CUBO A Mathematical Journal
Vol.10, N^{o}03, (137–144). October 2008

On Some Bitopological γ-Separation Axioms

M. Lellis Thivagar
Department of Mathematics, Arul Anandar College,
Karumathur, Madurai (Dt.)-625514, Tamilnadu, India
email: `mlthivagar@yahoo.co.in`

S. Athisaya Ponmani
Department of Mathematics, Jayaraj Annapackiam,
College for Women, Periyakulam, Theni (Dt.)-625601, Tamilnadu, India
email: `athisayaponmani@yahoo.co.in`

R. Raja Rajeswari
Department of Mathematics, Sri Parasakthi College,
Courtalam, Tirunelveli (Dt.)-627802, Tamilnadu, India
email: `raji_arul2000@yahoo.co.in`

and

Erdal Ekici
Department of Mathematics, Canakkale Onsekiz Mart University,
Terzioglu Campus, 17020 Canakkale, Turkey
email: `e.mail:eekici@comu.edu.tr`

ABSTRACT

The aim of this paper is to introduce the notions of (i,j)-γ-T_1, $(i,j)\gamma$-R_1, (i,j)-γ-T_2 and (i,j)-γ-US spaces, (i,j)-γ-open mappings and (i,j)-γ-irresolute mappings.

RESUMEN

El objetivo de este artículo es introducir las nociones de espacios (i,j)-γ-T_1, $(i,j)\gamma$-R_1, (i,j)-γ-T_2 y (i,j)-γ-US, aplicaciones (i,j)-abiertas y (i,j)-γ-irresolutas.

Key words and phrases: *(i,j)-γ-open set, (i,j)-γ-T_1 space, (i,j)-γ-R_1space, (i,j)-γ-US space, (i,j)-γ-open mapping, and (i,j)-γ-irresolute mapping.*

Math. Subj. Class.: *54C55.*

1 Introduction

In 1982, Mashhour et al. [11] introduced the notion of preopen sets, also called locally dense sets by Corson and Michael [4]. The class of preopen sets properly contains the class of open sets. As the intersection of two preopen sets may fail to be preopen, the class of preopen sets does not always form a topology. In a submaximal space i.e. a topological space X in which every dense subset is open, collection of all preopen sets form a topology. Indeed, many notions in Topology can be defined in terms of preopen sets (see [3], [5], [8], [12] and [13]). In 1987, Andrijevic [2] offered a new class of open sets called γ-open sets by utilizing preopen sets. Recently, Abd El Monsef et al. [1] have applied preopen sets in connection with the topological applications of rough set measures in information systems. Moreover, it has been shown in [6] that the notion preopen sets is important with respect to the digital topology. Many researchers also used the notion of preopen sets in fuzzy topological spaces which Professor El-Naschie has recently shown in [7] the importance of the notion of fuzzy topology which may be relevent to quantam particle physics in connection with string theory and ϵ^{∞}theory.

In a bitopological space (X,τ_1,τ_2), the γ-open set is generalized in the form of (i,j)-γ-open set, i, $j=1,2$ and $i \neq j$ [14] and these sets are used to define the separation axiom (i,j)-γ-T_0 [14].

In this paper we define $(i,j)\gamma$-T_1, (i,j)-γ-R_1, (i,j)-γ-T_2 and (i,j)-γ-US spaces and show that (i,j)-γ-US axiom is stronger than (i,j)-γ-T_1 axiom and is weaker than (i,j)-γ-T_2 axiom.

We recall some definitions and concepts which are useful in the following sections.

2 Preliminaries

In a topological space (X,τ), the interior and the closure of a subset A are denoted by $int(A)$ and $cl(A)$, respectively.

Definition 1 *A subset A of X is called pre-open set [11] if $A \subset int(cl(A))$.*

Definition 2 *A subset A of a topological space (X, τ) is called a γ-set [2] if $A \cap S \in PO(X)$ for each $S \in PO(X)$.*

In the above definition, $PO(X)$ is the family of all pre-open sets in X. The family of all γ-sets in X is denoted by $\gamma O(X)$.

In the following sections by a space X, we mean a bitopological space (X, τ_1, τ_2).

Definition 3 *A subset A of X is called (i, j)-pre-open [9] if $A \subset \tau_i$-int$(\tau_j$-cl$(A))$.*

Definition 4 *A subset A of X is called (i, j)-γ-open [14], if $A \cap B$ is (i, j)-pre-open for every (i, j)-pre-open set B in X.*

We denote the family of (i, j)-γ-open sets in X by (i, j)-$\gamma O(X)$.

Theorem 5 *[14] The family of all (i, j)-γ-open sets in X forms a topology on X.*

Definition 6 *A subset $A \subset X$ is called (i, j)-γ-closed [14] if its complement, A^c in X is (i, j)-γ-open.*

Definition 7 *For any $A \subset X$*

(i) (i, j)-γ-closure of A [14] is the intersection of all the (i, j)-γ-closed sets containing A and is written as (i, j)-γ-cl(A).

(ii) (i, j)-γ-kernal of A [14] is the intersection of all the (i, j)-γ-open sets containing A and is written as (i, j)-γ-ker(A).

Definition 8 *A space X is called (i, j)-γ-T_0[14] if for $x,\ y \in X,\ x \neq y$, there exists $U \in (i, j)$-$\gamma O(X)$ such that U contains only one of x and y but not the other where $i,\ j = 1, 2,\ i \neq j$.*

Definition 9 *A map f:$(X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called pairwise γ-continuous (briefly p.γ-continuous)[14] if the inverse image of each σ_i-open set of Y is (i, j)-γ-open in X for $i,\ j = 1, 2$ and $i \neq j$.*

In the following section the (i, j)-γ-open sets are used to define some separation axioms.

3 Some separation axioms

In this section we define the (i,j)-γ-T_1, (i,j)-γ-R_1, (i,j)-γ-T_2 and (i,j)-γ-US spaces and study some characterizations.

Definition 10 *A space X is called (i,j)-γ-T_1 if for x, y in X, $x \neq y$, there exist U, $V \in (i,j)$-$\gamma O(X)$ such that $x \in U$, $y \notin V$ and $y \in V$, $x \notin V$.*

Definition 11 *A space X is said to be (i,j)-γ-R_1 if for x, y in X, $x \neq y$ with (i,j)-γ-$cl(\{x\}) \neq (i,j)$-$\gamma cl(\{y\})$, there exist disjoint (i,j)-γ-open sets U, V such that (i,j)-γ-$cl(\{x\}) \subset U$ and (i,j)-γ-$cl(\{y\}) \subset V$.*

Theorem 12 *A space X is (i,j)-γ-T_1 if and only if the singletons in X are (i,j)-γ-closed sets.*

Proof. Proof is evident since the family (i,j)-$\gamma O(X)$ is a topology. ■

Theorem 13 *A space X is (i,j)-γ-R_1 if and only if (i,j)-γ-$ker(\{x\}) \neq (i,j)$-γ-$ker(\{y\})$ for any x, y in X , there exist disjoint (i,j)-γ-open sets U and V such that γ-$cl(\{x\}) \subset U$ and γ-$cl(\{y\}) \subset V$.*

Definition 14 *A space X is said to be (i,j)-γ-T_2 if for any two distinct points x, y in X, there exist disjoint (i,j)-γ-open sets U, V such that $x \in U$ and $y \in V$.*

Theorem 15 *A space X is (i,j)-γ-T_2 if and only if it is (i,j)-γ-T_0 and (i,j)-γ-R_1.*

Proof. Necessity. If X is (i,j)-γ-T_2 then it is (i,j)-γ-T_1 and then (i,j)-γ-T_0. Since X is (i,j)-γ-T_1, by Theorem 12, (i,j)-γ-$cl(\{x\}) = \{x\}$ and (i,j)-γ-$cl(\{y\}) = \{y\}$ for any two distinct points x, y in X. Therefore, (i,j)-γ-$cl(\{x\}) \neq (i,j)$-γ-$cl(\{y\})$ for any two distinct points x, y in X and hence X is (i,j)-γ-R_1.

Sufficiency. If X is (i,j)-γ-T_0 and if x, y are two distinct points in X, there exists an (i,j)-γ-open set U containing only one of x and y but not the other. Let $x \in U$ and $y \notin U$, say. Then $y \notin (i,j)$-γ-$ker(\{x\})$ and so (i,j)-γ-$ker(\{x\}) \neq (i,j)\gamma$-$ker(\{y\})$ for any two distinct points x, y in X. Since X is (i,j)-γ-R_1, by Theorem 13, there exist disjoint (i,j)-γ-open sets U and V such that (i,j)-$\gamma cl(\{x\}) \subset U$ and (i,j)-γ-$cl(\{y\}) \subset V$. Thus $x \in U$ and $y \in V$ and $U \cap V = \emptyset$. Hence X is (i,j)-γ-T_2. ■

Definition 16 *A net $\{x_\alpha{:}\alpha \in D, \geq\}$ is said to be bitopologically converges to a point $x \in X$, denoted by $\{x_\alpha{:}\alpha \in D, \geq\} \xrightarrow{\gamma} x$ if the net is eventually in every (i,j)-γ-open set containing x, i, $j = 1,2$, $i \neq j$.*

Theorem 17 *If a map* $f:(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is p.*γ*-continuous then for each* $x \in X$ *and each net* $\{x_\alpha : \alpha \in D, \geq\}$ *in* X*, bitopologically* γ*-converging to* x *the image net* $\{f(x_\alpha) : \alpha \in D, \geq\}$ *is bitopologically* γ*-convergent to* $f(x)$ *in* Y*.*

Proof. Let $V \subset Y$ be σ_i-open in Y containing $f(x)$, $i = 1, 2$. The bitopologically γ-convergence of the net $\{x_\alpha : \alpha \in D, \geq\}$ in X implies that there exists $\alpha_0 \in D$ such that for all $\alpha \geq \alpha_0$, $x_\alpha \in f^{-1}(V)$. Therefore, $f(x_\alpha) \in V$ for all $\alpha \geq \alpha_0$. Hence the net $\{f(x_\alpha) : \alpha \in D, \geq\} \xrightarrow{\gamma} f(x)$. ■

Definition 18 *A space* X *is said to be* (i, j)*-*γ*-*US *if every bitopologically* γ*-convergent net* $\{x_\alpha : \alpha \in D, \geq\}$ *in* X *is bitopologically* γ*-convergent to a unique pioint in* X*.*

Proposition 19 *Every* $(i, j)\gamma$*-*T_2 *space is* (i, j)*-*γ*-*US*.*

Proof. If possible, let the net $\{x_\alpha : \alpha \in D, \geq\}$ in a $(i, j)\gamma$-T_2 space X be bitopologically γ-convergent to two distinct points x, y in X. Then the net is eventually in every (i, j)-γ-open set containing x and also in every (i, j)-γ-open set containing y. This contradicts that X is (i, j)-γ-T_2. ■

Proposition 20 *Every* (i, j)*-*γ*-*US *space is* (i, j)*-*γ*-*T_1*.*

Proof. Let x, $y \in X$, $x \neq y$. If $x_n = x$ for every x in the net $\{x_\alpha : \alpha \in D, \geq\}$ then it is evident that the net is bitopologically γ-convergent to x. Since X is (i, j)-γ-US, the net $\{x_\alpha : \alpha \in D, \geq\}$ cannot be bitopologically γ-convergent to y and hence there exists an (i, j)-γ-open set containing y but not x. A similar argument gives an (i, j)-γ-open set containing x but not y. Hence X is (i, j)-γ-T_1. ■

Remark 21 *The following diagram holds for a space* X *as shown in the Proposition 19 and 20.*

$$(i,j)\text{-}\gamma\text{-}T_2 \textit{ space} \;\Rightarrow\; (i,j)\text{-}\gamma\text{-}US \textit{ space} \;\Rightarrow\; (i,j)\text{-}\gamma\text{-}T_1 \textit{ space}$$

Theorem 22 *A space* X *is* (i, j)*-*γ*-*T_2 *if and only if it is* (i, j)*-*γ*-*R_1 *and* (i, j)*-*γ*-*US*.*

Proof. If X is (i, j)-γ-T_2, then it is (i, j)-γ-R_1, by Theorem 15 and by Proposition 19, X is (i, j)-γ-US.

Conversely, if X is (i, j)-γ-R_1 and (i, j)-γ-US then by Proposition 20, X is (i, j)-γ-T_1. Thus X is $(i, j)\gamma$-T_1 and (i, j)-γ-R_1. Hence by Theorem 15, X is (i, j)-γ-T_2. ■

4 Some bitopolgical γ- mappings

In this section we define (i,j)-γ-open mappings and (i,j)-γ-irresolute mappings.

Definition 23 *A map $f{:}(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is called (i,j)-γ-open if the image of each τ_i-open set in X is (i,j)-γ-open in Y, $i,\ j = 1,2,\ i \neq j$.*

Recall that a map $f{:}(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is called (i,j)-pre-open if for each τ_i-open set in X, $f(U)$ is (i,j)-pre-open, $i,\ j = 1,2,\ i \neq j$.

Remark 24 *Every (i,j)-γ-open map is (i,j)-pre-open but the converse is not true in general as shown in the following example.*

Example 25 *Let $X = \{a,b,c\}$, $\tau_1 = \{\emptyset, \{b,c\}, X\}$, $Y = \{a,b,c\}$, $\sigma_1 = \{\emptyset, \{a\}, X\}$ and $\sigma_2 = \{\emptyset, \{a,b\}, X\}$. Define a map $f{:}X \to Y$ as follows $f(a) = b,\ f(b) = a,\ f(c) = c$. Then f is $(1,2)$-pre-open but not $(1,2)$-γ-open since $f(\{b,c\}) = \{a,c\}$ which is $(1,2)$-pre-open but not $(1,2)$-γ-open.*

Recall that a space X is said to be pairwise Hausdorff[12] if for $x,\ y \in X,\ x \neq y$, there exist open sets $U,\ V,\ U \in \tau_1,\ V \in \tau_2$ such that $x \in U$ and $y \in V$.

Theorem 26 *Let $f{:}(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ be a bijective (i,j)-γ-open map.If X be pairwise Hausdorff, then Y is (i,j)-γ-T_2.*

Proof. Let y_1 and y_2 be two distinct points in Y. Since f is bijective there exist x_1 and x_2 in X such that $f(x_1) \neq f(x_2)$. The space X is pairwise Hausdorff and so there exist disjoint sets $U,\ V$, $U \in \tau_1$ and $V \in \tau_2$ such that $x_1 \in U$and $x_2 \in V$. Then $f(x_1) \in f(U)$ and $f(x_2) \in f(V)$, $f(U)$ and $f(V)$ are (i,j)-γ-open sets and $f(U) \cap f(V) = \emptyset$. Thus Y is (i,j)-γ-T_2. ■

Definition 27 *A map $f{:}(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is called (i,j)-γ-irresolute if the inverse image of every (i,j)-γ-open set in Y is (i,j)-γ-open in X, $i,\ j = 1,2,\ i \neq j$.*

Theorem 28 *If $f{:}(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is an (i,j)-γ-irresolute bijective mapping and if Y is a (i,j)-γ-T_2 space then, X is (i,j)-γ-T_2.*

Proof. Let $x_1,\ x_2$ be two distinct points in X. Then there exist $y_1,\ y_2$ in Y such that $f(x_1) = y_1$ and $f(x_2) = y_2$ and $y_1 \neq y_2$. Since Y is (i,j)-γ-T_2, there exist disjoint (i,j)-γ-open sets $U,\ V$ such that $y_1 \in U$ and $y_2 \in V$. As f is (i,j)-γ-irresolute, $f^{-1}(U)$ and $f^{-1}(V)$ are (i,j)-γ-open sets in X containing x_1 and x_2 respectively. Hence X is (i,j)-γ-T_2. ■

Theorem 29 *Let* $f:(X,\tau_1,\tau_2) \rightarrow (Y,\sigma_1,\sigma_2)$ *and* $g:(Y,\sigma_1,\sigma_2) \rightarrow (Z,\varrho_1,\varrho_2)$ *be two maps. Then*

(i) If f *is* (i,j)-γ*-irresolute and* g *is* $p.\gamma$*-continuous then,* $g \circ f$ *is* $p.\gamma$*-continuous*

(ii) If both f *and* g *are* (i,j)-γ*-irresolute then,* $g \circ f$ *is* (i,j)-γ*-irresolute.*

Proof. Obvious. ■

ACKNOWLEDGEMENT

The second and third authors wish to acknowledge the support of University Grants Commission, New Delhi, India, FIP-X Plan.

Received: June 2008. Revised: August 2008.

References

[1] M.E. Abd El-Monsef, B.M. Taher and A.S. Salama, *Topological applications of rough set measures in information systems*, Preprint.

[2] D. Andrijevic, *On the Topology Generated by Pre-open sets*, Mate. Bech., **39** (1987), 367–376.

[3] A.V. Arhangel'Skii and P.J. Collins, *On submaximal spaces*, Topology Appl., **64** (1995), 219–241.

[4] H. Corson and E. Michael, *Metrizability of certain countable unions*, Illinois J. Math., **8** (1964), 351–360.

[5] S.N. El-Deeb, A. Hasanein, A.S. Mashhour and T. NOIRI, *On p-regular spaces*, Bull. Math. Soc. Sci. Math., R.S. Roumanie(N.S), **27** (75)(1983), 65-73.

[6] R. Devi, K. Bhuvaneswari and H. Maki, *Weak forms of* $g\varrho$*-closed sets, where* $\varrho \in \{\alpha, \alpha^\star, \alpha^{\star\star}\}$*, and digital plane*, Mem. Fac. Sci. Kochi Univ. Ser. A. Math., **25** (2004), 37–54.

[7] M.S. EL-NASCHIE, *On the certification of heterotic strings, M theory and* ϵ^∞ *theory*, Chaos, Solitons and Fractals, (2000), 2397–2408.

[8] J. Foran and P. Liebnitz, *A characterization of almost resolvable spaces*, Rend. Circ. Mat. Palermo, Serie II, Tomo XL, (1991), 136–141.

[9] A. Kar and P. Bhattacharyya, *Bitopological preopen sets, precontinuity and preopen mappings*, Indian J. Math.**34** (1992), 295–309.

[10] J.C. KELLY, *Bitopological Spaces*, Proc. Londo Math. Soc., **13** (1963), 71–89.

[11] A.S. Mashhour, I.A. Hasanein and S.N. El-Deeb, *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt, **53** (1982), 47–53.

[12] A.S. Mashhour, M.E. Abd El-Monsef, I.A. Hasanein and T. Noiri, *Strongly compact spaces*, Delta J. Sci., **8** (1984), 30–46.

[13] M.G. Murudeshwar and S.A. Naimpally, *Semi-Hausdorff spaces*, Canad. Math. Bull., **9** (1996), 353–356.

[14] Saeid Jafari, M. Lellis Thivagar, S. Athisaya Ponmani and R. Raja Rajeswari, *A Bitopological View of γ-open sets*, Communicated.