límite para el sistema polinomial de Liénard, generalizamos los resultados obtenidos para clases especiales de sistemas polinomiales, y completamos el analisis cualitativo global de un sistema dinamico linear por pedazos aproximando un sistema polinomial de tipo Liérnard con un número arbitrario finito de singularidades.

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Math. Subj. Class.: *34C05, 34C07, 34C23, 37G05, 37G10, 37G15.*

1 Introduction

We consider planar dynamical systems

$$\dot{x} = P_n(x, y), \quad \dot{y} = Q_n(x, y), \tag{1.1}$$

where $P_n(x, y)$ and $Q_n(x, y)$ are polynomials with real coefficients in the real variables x, y and not greater than n degree. First of all, we consider a special case of (1.1): a classical Liénard's polynomial system of the form

$$\dot{x} = y, \quad \dot{y} = -x + \mu_1\, y + \mu_2\, y^2 + \mu_3\, y^3 + \ldots + \mu_{2k}\, y^{2k} + \mu_{2k+1}\, y^{2k+1}. \tag{1.2}$$

The main problem of qualitative theory of such systems is *Hilbert's Sixteenth Problem* on the maximum number and relative position of their limit cycles, i. e., closed isolated trajectories of (1.1). This problem was formulated as one of the fundamental problems for mathematicians of the XX century, however it has not been solved even in the simplest (quadratic, cubic, etc.) cases of the polynomial systems. In this paper, we suggest a new geometric approach to solving the problem in the case of Liénard's system (1.2). In this special case, it is considered as *Smale's Thirteenth Problem* becoming one of the main problems for mathematicians of the XXI century [16], [20].

In Section 2 of this paper, applying a canonical system with field rotation parameters and using geometric properties of the spirals filling the interior and exterior domains of limit cycles, we present a solution of *Smale's Thirteenth Problem* for Liénard's polynomial system (1.2). In Section 3, by means of the same geometric approach, we generalize the obtained result and present a solution of *Hilbert's sixteenth problem* on the maximum number of limit cycles surrounding a singular point for an arbitrary polynomial system. In Section 4, we consider generalized Liénard's cubic system with three finite singularities, for which the developed geometric approach can complete its global qualitative analysis: in particular, it easily solves the problem on the maximum number of limit cycles in their different distribution. In this section, we give also an alternative proof of the main theorem for the generalized Liénard's system applying the Wintner–Perko termination principle for multiple limit cycles. In Section 5, by means of the same principle, we complete the global qualitative analysis of a piecewise linear dynamical system approximating a Liénard-type polynomial system with an arbitrary number of finite singularities.

2 Liénard's polynomial system

System (1.2) and more general Liénard's systems have been studied in numerous works (see, for example, [2], [16], [17], [19], [20]). It is easy to see that (1.2) has the only finite singularity: an anti-saddle at the origin. At infinity, system (1.2) for $k \geqslant 1$ has two singular points: a node at the "ends" of the y-axis and a saddle at the "ends" of the x-axis. For studying the infinite singularities, the methods applied in [2] for Rayleigh's and van der Pol's equations and also Erugin's two-isocline method developed in [10] can be used. Following [10], we will study limit cycle bifurcations of (1.2) by means of a canonical system containing only the field rotation parameters of (1.2). It is valid the following theorem.

Theorem 2.1. *Liénard's polynomial system* (1.2) *with limit cycles can be reduced to the canonical form*

$$\dot{x} = y \equiv P, \quad \dot{y} = -x + \mu_1\, y + y^2 + \mu_3\, y^3 + \ldots + y^{2k} + \mu_{2k+1}\, y^{2k+1} \equiv Q, \tag{2.1}$$

where $\mu_1,\ \mu_3, \ldots,\ \mu_{2k+1}$ *are field rotation parameters of* (2.1).

Proof. Vanish all odd parameters of (1.2),

$$\dot{x} = y, \quad \dot{y} = -x + \mu_2\, y^2 + \mu_4\, y^4 + \ldots + \mu_{2k}\, y^{2k}, \tag{2.2}$$

and consider the corresponding equation

$$\frac{dy}{dx} = \frac{-x + \mu_2\, y^2 + \mu_4\, y^4 + \ldots + \mu_{2k}\, y^{2k}}{y} \equiv F(x, y). \tag{2.3}$$

Since $F(x, -y) = -F(x, y)$, the direction field of (2.3) (and the vector field of (2.2) as well) is symmetric with respect to the x-axis. It follows that for arbitrary values of the parameters μ_2, $\mu_4, \ldots,\ \mu_{2k}$ system (2.2) has a center at the origin and cannot have a limit cycle surrounding this point. Therefore, without loss of generality, all even parameters of system (1.2) can be supposed to be equal, for example, to one: $\mu_2 = \mu_4 = \ldots = \mu_{2k} = 1$ (they could be also supposed to be equal to zero).

To prove that the rest (odd) parameters rotate the vector field of (2.1), let us calculate the following determinants:

$$\begin{gathered}
\Delta_{\mu_1} = PQ'_{\mu_1} - QP'_{\mu_1} = y^2 \geq 0, \\
\Delta_{\mu_3} = PQ'_{\mu_3} - QP'_{\mu_3} = y^2 \geq 0, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\Delta_{\mu_{2k+1}} = PQ'_{\mu_{2k+1}} - QP'_{\mu_{2k+1}} = y^2 \geq 0.
\end{gathered}$$

By definition of a field rotation parameter [4], for increasing each of the parameters $\mu_1,\ \mu_3, \ldots,$ μ_{2k+1}, under the fixed others, the vector field of system (2.1) is rotated in positive direction (counterclockwise) in the whole phase plane; and, conversely, for decreasing each of these parameters, the vector field of (2.1) is rotated in negative direction (clockwise).

Thus, for studying limit cycle bifurcations of (1.2), it is sufficient to consider canonical system (2.1) containing only its odd parameters, $\mu_1, \mu_3, \ldots, \mu_{2k+1}$, which rotate the vector field of (2.1). The theorem is proved. $\square$

By means of canonical system (2.1), let us study global limit cycle bifurcations of (1.2) and prove the following theorem.

Theorem 2.2. *Liénard's polynomial system* (1.2) *has at most k limit cycles.*

Proof. According to Theorem 2.1, for the study of limit cycle bifurcations of system (1.2), it is sufficient to consider canonical system (2.1) containing only the field rotation parameters of (1.2): $\mu_1, \mu_3, \ldots, \mu_{2k+1}$.

Vanish all these parameters:

$$\dot{x} = y, \quad \dot{y} = -x + y^2 + y^4 + \ldots + y^{2k}. \tag{2.4}$$

System (2.4) is symmetric with respect to the x-axis and has a center at the origin. Let us input successively the field rotation parameters into this system beginning with the parameters at the highest degrees of y and alternating with their signs. So, begin with the parameter μ_{2k+1} and let, for definiteness, $\mu_{2k+1} > 0$:

$$\dot{x} = y, \quad \dot{y} = -x + y^2 + y^4 + \ldots + y^{2k} + \mu_{2k+1}\, y^{2k+1}. \tag{2.5}$$

In this case, the vector field of (2.5) is rotated in positive direction (counterclockwise) turning the origin into a nonrough unstable focus.

Fix μ_{2k+1} and input the parameter $\mu_{2k-1} < 0$ into (2.5):

$$\dot{x} = y, \quad \dot{y} = -x + y^2 + y^4 + \ldots + \mu_{2k-1}\, y^{2k-1} + y^{2k} + \mu_{2k+1}\, y^{2k+1}. \tag{2.6}$$

Then the vector field of (2.6) is rotated in opposite direction (clockwise) and the focus immediately changes the character of its stability (since its degree of nonroughness decreases and the sign of the field rotation parameter at the lower degree of y changes) generating a stable limit cycle. Under further decreasing μ_{2k-1}, this limit cycle will expand infinitely, not disappearing at infinity (because of the parameter μ_{2k+1} at the higher degree of y).

Denote the limit cycle by Γ_1, the domain outside the cycle by D_1, the domain inside the cycle by D_2 and consider logical possibilities of the appearance of other (semi-stable) limit cycles from a "trajectory concentration" surrounding the origin. It is clear that, under decreasing the parameter μ_{2k-1}, a semi-stable limit cycle cannot appear in the domain D_2, since the focus spirals filling this domain will untwist and the distance between their coils will increase because of the vector field rotation.

By contradiction, we can also prove that a semi-stable limit cycle cannot appear in the domain D_1. Suppose it appears in this domain for some values of the parameters $\mu^*_{2k+1} > 0$ and $\mu^*_{2k-1} < 0$. Return to initial system (2.4) and change the inputting order for the field rotation parameters. Input first the parameter $\mu_{2k-1} < 0$:

$$\dot{x} = y, \quad \dot{y} = -x + y^2 + y^4 + \ldots + \mu_{2k-1}\, y^{2k-1} + y^{2k}. \tag{2.7}$$

Fix it under $\mu_{2k-1} = \mu^*_{2k-1}$. The vector field of (2.7) is rotated clockwise and the origin turns into a nonrough stable focus. Inputting the parameter $\mu_{2k+1} > 0$ into (2.7), we get again system (2.6), the vector field of which is rotated counterclockwise. Under this rotation, a stable limit cycle Γ_1 will immediately appear from infinity, more precisely, from a separatrix cycle of the Poincaré circle form containing infinite singularities of the saddle and node types [2]. This cycle will contract, the outside spirals winding onto the cycle will untwist and the distance between their coils will increase under increasing μ_{2k+1} to the value μ^*_{2k+1}. It follows that there are no values of $\mu^*_{2k-1} < 0$ and $\mu^*_{2k+1} > 0$, for which a semi-stable limit cycle could appear in the domain D_1.

This contradiction proves the uniqueness of a limit cycle surrounding the origin in system (2.6) for any values of the parameters μ_{2k-1} and μ_{2k+1} of different signs. Obviously, if these parameters have the same sign, system (2.6) has no limit cycles surrounding the origin at all.

Let system (2.6) have the unique limit cycle Γ_1. Fix the parameters $\mu_{2k+1} > 0$, $\mu_{2k-1} < 0$ and input the third parameter, $\mu_{2k-3} > 0$, into this system:

$$\dot{x} = y, \quad \dot{y} = -x + y^2 + \ldots + \mu_{2k-3}\, y^{2k-3} + y^{2k-2} + \ldots + \mu_{2k+1}\, y^{2k+1}. \tag{2.8}$$

The vector field of (2.8) is rotated counterclockwise, the focus at the origin changes the character of its stability and the second (unstable) limit cycle, Γ_2, immediately appears from this point. Under further increasing μ_{2k-3}, the limit cycle Γ_2 will join with Γ_1 forming a semi-stable limit cycle, Γ_{12}, which will disappear in a "trajectory concentration" surrounding the origin. Can another semi-stable limit cycle appear around the origin in addition to Γ_{12}? It is clear that such a limit cycle cannot appear either in the domain D_1 bounded on the inside by the cycle Γ_1 or in the domain D_3 bounded by the origin and Γ_2 because of the increasing distance between the spiral coils filling these domains under increasing the parameter μ_{2k-3}.

To prove impossibility of the appearance of a semi-stable limit cycle in the domain D_2 bounded by the cycles Γ_1 and Γ_2 (before their joining), suppose the contrary, i. e., for some set of values of the parameters, $\mu^*_{2k+1} > 0$, $\mu^*_{2k-1} < 0$, and $\mu^*_{2k-3} > 0$, such a semi-stable cycle exists. Return to system (2.4) again and input first the parameters $\mu_{2k-3} > 0$ and $\mu_{2k+1} > 0$:

$$\dot{x} = y, \quad \dot{y} = -x + y^2 + \ldots + \mu_{2k-3}\, y^{2k-3} + y^{2k-2} + y^{2k} + \mu_{2k+1}\, y^{2k+1}. \tag{2.9}$$

Both parameters act in a similar way: they rotate the vector field of (2.9) counterclockwise turning the origin into a nonrough unstable focus.

Fix these parameters under $\mu_{2k-3} = \mu^*_{2k-3}$, $\mu_{2k+1} = \mu^*_{2k+1}$ and input the parameter $\mu_{2k-1} < 0$ into (2.9) getting again system (2.8). Since, by our assumption, this system has two limit cycles for $\mu_{2k-1} > \mu^*_{2k-1}$, there exists some value of the parameter, μ^{12}_{2k-1} ($\mu^*_{2k-1} < \mu^{12}_{2k-1} < 0$), for which a semi-stable limit cycle, Γ_{12}, appears in system (2.8) and then splits into a stable cycle, Γ_1, and an unstable cycle, Γ_2, under further decreasing μ_{2k-1}. The formed domain D_2 bounded by the limit cycles Γ_1, Γ_2 and filled by the spirals will enlarge since, on the properties of a field rotation parameter, the interior unstable limit cycle Γ_2 will contract and the exterior stable limit cycle Γ_1 will expand under decreasing μ_{2k-1}. The distance between the spirals of the domain D_2 will naturally increase, what will prevent the appearance of a semi-stable limit cycle in this domain for $\mu_{2k-1} < \mu^{12}_{2k-1}$.

Thus, there are no such values of the parameters, $\mu^*_{2k+1} > 0$, $\mu^*_{2k-1} < 0$, and $\mu^*_{2k-3} > 0$, for which system (2.8) would have an additional semi-stable limit cycle. Obviously, there are no other values of the parameters μ_{2k+1}, μ_{2k-1}, and μ_{2k-3} for which system (2.8) would have more than two limit cycles surrounding the origin. Therefore, two is the maximum number of limit cycles for system (2.8). This result agrees with [19], where it was proved for the first time that the maximum number of limit cycles for Liénard's system of the form

$$\dot{x} = y, \quad \dot{y} = -x + \mu_1\, y + \mu_3\, y^3 + \mu_5\, y^5 \tag{2.10}$$

was equal to two.

Suppose that system (2.8) has two limit cycles, Γ_1 and Γ_2 (this is always possible if $\mu_{2k+1} \gg -\mu_{2k-1} \gg \mu_{2k-3} > 0$), fix the parameters μ_{2k+1}, μ_{2k-1}, μ_{2k-3} and consider a more general system than (2.8) (and (2.10)) inputting the fourth parameter, $\mu_{2k-5} < 0$, into (2.8):

$$\dot{x} = y, \quad \dot{y} = -x + y^2 + \ldots + \mu_{2k-5}\, y^{2k-5} + y^{2k-4} + \ldots + \mu_{2k+1}\, y^{2k+1}. \tag{2.11}$$

Under decreasing μ_{2k-5}, the vector field of (2.11) will be rotated clockwise and the focus at the origin will immediately change the character of its stability generating the third (stable) limit cycle, Γ_3. Under further decreasing μ_{2k-5}, Γ_3 will join with Γ_2 forming a semi-stable limit cycle, Γ_{23}, which will disappear in a "trajectory concentration" surrounding the origin; the cycle Γ_1 will expand infinitely tending to the Poincaré circle at infinity.

Let system (2.11) have three limit cycles: Γ_1, Γ_2, Γ_3. Could an additional semi-stable limit cycle appear under decreasing μ_{2k-5}, after splitting of which system (2.11) would have five limit cycles around the origin? It is clear that such a limit cycle cannot appear either in the domain D_2 bounded by the cycles Γ_1 and Γ_2 or in the domain D_4 bounded by the origin and Γ_3 because of the increasing distance between the spiral coils filling these domains under decreasing μ_{2k-5}. Consider two other domains: D_1 bounded on the inside by the cycle Γ_1 and D_3 bounded by the cycles Γ_2 and Γ_3. As before, we will prove impossibility of the appearance of a semi-stable limit cycle in these domains by contradiction.

Suppose that for some set of values of the parameters $\mu^*_{2k+1} > 0$, $\mu^*_{2k-1} < 0$, $\mu^*_{2k-3} > 0$, and $\mu^*_{2k-5} < 0$, such a semi-stable cycle exists. Return to system (2.4) again, input first the parameters $\mu_{2k-5} < 0$, $\mu_{2k-1} < 0$ and then the parameter $\mu_{2k+1} > 0$:

$$\dot{x} = y, \quad \dot{y} = -x + y^2 + \ldots + \mu_{2k-5} y^{2k-5} + \ldots + \mu_{2k-1} y^{2k-1} + y^{2k} + \mu_{2k+1} y^{2k+1}. \tag{2.12}$$

Fix the parameters μ_{2k-5}, μ_{2k-1} under the values μ^*_{2k-5}, μ^*_{2k-1}, respectively. Under increasing μ_{2k+1}, the node at infinity will change the character of its stability, the separatrix behavior of the infinite saddle will be also changed and a stable limit cycle, Γ_1, will immediately appear from the Poincaré circle at infinity [2]. Fix μ_{2k+1} under the value μ^*_{2k+1} and input the parameter $\mu_{2k-3} > 0$ into (2.12) getting system (2.11).

Since, by our assumption, (2.11) has three limit cycles for $\mu_{2k-3} < \mu^*_{2k-3}$, there exists some value of the parameter μ^{23}_{2k-3} $(0 < \mu^{23}_{2k-3} < \mu^*_{2k-3})$ for which a semi-stable limit cycle, Γ_{23}, appears in this system and then splits into an unstable cycle, Γ_2, and a stable cycle, Γ_3, under further increasing μ_{2k-3}. The formed domain D_3 bounded by the limit cycles Γ_2, Γ_3 and also the domain D_1 bounded on the inside by the limit cycle Γ_1 will enlarge and the spirals filling these domains will untwist excluding a possibility of the appearance of a semi-stable limit cycle there.

All other combinations of the parameters μ_{2k+1}, μ_{2k-1}, μ_{2k-3}, and μ_{2k-5} are considered in a similar way. It follows that system (2.11) has at most three limit cycles. If we continue the procedure of successive inputting the odd parameters, $\mu_{2k-7}, \ldots, \mu_3, \mu_1$, into system (2.4), it is possible first to obtain k limit cycles $(\mu_{2k+1} \gg -\mu_{2k-1} \gg \mu_{2k-3} \gg -\mu_{2k-5} \gg \mu_{2k-7} \gg \ldots)$ and then to conclude that canonical system (2.1) (i. e., Liénard's polynomial system (1.2) as well) has at most k limit cycles. The theorem is proved. $\square$

3 An arbitrary polynomial system

Let us consider an arbitrary polynomial system

$$\dot{x} = P_n(x, y, \mu_1, \ldots, \mu_k), \quad \dot{y} = Q_n(x, y, \mu_1, \ldots, \mu_k), \tag{3.1}$$

where P_n and Q_n are polynomials in the real variables x, y and not greater than n degree containing k field rotation parameters, $\mu_1, \ldots, \mu_k$, and having an anti-saddle at the origin. Generalizing the main result of the previous section, we will prove the following theorem.

Theorem 3.1. *Polynomial system* (3.1) *containing k field rotation parameters and having a singular point of the center type at the origin for the zero values of these parameters can have at most $k-1$ limit cycles surrounding the origin.*

Proof. Vanish all parameters of (3.1) and suppose that the obtained system

$$\dot{x} = P_n(x, y, 0, \ldots, 0), \quad \dot{y} = Q_n(x, y, 0, \ldots, 0) \tag{3.2}$$

has a singular point of the center type at the origin. Let us input successively the field rotation parameters, $\mu_1, \ldots, \mu_k$, into this system.

Suppose, for example, that $\mu_1 > 0$ and that the vector field of the system

$$\dot{x} = P_n(x, y, \mu_1, 0, \ldots, 0), \quad \dot{y} = Q_n(x, y, \mu_1, 0, \ldots, 0) \tag{3.3}$$

is rotated counterclockwise turning the origin into a stable focus under increasing μ_1.

Fix μ_1 and input the parameter μ_2 into (3.3) changing it so that the field of the system

$$\dot{x} = P_n(x, y, \mu_1, \mu_2, 0, \ldots, 0), \quad \dot{y} = Q_n(x, y, \mu_1, \mu_2, 0, \ldots, 0) \tag{3.4}$$

would be rotated in opposite direction (clockwise). Let be so for $\mu_2 < 0$. Then, for some value of this parameter, a limit cycle will appear in system (3.4). There are three logical possibilities for such a bifurcation: 1) the limit cycle appears from the focus at the origin; 2) it can also appear from some separatrix cycle surrounding the origin; 3) the limit cycle appears from a so-called "trajectory concentration". In the last case, the limit cycle is semi-stable and, under further decreasing μ_2, it splits into two limit cycles (stable and unstable), one of which then disappears at (or tends to) the origin and the other disappears on (or tends to) some separatrix cycle surrounding this point. But since the stability character of both a singular point and a separatrix cycle is quite easily controlled [10], this logical possibility can be excluded. Let us choose one of the two other possibilities: for example, the first one, the so-called Andronov–Hopf bifurcation. Suppose that, for some value of μ_2, the focus at the origin becomes non-rough, changes the character of its stability and generates a stable limit cycle, Γ_1.

Under further decreasing μ_2, three new logical possibilities can arise: 1) the limit cycle Γ_1 disappears on some separatrix cycle surrounding the origin; 2) a separatrix cycle can be formed earlier than Γ_1 disappears on it, then it generates one more (unstable) limit cycle, Γ_2, which joins with Γ_1 forming a semi-stable limit cycle, Γ_{12}, disappearing in a "trajectory concentration" under further decreasing μ_2; 3) in the domain D_1 outside the cycle Γ_1 or in the domain D_2 inside Γ_1, a semi-stable limit cycle appears from a "trajectory concentration" and then splits into two limit cycles (logically, the appearance of such semi-stable limit cycles can be repeated).

Let us consider the third case. It is clear that, under decreasing μ_2, a semi-stable limit cycle cannot appear in the domain D_2, since the focus spirals filling this domain will untwist and the distance between their coils will increase because of the vector field rotation. By contradiction, we can prove that a semi-stable limit cycle cannot appear in the domain D_1. Suppose it appears in this domain for some values of the parameters $\mu_1^* > 0$ and $\mu_2^* < 0$. Return to initial system (3.2) and change the inputting order for the field rotation parameters. Input first the parameter $\mu_2 < 0$:

$$\dot{x} = P_n(x, y, \mu_2, 0, \ldots, 0), \quad \dot{y} = Q_n(x, y, \mu_2, 0, \ldots, 0). \tag{3.5}$$

Fix it under $\mu_2 = \mu_2^*$. The vector field of (3.5) is rotated clockwise and the origin turns into a unstable focus. Inputting the parameter $\mu_1 > 0$ into (3.5), we get again system (3.4), the vector field of which is rotated counterclockwise. Under this rotation, a stable limit cycle, Γ_1, will appear from some separatrix cycle. The limit cycle Γ_1 will contract, the outside spirals winding onto this cycle will untwist and the distance between their coils will increase under increasing μ_1 to the value μ_1^*. It follows that there are no values of $\mu_2^* < 0$ and $\mu_1^* > 0$, for which a semi-stable limit cycle could appear in the domain D_1.

The second logical possibility can be excluded by controlling the stability character of the separatrix cycle [10]. Thus, only the first possibility is valid, i. e., system (3.4) has at most one limit cycle.

Let system (3.4) have the unique limit cycle Γ_1. Fix the parameters $\mu_1 > 0$, $\mu_2 < 0$ and input the third parameter, $\mu_3 > 0$, into this system supposing that μ_3 rotates its vector field counterclockwise:

$$\dot{x} = P_n(x, y, \mu_1, \mu_2, \mu_3, 0, \ldots, 0), \quad \dot{y} = Q_n(x, y, \mu_1, \mu_2, \mu_3, 0, \ldots, 0). \tag{3.6}$$

Here we can have two basic possibilities: 1) the limit cycle Γ_1 disappears at the origin; 2) the second (unstable) limit cycle, Γ_2, appears from the origin and, under further increasing the parameter μ_3, the cycle Γ_2 joins with Γ_1 forming a semi-stable limit cycle, Γ_{12}, which disappears in a "trajectory concentration" surrounding the origin. Besides, we can also suggest that: 3) in the domain D_2 bounded by the origin and Γ_1, a semi-stable limit cycle, Γ_{23}, appears from a "trajectory concentration", splits into an unstable cycle, Γ_2, and a stable cycle, Γ_3, and then the cycles Γ_1, Γ_2 disappear through a semi-stable limit cycle, Γ_{12}, and the cycle Γ_3 disappears through the Andronov–Hopf bifurcation; 4) a semi-stable limit cycle, Γ_{34}, appears in the domain D_2 bounded by the cycles Γ_1, Γ_2 and, for some set of values of the parameters, μ_1^*, μ_2^*, μ_3^*, system (3.6) has at least four limit cycles.

Let us consider the last, fourth, case. It is clear that a semi-stable limit cycle cannot appear either in the domain D_1 bounded on the inside by the cycle Γ_1 or in the domain D_3 bounded by the origin and Γ_2 because of the increasing distance between the spiral coils filling these domains under increasing the parameter μ_3. To prove impossibility of the appearance of a semi-stable limit cycle in the domain D_2, suppose the contrary, i. e., for some set of values of the parameters, $\mu_1^* > 0$, $\mu_2^* < 0$, and $\mu_3^* > 0$, such a semi-stable cycle exists. Return to system (3.2) again and input first the parameters $\mu_3 > 0$, $\mu_1 > 0$:

$$\dot{x} = P_n(x, y, \mu_1, \mu_3, 0, \ldots, 0), \quad \dot{y} = Q_n(x, y, \mu_1, \mu_3, 0, \ldots, 0). \tag{3.7}$$

Fix these parameters under $\mu_3 = \mu_3^*$, $\mu_1 = \mu_1^*$ and input the parameter $\mu_2 < 0$ into (3.7) getting again system (3.6). Since, by our assumption, this system has two limit cycles for $\mu_2 > \mu_2^*$, there exists some value of the parameter, μ_2^{12} ($\mu_2^* < \mu_2^{12} < 0$), for which a semi-stable limit cycle, Γ_{12}, appears in system (3.6) and then splits into a stable cycle, Γ_1, and an unstable cycle, Γ_2, under further decreasing μ_2. The formed domain D_2 bounded by the limit cycles Γ_1, Γ_2 and filled by the spirals will enlarge, since, on the properties of a field rotation parameter, the interior unstable limit cycle Γ_2 will contract and the exterior stable limit cycle Γ_1 will expand under decreasing μ_2. The distance between the spirals of the domain D_2 will naturally increase, what will prevent the appearance of a semi-stable limit cycle in this domain for $\mu_2 < \mu_2^{12}$.

Thus, there are no such values of the parameters, $\mu_1^* > 0$, $\mu_2^* < 0$, $\mu_3^* > 0$, for which system (3.6) would have an additional semi-stable limit cycle. Therefore, the fourth case cannot be realized. The third case is considered absolutely similarly. It follows from the first two cases that system (3.6) can have at most two limit cycles.

Suppose that system (3.6) has two limit cycles, Γ_1 and Γ_2, fix the parameters $\mu_1 > 0$, $\mu_2 < 0$, $\mu_3 > 0$ and input the fourth parameter, $\mu_4 < 0$, into this system supposing that μ_4 rotates its vector field clockwise:

$$\dot{x} = P_n(x, y, \mu_1, \dots, \mu_4, 0, \dots, 0), \quad \dot{y} = Q_n(x, y, \mu_1, \dots, \mu_4, 0, \dots, 0). \tag{3.8}$$

The most interesting logical possibility here is that when the third (stable) limit cycle, Γ_3, appears from the origin and then, under preservation of the cycles Γ_1 and Γ_2, in the domain D_3 bounded on the inside by the cycle Γ_3 and on the outside by the cycle Γ_2, a semi-stable limit cycle, Γ_{45}, appears and then splits into a stable cycle, Γ_4, and an unstable cycle, Γ_5, i.e., when system (3.8) for some set of values of the parameters, μ_1^*, μ_2^*, μ_3^*, μ_4^*, has at least five limit cycles. Logically, such a semi-stable limit cycle could also appear in the domain D_1 bounded on the inside by the cycle Γ_1, since, under decreasing μ_4, the spirals of the trajectories of (3.8) will twist and the distance between their coils will decrease. On the other hand, in the domain D_2 bounded on the inside by the cycle Γ_2 and on the outside by the cycle Γ_1 and also in the domain D_4 bounded by the origin and Γ_3, a semi-stable limit cycle cannot appear, since, under decreasing μ_4, the spirals will untwist and the distance between their coils will increase. To prove impossibility of the appearance of a semi-stable limit cycle in the domains D_3 and D_1, suppose the contrary, i.e., for some set of values of the parameters, $\mu_1^* > 0$, $\mu_2^* < 0$, $\mu_3^* > 0$, and $\mu_4^* < 0$, such a semi-stable cycle exists. Return to system (3.2) again, input first the parameters $\mu_4 < 0$, $\mu_2 < 0$ and then the parameter $\mu_1 > 0$:

$$\dot{x} = P_n(x, y, \mu_1, \mu_2, \mu_4, 0, \dots, 0), \quad \dot{y} = Q_n(x, y, \mu_1, \mu_2, \mu_4, 0, \dots, 0). \tag{3.9}$$

Fix the parameters μ_4, μ_2 under the values μ_4^*, μ_2^*, respectively. Under increasing μ_1, a separatrix cycle is formed around the origin generating a stable limit cycle, Γ_1. Fix μ_1 under the value μ_1^* and input the parameter $\mu_3 > 0$ into (3.9) getting system (3.8).

Since, by our assumption, system (3.8) has three limit cycles for $\mu_3 < \mu_3^*$, there exists some value of the parameter μ_3^{23} $(0 < \mu_3^{23} < \mu_3^*)$ for which a semi-stable limit cycle, Γ_{23}, appears in this system and then splits into an unstable cycle, Γ_2, and a stable cycle, Γ_3, under further increasing μ_3. The formed domain D_3 bounded by the limit cycles Γ_2, Γ_3 and also the domain D_1 bounded on the inside by the limit cycle Γ_1 will enlarge and the spirals filling these domains will untwist excluding a possibility of the appearance of a semi-stable limit cycle there.

All other combinations of the parameters μ_1, μ_2, μ_3, and μ_4 are considered in a similar way. It follows that system (3.8) has at most three limit cycles. If we continue the procedure of successive inputting the field rotation parameters, $\mu_5, \mu_6, \dots, \mu_k$, into system (3.2), it is possible to conclude that system (3.1) can have at most $k - 1$ limit cycles surrounding the origin. The theorem is proved. $\square$

4 Generalized liénard's cubic system

In [13], we considered generalized Liénard's cubic system of the form:

$$\dot{x} = y, \quad \dot{y} = -x + (\lambda - \mu)\, y + (3/2)\, x^2 + \mu\, xy - (1/2)\, x^3 + \alpha\, x^2 y\,. \tag{4.1}$$

This system has three finite singularities: a saddle $(1, 0)$ and two antisaddles — $(0, 0)$ and $(2, 0)$. At infinity system (4.1) can have either the only nilpotent singular point of fourth order with two closed elliptic and four hyperbolic domains or two singular points: one of them is a hyperbolic saddle and the other is a triple nilpotent singular point with two elliptic and two hyperbolic domains. We studied global bifurcations of limit and separatrix cycles of (4.1), found possible distributions of its limit cycles and carried out a classification of its separatrix cycles. We proved also the following theorems.

Theorem 4.1. *The foci of system* (4.1) *can be at most of second order.*

Theorem 4.2. *System* (4.1) *has at least three limit cycles.*

Using the results obtained in [13] and applying the approach developed in this paper, we can easily prove a much stronger theorem.

Theorem 4.3. *System* (4.1) *has at most three limit cycles with the following their distributions*: $((1, 1), 1)$, $((1, 2), 0)$, $((2, 1), 0)$, $((1, 0), 2)$, $((0, 1), 2)$, *where the first two numbers denote the numbers of limit cycles surrounding each of two anti-saddles and the third one denotes the number of limit cycles surrounding simultaneously all three finite singularities.*

Theorem 4.3 agrees, for example, with the earlier results by Iliev and Perko [15], but it does not agree with a quite recent result by Dumortier and Li [5] published in the same journal. The authors of both papers use very similar methods: small perturbations of a Hamiltonian system. In [15], the zeros of the Melnikov functions are studied and, in particular, it is proved that at most two limit cycles can bifurcate from either the interior or exterior period annulus of the Hamiltonian under small parameter perturbations giving a generalized Liénard system. In [5], zeros of the Abelian integrals are studied and it is "proved" that at most four limit cycles can bifurcate from the exterior period annulus. Thus, Dumortier and Li "obtain" a configuration of four big limit cycles surrounding three finite singularities together with the fifth small limit cycle which surrounds one of the anti-saddles.

The result by Dumortier and Li [5] also does not agree with the Wintner–Perko termination principle for multiple limit cycles [10], [18]. Applying the method as developed in [3], [7]–[13], we can show that system (4.1) cannot have either a multiplicity-three limit cycle or more than three limit cycles in any configuration. That will be another proof of Theorem 4.3 (the same approach can be applied to proving Theorems 2.2 and 3.1 as well). But first let us formulate the Wintner–Perko termination principle [18] for the polynomial system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\mu}), \tag{4.2$\boldsymbol{\mu}$}$$

where $\boldsymbol{x} \in \mathbf{R}^2$; $\boldsymbol{\mu} \in \mathbf{R}^n$; $\boldsymbol{f} \in \mathbf{R}^2$ ($\boldsymbol{f}$ is a polynomial vector function).

Theorem 4.4 (Wintner–Perko termination principle). *Any one-parameter family of multiplicity-m limit cycles of relatively prime polynomial system* $(4.2_{\boldsymbol{\mu}})$ *can be extended in a unique way to a maximal one-parameter family of multiplicity-m limit cycles of* $(4.2_{\boldsymbol{\mu}})$ *which is either open or cyclic.*

If it is open, then it terminates either as the parameter or the limit cycles become unbounded; or, the family terminates either at a singular point of $(4.2_{\boldsymbol{\mu}})$, *which is typically a fine focus of multiplicity m, or on a* (*compound*) *separatrix cycle of* $(4.2_{\boldsymbol{\mu}})$, *which is also typically of multiplicity m.*

The proof of this principle for general polynomial system $(4.2_{\boldsymbol{\mu}})$ with a vector parameter $\boldsymbol{\mu} \in \mathbf{R}^n$ parallels the proof of the planar termination principle for the system

$$\dot{x} = P(x, y, \lambda), \quad \dot{y} = Q(x, y, \lambda) \tag{4.2_λ}$$

with a single parameter $\lambda \in \mathbf{R}$ (see [10], [18]), since there is no loss of generality in assuming that system $(4.2_{\boldsymbol{\mu}})$ is parameterized by a single parameter λ; i. e., we can assume that there exists an analytic mapping $\boldsymbol{\mu}(\lambda)$ of $\mathbf{R}$ into $\mathbf{R}^n$ such that $(4.2_{\boldsymbol{\mu}})$ can be written as $(4.2\,_{\boldsymbol{\mu}(\lambda)})$ or even (4.2_λ) and then we can repeat everything, what had been done for system (4.2_λ) in [18]. In particular, if λ is a field rotation parameter of (4.2_λ), the following Perko's theorem on monotonic families of limit cycles is valid.

Theorem 4.5. *If L_0 is a nonsingular multiple limit cycle of* (4.2_0), *then L_0 belongs to a one-parameter family of limit cycles of* (4.2_λ); *furthermore*:

1) *if the multiplicity of L_0 is odd, then the family either expands or contracts monotonically as λ increases through λ_0*;

2) *if the multiplicity of L_0 is even, then L_0 bifurcates into a stable and an unstable limit cycle as λ varies from λ_0 in one sense and L_0 disappears as λ varies from λ_0 in the opposite sense; i. e., there is a fold bifurcation at λ_0.*

Proof of Theorem 4.3. The proof is carried out by contradiction. Suppose that system (4.1) with three field rotation parameters, λ, μ, and α, has three limit cycles around, for example, the origin (the case when limit cycles surround another focus is considered in a similar way). Then we get into some domain in the space of these parameters which is bounded by two fold bifurcation surfaces forming a cusp bifurcation surface of multiplicity-three limit cycles.

The corresponding maximal one-parameter family of multiplicity-three limit cycles cannot be cyclic, otherwise there will be at least one point corresponding to the limit cycle of multiplicity four (or even higher) in the parameter space. Extending the bifurcation curve of multiplicity-four limit cycles through this point and parameterizing the corresponding maximal one-parameter family of multiplicity-four limit cycles by a field rotation parameter, according to Theorem 4.5, we will obtain a monotonic curve which, by the Wintner–Perko termination principle (Theorem 4.4), terminates either at the origin or on some separatrix cycle surrounding the origin. Since we know absolutely precisely at least the cyclicity of the singular point (Theorem 4.1) which is equal to two, we have got a contradiction with the termination principle stating that the multiplicity of limit cycles cannot be higher than the multiplicity (cyclicity) of the singular point in which they terminate.

If the maximal one-parameter family of multiplicity-three limit cycles is not cyclic, on the same principle (Theorem 4.4), this again contradicts to Theorem 4.1 not admitting the multiplicity of limit cycles higher than two. Moreover, it also follows from the termination principle that either the ordinary separatrix loop or the eight-loop cannot have the multiplicity (cyclicity) higher than two (in that way, it can be proved that the cyclicity of three other separatrix cycles [13] is at most two). Therefore, according to the same principle, there are no more than two limit cycles in the exterior domain surrounding all three finite singularities of (4.1). Thus, system (4.1) cannot have either a multiplicity-three limit cycle or more than three limit cycles in any configuration. The theorem is proved. □

5 A piecewise linear dynamical system

Consider a Liénard-type dynamical system

$$\dot{x} = y - \varphi(x), \;\; \dot{y} = \beta - \alpha x - y, \;\; \alpha > 0, \;\; \beta > 0, \tag{5.1}$$

where $\varphi(x)$ is a piecewise linear function containing k dropping sections and approximating an arbitrary polynomial of degree $2k+1$. The line $\beta - \alpha x - y = 0$ and the curve $y = \varphi(x)$ can be considered as the isoclines of zero and infinity, respectively, for the corresponding equation. Such systems and equations may occur, for example, when tunnel diode circuits and some other problems are studied (see [1], [2], [6], [14]).

Suppose that the ascending sections of system (5.1) have an inclination $k_1 > 0$ and the descending (dropping) sections have an inclination $k_2 < 0$. Then the phase plane of (5.1) can be divided onto $2k+1$ parts in every of which (5.1) is a linear system: the ascending sections are in $k+1$ strip regions $(I, III, V, \ldots, 2K+1)$ and the descending sections are in other k such regions $(II, IV, VI, \ldots, 2K)$. The parameters k_1, k_2, and also α can be considered as rotation parameters for the sewed vector field of (5.1) (see [2], [10]).

System (5.1) can have an odd number of simple singular points: $1, 3, 5, \ldots, 2k+1$. If (5.1) has the only singular point, this point will be always an antisaddle (center, focus or node). A focus (node) will be always stable in odd regions and unstable in even regions if $k_2 > 1$. If system (5.1) has $2k+1$ singularities, then k of them are saddles (they are in even regions) and $k+1$ others are antisaddles (foci or nodes) which are always stable (they are in odd regions). The pieces of the straight lines $\beta = x_{2i-1}\alpha + y_{2i-1}$ and $\beta = x_{2i}\alpha + y_{2i}$ $(i = 1, 2, \ldots, k)$, where (x_{2i-1}, y_{2i-1}) and

(x_{2i}, y_{2i}) are the coordinates of the upper and lower corner points of the curve $\varphi(x)$, respectively, form a discriminant curve separating the domains in the plane (α, β), where $\alpha \leq k_2$, with different numbers of singular points. The points of the discriminant curve correspond to the sewed singularities of saddle-focus or saddle-node type ($\alpha < k_2$) and its corner points correspond to the unstable equilibrium segments ($\alpha = k_2$) which coincide with the dropping sections of the curve $y = \varphi(x)$.

In the case when $k_2 < 1$, closed trajectories cannot exist and only bifurcations of singular points are possible in system (5.1). Therefore, we will consider further only the case when $k_2 > 1$ and $(k_1 - 1)^2 < 4k_2$ giving various bifurcations and, first of all, the bifurcations of limit cycles. Studying all such bifurcations (local and global), we will give a proof of the following theorem.

Theorem 5.1. *System* (5.1) *with k dropping sections and $2k+1$ singular points can have at most $k+2$ limit cycles, $k+1$ of which surround the foci one by one and the last, $(k+2)$-th, limit cycle surrounds all of the singular points of* (5.1).

Proof of Theorem 5.1. To prove the theorem, we will study both local and global bifurcations of limit cycles. The limit cycle of system (5.1) will be called *small* if it belongs to at most two adjoining regions; the cycle will be called *big* if it belongs to at least three adjoining regions.

5.1 Local bifurcations

Following [1], we will study first stability of the singular points on the line of sewing. Suppose that the straight line $\beta - \alpha x - y = 0$ passes through the corner point (x_1, y_1) of the curve $y = \varphi(x)$ on the boundary of regions I, II and that $\alpha > (k_2 + 1)^2/4$. Then the region I (II) will be filled by the pieces of trajectories of the stable (unstable) focus.

Introduce positive coordinates S_0 (lower (x_1, y_1)) and S_1 (upper (x_1, y_1)) on the line of sewing of regions I and II; S_2 (lower (x_2, y_2)) and S_3 (upper (x_2, y_2)) on the line of sewing of regions II and III, etc. The maps $S_0 \to S_1$ along the trajectories of region I and $S_1 \to S_0$ along the trajectories of region II are written as follows:

$$S_1 = S_0 e^{\pi\sigma_1/\omega_1}, \quad \bar{S}_0 = S_1 e^{\pi\sigma_2/\omega_2}, \tag{5.2}$$

where σ_i, ω_i $(i = 1, 2)$ are the real and imaginary parts of the roots of the characteristic equation for a singular point of regions I, II, respectively.

The singular point (x_1, y_1) will be a sewed center ($\bar{S}_0 = S_0$) iff $\sigma_1/\omega_1 + \sigma_2/\omega_2 = 0$, i. e., when $\alpha = \alpha^* \equiv (1 - k_1/k_2)/(k_2 - k_1 + 2)$. The sewed focus (x_1, y_1) will be stable ($\bar{S}_0 < S_0$) when $\alpha > \alpha^*$ and unstable ($\bar{S}_0 > S_0$) when $\alpha < \alpha^*$.

Consider the return map $S_0 \to \bar{S}_0$ along the trajectories of regions I and II. For region I, we will have

$$\begin{aligned} S_0 &= \frac{\delta_0}{\sin\omega_1\tau_1}(\omega_1\cos\omega_1\tau_1 - \sigma_1\sin\omega_1\tau_1 - \omega_1 e^{-\sigma_1\tau_1}) \equiv \delta_0\zeta(\tau_1), \\ S_1 &= \frac{\delta_0}{\sin\omega_1\tau_1}(\omega_1\cos\omega_1\tau_1 + \sigma_1\sin\omega_1\tau_1 - \omega_1 e^{\sigma_1\tau_1}) \equiv \delta_0\chi(\tau_1), \end{aligned} \tag{5.3}$$

where δ_0 is the distance from the boundary of regions I, II to the singular point; ζ and χ are monotonic functions. The return map along the trajectories of region II has a similar form.

Calculation of the first derivative for the return map gives

$$\frac{\mathrm{d}\bar{S}_0}{\mathrm{d}S_0} = \frac{S_0}{\bar{S}_0}\, e^{2(\sigma_1\tau_1+\sigma_2\tau_2)}, \tag{5.4}$$

where τ_i $(i = 1, 2)$ is motion time along the trajectories of regions I, II, respectively; $\sigma_i = (1+k_i)/2$ $(i = 1, 2)$.

Studying the return map $S_0 \to \bar{S}_0$ by means of (5.4), we prove that at most one limit cycle can exist in regions I and II (see also [1]). The same result can be obtained for regions III and $IV, \ldots, 2K-1$ and $2K$.

Consider now the map $\bar{S}_0 = f(S_0)$ sewed of two pieces: $\bar{S}_0 = \xi(S_0)$ along the trajectories in regions I, $II, \ldots, 2K$ and $\bar{S}_0 = \psi(S_0)$ along the trajectories in all regions, I, $II, \ldots, 2K$, $2K+1$. The map $S_0 \to S_1$ in region I is given by (5.3). The maps $S_1 \to S_3$, $S_3 \to S_5, \ldots, S_{2k-1} \to S_{2k-2}$ $(S_{2k-1} \to S_{2k+1}$, $S_{2k+1} \to S_{2k}$, $S_{2k} \to S_{2k-2})$, $S_{2k-2} \to S_{2k-4}, \ldots, S_2 \to S_0$ have similar forms.

The derivatives for the functions $\xi(S_0)$, $\psi(S_0)$ are given by the following expressions, respectively:

$$\frac{\mathrm{d}\bar{S}_0}{\mathrm{d}S_0} = \frac{S_0}{\bar{S}_0}\, e^{2(\sigma_1(\tau_1+\tau_3^+ + \tau_3^- + \ldots + \tau_{2k-1}) + \sigma_2(\tau_2^+ + \tau_2^- + \ldots + \tau_{2k-2}^+ + \tau_{2k-2}^-))}, \tag{5.5}$$

$$\frac{\mathrm{d}\bar{S}_0}{\mathrm{d}S_0} = \frac{S_0}{\bar{S}_0}\, e^{2(\sigma_1(\tau_1+\tau_3^+ + \tau_3^- + \ldots + \tau_{2k+1}) + \sigma_2(\tau_2^+ + \tau_2^- + \ldots + \tau_{2k}^+ + \tau_{2k}^-))}, \tag{5.6}$$

where τ_1, τ_{2k-1}, τ_{2k+1} are motion times in regions I, $2K-1$, $2K+1$ and τ_{2i}^+ (τ_{2i}^-), τ_{2i+1}^+ (τ_{2i+1}^-), $i = 1, 2, \ldots, k$, are motion times in the upper (lower) parts of regions $II, III, \ldots, 2K$, respectively.

Studying the return map $\bar{S}_0 = f(S_0)$ by means of (5.5) and (5.6), we prove that at most two limit cycles can be generated by the boundary of the domain filled by closed trajectories of (5.1) and that these two limit cycles can be only outside the boundary.

Suppose that a part of the straight line $\beta - \alpha x - y = 0$ coincides with a dropping section of (5.1), for example, with the first one ($\alpha = k_2$). The dropping section of (5.1) will be an unstable equilibrium segment and regions I, II (because of the condition $(k_1-1)^2 < 4k_2$) will be filled by trajectories of the stable foci. It is easy to obtain an explicit expression for the map of the half-line S_0 into itself:

$$\bar{S}_0 = S_0\, e^{2\pi\sigma_1/\omega_1} + \delta(k_2 - 1)(1 + e^{\pi\sigma_1/\omega_1}), \tag{5.7}$$

where δ is the width of regions II.

This map has the only stable fixed point, and we can show that two stable foci surrounded by unstable limit cycles (one by one) are generated from the ends of the equilibrium segment under the rotation of the line $\beta - \alpha x - y = 0$ (see also [1]).

The simplest type of separatrix cycles of (5.1) is a so-called eight-loop formed by two ordinary saddle loops. In the case of $2k + 1$ simple singular points, a separatrix cycle can contain $k + 1$ saddle loops, the first and the last of which are ordinary loops with one rough saddle on each and the $k - 1$ others are separatrix digons with two rough saddles on each. Such a separatrix cycle will be called *nondegenerate.* In the cases when the straight line $\beta - \alpha x - y = 0$ passes through the corner points of the curve $y = \varphi(x)$, we will have *degenerate* separatrix cycles of lips-type containing one or two sewed saddle-nodes. It is clear that the bifurcations of separatrix cycles do not depend on the parameter β (see [1]). The separatrix cycles can be formed or destroyed only under a variation of the parameter α. The character of their stability will be determined by the sign of the saddle quantities which are always positive in our case, when the saddles are inside or on the boundary of even regions II, $IV, \ldots, 2K$ and $k_2 > 1$ (the corresponding theorems are valid for the piecewise linear dynamical systems as well [2]). It follows that the separatrix cycles of (5.1) are always unstable (inside and outside) and, under a variation of α, a nondegenerate separatrix cycle can generate at most $k + 1$ small unstable limit cycles inside its loops (digons) or the only big unstable limit cycle outside it.

5.2 Global bifurcations

Now we are able to consider the global bifurcations of limit cycles. Suppose again that the zero isocline $\beta - \alpha x - y = 0$ passes through the corner point (x_1, y_1) of the infinite isocline $y = \varphi(x)$ and that $\alpha > \alpha^*$. In this case, the only singular point in the phase plane is a sewed stable focus and all trajectories of (5.1) tend to it when $t \to +\infty$. For decreasing α ($k_2 < \alpha < \alpha^*$), the sewed focus becomes unstable and a stable limit cycle is generated from the boundary curve of the domain filled by closed trajectories (immediately after passing the value α^* by the parameter α).

For $\alpha = k_2$, the first dropping section of (5.1) will coincide with a part of the straight line $\beta - \alpha x - y = 0$ and an unstable equilibrium segment will appear inside the stable limit cycle. If we rotate the line $\beta - \alpha x - y = 0$ around an interior point of the segment (changing both of the parameters, α and β), two unstable limit cycles surrounding stable foci (one by one) will be generated from the ends (x_1, y_1) and (x_2, y_2) of the equilibrium segment. Under the further rotation of the line $\beta - \alpha x - y = 0$, it will pass first through the next corner point, (x_4, y_4), and then, successively, through the points $(x_6, y_6), \ldots, (x_{2k}, y_{2k})$. Every time, the corner point becomes a sewed saddle-node generating an unstable limit cycle surrounding a stable focus. So, we will get a piecewise linear system with $2k + 1$ singular points having at least $k + 1$ small unstable limit cycles surrounding the stable foci (one by one) inside a big stable limit cycle, $k + 2$, surrounding all of the singular points.

Under the further rotation of the zero isocline, all $k + 1$ small limit cycles simultaneously disappear in a separatrix cycle consisting of $k+1$ loops (digons), this separatrix cycle generates a big (unstable) limit cycle which combines with another big (stable) limit cycle of (5.1) forming a semi-stable (double) limit cycle which finally disappears in a so-called trajectory condensation.

Let us prove that system (5.1) cannot have more than $k+2$ limit cycles. The proof is carried out by contradiction by means of the Wintner–Perko termination principle [2], [10], [18]. Since a small limit cycle is always unique in the corresponding strip regions, suppose that system (5.1) with three field rotation parameters, k_1, k_2, and α, has three big limit cycles. Then we get into some domain in the space of these parameters which is bounded by two fold bifurcation surfaces forming a cusp bifurcation surface of multiplicity-three limit cycles [10], [18].

The corresponding maximal one-parameter family of multiplicity-three limit cycles cannot be cyclic, otherwise there will be at least one point corresponding to the limit cycle of multiplicity four (or even higher) in the parameter space. Extending the bifurcation curve of multiplicity-four limit cycles through this point and parameterizing the corresponding maximal one-parameter family of multiplicity-four limit cycles by a field rotation parameter, for example, by the parameter α, we will obtain a monotonic curve which, by the Wintner–Perko termination principle, terminates either at the boundary curve of the domain filled by closed trajectories of (5.1) or on some degenerate separatrix cycle of (5.1) [10], [18].

Since we know at least the cyclicity of the boundary curve which is equal to two, we have got a contradiction with the termination principle stating that the multiplicity of limit cycles cannot be higher than the multiplicity (cyclicity) of the end bifurcation points in which they terminate [10], [18].

If the maximal one-parameter family of multiplicity-three limit cycles is not cyclic, using the same principle, this again contradicts with the cyclicity result for the boundary curve not admitting the multiplicity of limit cycles to be higher than two. Moreover, it also follows from the termination principle that the degenerate separatrix cycles of (5.1) cannot have the multiplicity (cyclicity) higher than two. Therefore, according to the same principle, there are no more than two big limit cycles in the exterior domain outside the boundary curve of (5.1).

The same results can be obtained by means of the new geometric methods developed in [12]. The phase portraits and bifurcation diagrams for system (5.1) will be similar to that which were constructed in [1], [2]. Thus, system (5.1) with $2k+1$ singular points cannot have more than $k+2$ limit cycles, i. e., $k+2$ is the maximum number of limit cycles of such system and the obtained distribution ($k+1$ small limit cycles plus a big limit cycle) is the only possibility for their distribution. The theorem is proved. □

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