CUBO A Mathematical Journal
Vol.10, N$^{\underline{o}}$03, (145–159). October 2008

Asymptotic Constancy and Stability in Nonautonomous Stochastic Differential Equations

John A.D. Appleby*
School of Mathematical Sciences, Dublin City University,
Dublin 9, Ireland
email: `john.appleby@dcu.ie`
URL: `http://webpages.dcu.ie/~applebyj`

James P. Gleeson
Department of Applied Mathematics, University College Cork,
Cork, Ireland
email: `j.gleeson@ucc.ie`

and

Alexandra Rodkina
Department of Mathematics and Computer Science,
The University of the West Indies, Mona, Kingston 7, Jamaica
email: `alexandra.rodkina@uwimona.edu.jm`

ABSTRACT

This paper considers the asymptotic behaviour of a scalar non-autonomous stochastic differential equation which has zero drift, and whose diffusion term is a product of a function of time and space dependent function, and which has zero as a unique

*The first author is partially supported by an Albert College Fellowship awarded by Dublin City University's Research Advisory Panel. The third author is supported by the Mona Research Fellowship Programm awarded by the University of the West Indies, Jamaica and by London Mathematical Society

equilibrium solution. We classify the pathwise limiting behaviour of solutions; solution either tends to a non-trivial, non-equilibrium and random limit, or the solution hits zero in finite time. In the first case, the exact rate of decay can always be computed. These results can be inferred from the square integrability of the time dependent factor, and the asymptotic behaviour of the corresponding autonomous stochastic equation, where the time dependent multiplier is unity.

RESUMEN

Este artículo considera el comportamiento asintótico de una ecuación diferencial estocastica escalar no-autónoma la cual tiene cero desviación y cujo término de difusión es un producto de una función de equilibrio. Nosotros clasificamos el comportamiento limite por caminos de las soluciones; la solución atiende a un no equilibrio y límite randon no trivial, o la solución encuentra cero en tiempo finito. En el primer caso, las tasas de decaimiento siempre pueden ser calculadas. Estos resultados pueden ser inferidos de la integrabilidad al cuadrado del factor dependiente del tiempo, y el comportamiento asintótico de la correspondiente ecuación estocatica autónoma, donde el multiplicador dependiente del tiempo es la unidad.

Key words and phrases: *Brownian motion, almost sure asymptotic stability, asymptotic constancy, stochastic differential equation, nonautonomous, Feller's test, explosions.*

Math. Subj. Class.: *60H10, 93E15.*

1 Introduction

This note considers the asymptotic behaviour of solutions of the "separable" stochastic differential equation

$$dX(t) = \sigma(t)g(X(t))\,dB(t). \tag{1.1}$$

A solution of this equation with initial condition ξ is denoted by $X(\cdot,\xi)$. It is presumed that zero is a point equilibrium, so $X(t,0) = 0$ is a solution of (1.1). A standard deterministic change of time scale reduces this equation to an autonomous equation

$$d\tilde{X}(t) = g(\tilde{X}(t))\,d\tilde{B}(t), \tag{1.2}$$

from which it can be shown that the condition that $\sigma \in L^2([0,\infty);\mathbb{R})$ largely determines whether the solution tends to the equilibrium or to a non–trivial and non–equilibrium limit. Another feature which is examined is the relationship between the process $\tilde{X}$ hitting zero in a finite amount of time, or tending to zero as $t \to \infty$ (in the case when $\tilde{X}$ remains strictly positive) and the corresponding properties of X. As will be seen, a complete picture of the dynamics of (1.1) can be deduced in terms of conditions on g and σ.

Even though the time–change technique employed is well–known, some novel features appear in the analysis. First, we are unaware of an extensive literature concerning the pathwise convergence of solutions of stochastic differential equations to non–equilibrium limits. Second, we determine here sharp upper and lower estimates in terms of the rate at which the noise intensity fades on the almost sure rate of convergence of the solution to this non–equilibrium limit, in the case when $\sigma \in L^2([0,\infty);\mathbb{R})$. This requires a delicate use of the law of the iterated logarithm, partly correcting an error on the asymptotic behaviour of a tail martingale established in [1] and used in [2]. Finally, in the case where $\sigma \notin L^2([0,\infty);\mathbb{R})$, the results here, taken in conjunction with work in [3, 4] would enable exact almost sure rates of convergence to zero of solutions of (1.1) to be established.

2 Existence of solutions

In this paper we deal with highly nonlinear stochastic differential equations, SDE, whose solutions can hit zero at finite time, due to the non-Lipshitz behavior of the diffusion coefficients. Moreover, for non-autonomous equations, it is convenient for the completeness of our exposition, to state carefully and to prove an existence result. This result is a corollary of well-known existence result and martingale time changing theorem.

Let $(\Omega, \mathcal{F}, (\mathcal{F}(t))_{t\geq 0}, \mathbb{P})$ be a complete probability space with filtration $(\mathcal{F}(t))_{t\geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}(0)$ contains all $\mathbb{P}$-null sets). Let $(B(t))_{t\geq 0}$ be a scalar standard Brownian motion, defined on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}(t))_{t\geq 0}, \mathbb{P})$. Since we will consider equations with deterministic initial conditions, it is enough to work with the natural filtration of B: that is $\mathcal{F}(t) \equiv \mathcal{F}^B(t)$ where $\mathcal{F}^B(t) = \sigma(\tilde{B}(s) : 0 \leq s \leq t)$.

Suppose that the function σ obeys

$$\sigma \in C([0,\infty);\mathbb{R}). \tag{2.1}$$

We define the local martingale $M = \{M(t), 0 \leq t < \infty, \mathcal{F}^B(t)\}$ by

$$M(t) = \int_0^t \sigma(s)\, dB(s), \quad t \geq 0, \tag{2.2}$$

with square variation $\langle M \rangle$ given by

$$\langle M \rangle(t) = \int_0^t \sigma^2(s)\, ds.$$

Let T^* be given by

$$T^* = \int_0^\infty \sigma^2(s)\, ds, \tag{2.3}$$

where we define $T^* = \infty$ if $\sigma \notin L^2([0,\infty);\mathbb{R})$.

Define also, for each $0 \le s \le T^*$, the $\mathcal{F}^B$—stopping time

$$T(s) = \inf\{t \ge 0 : \langle M\rangle(t) \ge s\}. \tag{2.4}$$

By the martingale time change theorem, there exists a standard Brownian motion $\{\tilde{B}(t); 0 \le t \le T^*; \mathcal{G}(t)\}$, such that

$$\tilde{B}(s) = M(T(s)), \quad \mathcal{G}(s) = \mathcal{F}^B(T(s)), \quad 0 \le s \le T^*,$$

and moreover

$$M(t) = \tilde{B}(\langle M\rangle(t)), \quad \mathcal{F}^B(t) = \mathcal{G}(\langle M\rangle(t)), \quad t \ge 0.$$

In what follows we presume that $g : \mathbb{R} \to \mathbb{R}$ obeys

$$g(0) = 0. \tag{2.5}$$

Since we do not want the equation to have any other equilibria in $(0, \infty)$ we ask that the non–degeneracy condition

$$g(x) > 0, \quad \text{for all } x \ne 0, \tag{2.6}$$

also be satisfied.

We are now in a position to state an existence result for the solution of the autonomous stochastic differential equation.

Proposition 2.1. *Suppose that g obeys (2.5) and (2.6) and that*

$$\begin{gathered}\textit{there exists a strictly increasing function}\\ q : [0,\infty) \to [0,\infty) \textit{ with } q(0) = 0 \textit{ such that}\\ \int_0^\epsilon \frac{1}{q^2(u)}\,du = \infty, \quad \textit{for all } \epsilon > 0,\end{gathered} \tag{2.7a}$$

and that g and q both obey

$$|g(x) - g(y)| \le q(|x - y|), \quad x, y \in \mathbb{R}. \tag{2.7b}$$

Then there exists a unique strong non-exploding solution $\tilde{X}$ of

$$d\tilde{X}(t) = g(\tilde{X}(t))\,d\tilde{B}(t), \quad t \ge 0, \quad X(0) = \xi > 0, \tag{2.8}$$

on the complete probability space $(\Omega, \mathcal{F}, (\mathcal{G}(t))_{t\ge0}, \mathbb{P})$.

This result is a corollary of the result of Yamada and Watanabe (see e.g. [7], Proposition 2.13, page 291, and [7], Theorem 5.4., page 332).

The main concern of this paper is the asymptotic behaviour of solutions of the nonautonomous equations

$$dX(t) = \sigma(t)g(X(t))\,dB(t), \quad t \ge 0, \quad X(0) = \xi > 0. \tag{2.9}$$

Before conducting this asymptotic analysis however, we must verify that this equation has a well-defined solution. This is accomplished by the following result.

Proposition 2.2. *Suppose that g and q obey (2.5), (2.6), (2.7a) and (2.7b) and σ obeys (2.1).*

Then there exists a unique strong non-exploding solution X of (2.9) on the complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}^B(t))_{t\geq 0}, \mathbb{P})$.

Proof. By Proposition 2.1, $\tilde{X}$ is the unique strong solution of (2.8). Consider $\tilde{X}(t)$ for $t \in [0, T^*)$, where T^* is defined as in (2.3), and define

$$X(t) = \tilde{X}\left(\int_0^t \sigma^2(s)ds\right), \quad t \geq 0. \tag{2.10}$$

Then, as $\tilde{X}(s)$ is $\mathcal{G}(s)$-measurable, $\tilde{X}(t)$ is $\mathcal{G}\left(\int_0^t \sigma^2(s)ds\right)$-measurable. But $\mathcal{G}(\langle M\rangle(t)) = \mathcal{F}^B(t)$, so $X(t)$ is $\mathcal{F}^B(t)$-measurable. Now, by [7], Proposition 3.4.8, we get for $t \geq 0$

$$\int_0^{\langle M\rangle(t)} g(\tilde{X}(u))d\tilde{B}(u) = \int_0^t g(X(s))dM(s) = \int_0^t \sigma(s)g(X(s))dB(s),$$

and, therefore, as $\langle M\rangle(t) = \int_0^t \sigma^2(s)ds$, we get

$$\begin{aligned}
X(t) &= \tilde{X}\left(\int_0^t \sigma^2(s)ds\right) = \tilde{X}\left(\langle M\rangle(t)\right) \\
&= \tilde{X}(0) + \int_0^{\langle M\rangle(t)} g(\tilde{X}(u))d\tilde{B}(u) = \xi + \int_0^t \sigma(s)g(X(s))dB(s).
\end{aligned}$$

Therefore X defined by (2.10) is a strong solution of (2.9) on $(\Omega, \mathcal{F}, (\mathcal{F}^B(t))_{t\geq 0}, \mathbb{P})$.

To show uniqueness, suppose that there is another solution of (2.9), Y. Then $\tilde{Y}$ defined by $\tilde{Y}(s) = Y(T(s))$ obeys (2.8). Hence, as (2.8) has a unique solution, we have $\tilde{Y} = \tilde{X}$, and therefore it follows that $Y = X$.

This completes the proof. □

In this paper, we choose to write explicitly the dependence of solutions on their initial conditions, which are always assumed to be deterministic. Thus, the value at time $t \geq 0$ of the process Y with initial condition $Y(0) = \xi$ is denoted by $Y(t, \xi)$.

3 Main result

In this section, we state and discuss the main results of the paper concerning the asymptotic behaviour of non–autonomous equation. At the end of the section we present an example of a non–autonomous linear equation which can be analysed without the use of the theorems established here, but whose behaviour illustrates the results proven.

As seen in the proof of Proposition 2.2 the non–autonomous equation (2.9) is equivalent to (2.8), under a deterministic time change. However the subject of Proposition 2.2 is the existence for non–autonomous equation. The following Proposition by contrust, focusses on the relation between the solutions of the two equations.

Proposition 3.1. *Let $\xi > 0$ be deterministic. Suppose that g obeys (2.5), (2.6). Let σ obey (2.1). Suppose that $X(\cdot,\xi)$ is the unique strong solution of (2.9) with $X(0,\xi) = \xi$. Let T^* be given by* (2.3)*, and T be defined by* (2.4)*.*

(i) If $\sigma \in L^2([0,\infty);\mathbb{R})$, then there exists a standard Brownian motion $\tilde{B} = \{\tilde{B}(t); 0 \le t \le T^*; \mathcal{G}(t)\}$ where $\mathcal{G}(t) = \mathcal{F}^B(T(t))$ and $\tilde{B}(t) = B(T(t))$ such that the process $\tilde{X} = \{\tilde{X}(t); 0 \le t \le T^*; \mathcal{G}(t)\}$ defined by $\tilde{X}(t) = X(T(t))$ obeys (2.8)*.*

(ii) If $\sigma \notin L^2([0,\infty);\mathbb{R})$, then there exists a standard Brownian motion $\tilde{B} = \{\tilde{B}(t); 0 \le t < \infty; \mathcal{G}(t)\}$ where $\mathcal{G}(t) = \mathcal{F}^B(T(t))$ and $\tilde{B}(t) = B(T(t))$ such that the process $\tilde{X} = \{\tilde{X}(t); 0 \le t < \infty; \mathcal{G}(t)\}$ defined by $\tilde{X}(t) = X(T(t))$ obeys (2.8)*.*

Before stating the first result on asymptotic behaviour we present some notation and an important auxiliary result.

Suppose that $\tilde{X}(\cdot,\xi)$ is the solution of (2.8), where $\xi > 0$. Define

$$\tilde{S}_0(\xi) = \inf\{t \ge 0 : \tilde{X}(t,\xi) = 0\}. \tag{3.1}$$

Let us suppose that for $\delta > 0$ we may define the function $v : (0,\infty) \to (0,\infty)$ by

$$v(x) = 2\int_x^\delta \int_y^\delta \frac{dz}{g^2(z)}\,dy, \quad x > 0. \tag{3.2}$$

The following result is due to Feller (see e.g. [7], Theorem 5.5.29, page 348).

Proposition 3.2. *Let $\xi > 0$ be deterministic, and $\tilde{X}(\cdot,\xi)$ be a strong solution of (2.8). If $\tilde{S}_0(\xi)$ is as defined in (3.1), then*

$$\lim_{t\to\tilde{S}_0(\xi)} \tilde{X}(t,\xi) = 0, \qquad \sup_{0\le t<\tilde{S}_0(\xi)} \tilde{X}(t,\xi) < \infty, \quad a.s.$$

Let v be defined by (3.2). Then

(i) $\lim_{x\to 0^+} v(x) < \infty$ implies $\tilde{S}_0(\xi) < \infty$, a.s.;

(ii) $\lim_{x\to 0^+} v(x) = \infty$ implies $\tilde{S}_0(\xi) = \infty$, a.s.

We define also

$$S_0(\xi) = \inf\{t \ge 0 : X(t,\xi) = 0\}. \tag{3.3}$$

We can determine whether $S_0(\xi)$ is finite or infinite with the help of Proposition 2.2.

We may now state the first main result on the asymptotic behaviour in this paper. It is a direct consequence of Proposition 3.1 and Proposition 3.2.

Theorem 3.3. *Let $\xi > 0$ be deterministic. Suppose that g obeys (2.5), (2.6). Let σ obey (2.1). Suppose that $X(\cdot,\xi)$ is the unique strong solution of (2.9) with $X(0,\xi) = \xi$.*

Let v be defined as in (3.2), and suppose that

$$\lim_{x\to 0^+} v(x) = \infty.$$

Then we have the following case distinction:

(a) If $\sigma \in L^2((0,\infty);\mathbb{R})$, then there exists an almost surely positive and $\mathcal{F}^B(\infty)$–measurable random variable $L = L(\xi,\omega)$ such that

$$\lim_{t\to\infty} X(t,\xi) = L(\xi) > 0, \quad a.s. \tag{3.4}$$

(b) If $\sigma \notin L^2((0,\infty);\mathbb{R})$, then

$$\lim_{t\to\infty} X(t,\xi) = 0, \quad a.s.$$

and $S_0(\xi)$ defined by (3.3) obeys $S_0(\xi) = \infty$, a.s.

The proof of this result and proofs of subsequent results in the this section, are postponed to the final section of the paper.

When $\sigma \in L^2([0,\infty);\mathbb{R})$ the rate at which convergence to $L(\xi)$ occurs can be determined exactly.

Theorem 3.4. *Let $\xi > 0$ be deterministic. Suppose that g obeys (2.5), (2.6), and let $g \in C^1((0,\infty);(0,\infty))$. Let σ obey (2.1) and $\sigma \in L^2([0,\infty);\mathbb{R})$. Suppose that $X(\cdot,\xi)$ is the unique strong solution of (2.9) with $X(0,\xi) = \xi$.*

Let v be defined as in (3.2), and suppose that

$$\lim_{x\to 0^+} v(x) = \infty.$$

Let $L(\xi)$ be the almost surely positive and $\mathcal{F}^B(\infty)$–measurable random variable defined by (3.4). *Then:*

(i) If σ obeys

$$\int_t^\infty \sigma^2(s)\,ds > 0, \quad \text{for all } t \geq 0, \tag{3.5}$$

then

$$\limsup_{t\to\infty} \frac{X(t,\xi) - L(\xi)}{\sqrt{2\int_t^\infty \sigma^2(s)\,ds \log\log\left(\int_t^\infty \sigma^2(s)\,ds\right)^{-1}}} = g(L(\xi)), \quad a.s., \tag{3.6a}$$

$$\liminf_{t\to\infty} \frac{X(t,\xi) - L(\xi)}{\sqrt{2\int_t^\infty \sigma^2(s)\,ds \log\log\left(\int_t^\infty \sigma^2(s)\,ds\right)^{-1}}} = -g(L(\xi)), \quad a.s. \tag{3.6b}$$

(ii) If σ does not obey (3.5) *i.e., if there exists $\tau \geq 0$ such that $\int_t^\infty \sigma^2(s)\,ds = 0$ for all $t \geq \tau$, then*

$$X(t,\xi) = X(\tau,\xi) = L(\xi), \quad \textit{for all } t \geq \tau, \textit{ a.s.}$$

Of course, in case (b) in Theorem 3.3, the rate of convergence cannot be so easily computed. However pathwise rates of decay to zero for nonlinear autonomous stochastic differential equations have been found in [3, 4], and could readily be applied here.

Finally, the distribution of the random limit L is in principle well–understood, by using the forward Kolmogorov equation for the process $\tilde{X}$.

Theorem 3.5. *Let $\xi > 0$ be deterministic. Suppose that g obeys (2.5), (2.6). Let σ obey (2.1) and $\sigma \in L^2([0,\infty);\mathbb{R})$. Let T^* be given by* (2.3)*. Suppose that $X(\cdot,\xi)$ is the unique strong solution of (2.9) with $X(0,\xi) = \xi$.*

Let v be defined as in (3.2), and suppose that

$$\lim_{x\to 0^+} v(x) = \infty,$$

and let $L(\xi)$ be the almost surely positive and $\mathcal{F}^B(\infty)$–measurable random variable defined by (3.4)*. Then*

$$\mathbb{P}[L(\xi) \leq x] = \int_0^x \Gamma(T^*;y)\,dy, \quad x \geq 0,$$

where

$$\frac{\partial \Gamma}{\partial t}(t;y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(g^2(y)\Gamma(t;y)\right), \quad (t,y) \in [0,T^*] \times (0,\infty),$$

and $\Gamma(0;y) = \delta_\xi(y)$, $y \in \mathbb{R}$, where δ_ξ is the δ-function.

The result holds because $L(\xi) = \lim_{t\to T^*+} \tilde{X}(t,\xi) = \tilde{X}(T^*,\xi)$. Moreover, as $\tilde{X}$ is a diffusion process with known infinitesimal generator and deterministic initial condition ξ, we can deduce its distribution function from the forward Kolmogorov equation, and therefore, the distribution of $L(\xi)$ is also known.

It remains merely to classify the behaviour in the case when $\lim_{x\to 0} v(x) < \infty$.

Theorem 3.6. *Let $\xi > 0$ be deterministic. Suppose that g obeys (2.5), (2.6). Let σ obey (2.1). Suppose that $X(\cdot,\xi)$ is the unique strong solution of (2.9) with $X(0,\xi) = \xi$.*

Let v be defined as in (3.2), and suppose that

$$\lim_{x\to 0^+} v(x) < \infty.$$

Then we have the following case distinction:

(a) If $\sigma \in L^2((0,\infty);\mathbb{R})$, then there exists an almost surely positive and $\mathcal{F}^B(\infty)$–measurable random variable $L = L(\xi,\omega)$ such that

$$\lim_{t\to\infty} X(t,\xi) = L(\xi) > 0, \quad \textit{a.s. on } \{\tilde{S}_0(\xi) \geq T^*\},$$

and

$$\lim_{t\to S_0(\xi)^-} X(t,\xi) = 0, \quad a.s.\ on\ \{\tilde{S}_0(\xi) < T^*\},$$

where $\tilde{S}_0(\xi)$ *defined by* (3.1) *and* $S_0(\xi)$ *is defined by* (3.3).

(b) If $\sigma \notin L^2((0,\infty);\mathbb{R})$*, then*

$$\lim_{t\to S_0(\xi)^-} X(t,\xi) = 0, \quad a.s.,$$

where $S_0(\xi)$ *defined by (3.3) obeys* $S_0(\xi) < +\infty$*, a.s.*

In case (a) the rate of convergence to the non–trivial random limit is the same as given in Theorem 3.4, but only a.s. on the event $\{\tilde{S}_0(\xi) > T^*\}$.

The probability of the event $\{\tilde{S}_0(\xi) < T^*\}$ can be computed for the process $\tilde{X}$ obeying (2.8), by considering the limit

$$\mathbb{P}[\tilde{S}_0(\xi) < T^*] = \lim_{a\to 0^+} \mathbb{P}[\tilde{S}_a(\xi) < T^*]$$

where for $\xi > a > 0$ we define $\tilde{S}_a(\xi) = \inf\{t \geq 0 : \tilde{X}(t,\xi) = a\}$. It is possible to compute the moment generating function of $\tilde{S}_a(\xi)$, $\lambda \mapsto \mathbb{E}[e^{-\lambda \tilde{S}_a(\xi)}]$ for $\lambda \geq 0$, by solving an appropriate Sturm–Liouville problem, from which the probability $\mathbb{P}[\tilde{S}_a(\xi) < T^*]$ can in principle be determined by inverse transform methods. The interested reader can refer to [6, Chapter 4.11] for further details on computation of the moment generating function.

3.1 An example

A simple example of a process which can be analysed completely *without* appealing to these results (but which is consistent with them) is the unique strong solution of

$$X(t) = \xi + \int_0^t \sigma(s)X(s)\,dB(s), \quad t \geq 0,$$

where $\xi > 0$ and $\sigma \in C([0,\infty);\mathbb{R})$. This equation has explicit solution

$$X(t,\xi) = \xi \exp\left(\int_0^t \sigma(s)\,dB(s) - \frac{1}{2}\int_0^t \sigma^2(s)\,ds\right), \quad t \geq 0.$$

Here we identify $g(x) = x$, $x \geq 0$, and have $\lim_{x\to 0^+} v(x) = \infty$. Hence Theorems 3.3, 3.4 and 3.5 can be applied to this stochastic differential equation.

In the case when $\sigma \in L^2([0,\infty);\mathbb{R})$, the martingale convergence theorem (cf., e.g., [8, Proposition IV.1.26]) ensures that $\lim_{t\to\infty} \int_0^t \sigma(s)\,dB(s)$ exists and is almost surely finite. Therefore, there is an almost surely positive and almost surely finite $\mathcal{F}^B(\infty)$—measurable random variable $L(\xi)$ given by

$$L(\xi) = \xi \exp\left(\int_0^\infty \sigma(s)\,dB(s) - \frac{1}{2}\int_0^\infty \sigma^2(s)\,ds\right)$$

such that

$$\lim_{t\to\infty} X(t,\xi) = L(\xi) > 0, \quad \text{a.s.}$$

This chimes with part (a) of Theorem 3.3.

In the case when $\sigma \notin L^2([0,\infty);\mathbb{R})$, we have that

$$\lim_{t\to\infty} \int_0^t \sigma^2(s)\,ds = +\infty, \quad \lim_{t\to\infty} \frac{\int_0^t \sigma(s)\,dB(s)}{\int_0^t \sigma^2(s)\,ds} = 0, \quad \text{a.s.}$$

The latter fact resulting from the Strong Law of Large Numbers for martingales (cf., e.g., [8, Exercise V.I.16]). Therefore

$$\lim_{t\to\infty} \frac{1}{\int_0^t \sigma^2(s)\,ds} \log X(t) = -\frac{1}{2}, \quad \text{a.s.}$$

Hence $\lim_{t\to\infty} X(t,\xi) = 0$, a.s., which agrees with part (b) of Theorem 3.3.

Moreover, in the case when $\sigma \in L^2([0,\infty);\mathbb{R})$, $L(\xi)$ is lognormally distributed; this is obvious by observation of the formula for $L(\xi)$, but can also be confirmed by solving the partial differential equation for the transition density Γ in Theorem 3.5.

In the case when $\sigma \in L^2([0,\infty);\mathbb{R})$, the rate of convergence in Theorem 3.4 can be obtained, if it is shown that

$$\limsup_{t\to\infty} \frac{\int_t^\infty \sigma(s)\,dB(s)}{\sqrt{2\int_t^\infty \sigma^2(s)\,ds \log\log\left(\int_t^\infty \sigma^2(s)\,ds\right)^{-1}}} = 1, \quad \text{a.s.,} \tag{3.7a}$$

$$\liminf_{t\to\infty} \frac{\int_t^\infty \sigma(s)\,dB(s)}{\sqrt{2\int_t^\infty \sigma^2(s)\,ds \log\log\left(\int_t^\infty \sigma^2(s)\,ds\right)^{-1}}} = -1, \quad \text{a.s.} \tag{3.7b}$$

This can be established as follows: define $T^* = \int_0^\infty \sigma^2(s)\,ds$, and let M be the local martingale defined in (2.2), and with square variation $\langle M\rangle$. Then, by the martingale time change theorem there exists a Brownian motion $\tilde{B}$ such that $M(t) = \tilde{B}(\langle M\rangle(t))$, $0 \le t < \infty$. Thus

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{\int_t^\infty \sigma(s)\,dB(s)}{\sqrt{2\int_t^\infty \sigma^2(s)\,ds \log\log\left(\int_t^\infty \sigma^2(s)\,ds\right)^{-1}}}\\
&= \limsup_{t\to\infty} \frac{M(\infty) - M(t)}{\sqrt{2(\langle M\rangle(\infty) - \langle M\rangle(t)) \log\log\left(\langle M\rangle(\infty) - \langle M\rangle(t)\right)^{-1}}}\\
&= \limsup_{t\to\infty} \frac{\tilde{B}(\langle M\rangle(\infty)) - \tilde{B}(\langle M\rangle(t))}{\sqrt{2(\langle M\rangle(\infty) - \langle M\rangle(t)) \log\log\left(\langle M\rangle(\infty) - \langle M\rangle(t)\right)^{-1}}}\\
&= \limsup_{s\to T^{*-}} \frac{\tilde{B}(T^*) - \tilde{B}(s)}{\sqrt{2(T^*-s)\log\log(T^*-s)^{-1}}}\\
&= \limsup_{u\to 0^+} \frac{\tilde{B}(T^*) - \tilde{B}(T^*-u)}{\sqrt{2u\log\log(u)^{-1}}} = 1, \quad \text{a.s.,}
\end{aligned}$$

since $\bar{B}$ defined by $\bar{B}(t) = \tilde{B}(T^*) - \tilde{B}(T^* - t)$, $0 \le t \le T^*$, is also a standard Brownian motion, and therefore subject to the Law of the Iterated Logarithm. (3.7) was stated in [2], but a proof was not supplied there. A variant of this result is proven in [1] but this proof contains an error as it is *incorrectly* stated there that

$$\int_t^\infty X(s)\,dB(s) = \int_0^{1/t} \frac{1}{s} X(1/s)\,dW(s)$$

for a process $X \in L^2([0,\infty);\mathbb{R})$ a.s., where W is the standard Brownian motion given by $W(t) = tB(1/t)$ for $t > 0$ and $W(0) = 0$.

4 Proofs

4.1 Proof of Theorem 3.3

By Proposition 3.2, it follows that $\lim_{x\to 0^+} v(x) = \infty$ implies $\tilde{S}_0(\xi) = +\infty$, a.s.

In case (a), when $\sigma \in L^2([0,\infty);\mathbb{R})$, $T(s) \to \infty$ as $s \to T^{*-}$. Hence, as $\tilde{X}(s) = X(T(s))$ for $s \in [0, T^*]$,

$$\lim_{t\to\infty} X(t) = \lim_{s\to T^{*-}} X(T(s)) = \lim_{s\to T^{*-}} \tilde{X}(s) = \tilde{X}(T^*) > 0, \quad \text{a.s.},$$

because $T^* < +\infty = \tilde{S}_0(\xi)$, a.s.

In case (b), when $\sigma \notin L^2([0,\infty);\mathbb{R})$, $T(s) \to \infty$ as $s \to \infty$. Hence, as $\tilde{X}(s) = X(T(s))$ for $s \in [0,\infty)$, we have $S_0(\xi) = T(\tilde{S}_0(\xi)) = +\infty$, a.s., and

$$\lim_{t\to\infty} X(t) = \lim_{s\to\infty} X(T(s)) = \lim_{s\to\infty} \tilde{X}(s) = \lim_{s\to\tilde{S}_0(\xi)} \tilde{X}(s) = 0, \quad \text{a.s.},$$

because $\tilde{S}_0(\xi) = \infty$, a.s.

4.2 Proof of Theorem 3.4

The proof of part (ii) is straightforward, because for $t \ge \tau$ we have

$$X(t) = X(\tau) + \int_\tau^t \sigma(s) g(X(s))\,dB(s) = X(\tau),$$

as $\int_\tau^\infty \sigma^2(s)\,ds = 0$ and the continuity of σ imply that $\sigma(t) = 0$ for all $t \in [\tau,\infty)$.

To prove part (i) we proceed as follows. Because $\sigma \in L^2([0,\infty);\mathbb{R})$, and $\lim_{x\to 0^+} v(x) = \infty$, by Proposition 3.1 and Proposition 3.2, the process $\tilde{X} = \{X(t); 0 \le t \le T^*; \mathcal{G}(t)\}$ defined by $\tilde{X}(t) = X(T(t))$ is strictly positive a.s. and obeys

$$\tilde{X}(t) = \xi + \int_0^t g(\tilde{X}(s))\,d\tilde{B}(s), \quad 0 \le t \le T^*,$$

where T^* and T are defined by (2.3) and (2.4).

Since g in $C^1((0,\infty);(0,\infty))$ we may define the function $h \in C^2((0,\infty);\mathbb{R})$ by

$$h(x) = \int_1^x \frac{1}{g(u)}\,du, \quad x \in \mathbb{R}.$$

Then the process $\tilde{Y} = \{\tilde{Y}(t); 0 \le t \le T^*; \mathcal{G}(t)\}$ defined by $\tilde{Y}(t) = h(\tilde{X}(t))$ is well–defined. Since $h \in C^2((0,\infty);\mathbb{R})$ and $\tilde{X}(t) > 0$ for all $t \in [0,T^*]$ a.s. $\tilde{Y}$ is an Itô–process, which by Itô's rule, has decomposition for $0 \le t \le T^*$ given by

$$\tilde{Y}(t) = h(\tilde{X}(t)) = h(\xi) + \tilde{B}(t) - \frac{1}{2}\int_0^t g'(\tilde{X}(s))\,ds.$$

Since $\tilde{X}$ is almost surely positive and continuous, $g \in C^1((0,\infty);(0,\infty))$, it follows that

$$\lim_{t\to T^*-} \frac{1}{T^*-t}\int_t^{T^*} g'(\tilde{X}(s))\,ds = g'(\tilde{X}(T^*)), \quad \text{a.s.}$$

Therefore

$$\lim_{t\to T^*-} \frac{1}{\sqrt{2(T^*-t)\log\log(1/(T^*-t))}}\int_t^{T^*} g'(\tilde{X}(s))\,ds = 0, \quad \text{a.s.}$$

Then for $t \in [0,T^*]$

$$h(\tilde{X}(T^*)) - h(\tilde{X}(t)) = \tilde{B}(T^*) - \tilde{B}(t) - \frac{1}{2}\int_t^{T^*} g'(\tilde{X}(s))\,ds,$$

and

$$\begin{aligned}
&\limsup_{t\to T^*-} \frac{h(\tilde{X}(T^*)) - h(\tilde{X}(t))}{\sqrt{2(T^*-t)\log\log(1/(T^*-t))}} \\
&= \limsup_{t\to T^*-} \frac{\tilde{B}(T^*) - \tilde{B}(t)}{\sqrt{2(T^*-t)\log\log(1/(T^*-t))}} \\
&= \limsup_{s\to 0+} \frac{\tilde{B}(T^*) - \tilde{B}(T^*-s)}{\sqrt{2s\log\log(1/s)}}.
\end{aligned}$$

Now, because $\bar{B} = \{\bar{B}(t) : 0 \le t \le T^*; \mathcal{F}^{\bar{B}}(t)\}$ defined by $\bar{B}(t) = \tilde{B}(T^*) - \tilde{B}(T^*-t)$ is a standard Brownian motion, by the Law of the Iterated Logarithm for Brownian motion we have

$$\limsup_{s\to 0^*+} \frac{\tilde{B}(T^*) - \tilde{B}(T^*-s)}{\sqrt{2s\log\log(1/s)}} = \limsup_{s\to 0^*+} \frac{\bar{B}(s)}{\sqrt{2s\log\log(1/s)}} = 1, \quad \text{a.s.}$$

Hence

$$\limsup_{t\to T^*-} \frac{h(\tilde{X}(T^*)) - h(\tilde{X}(t))}{\sqrt{2(T^*-t)\log\log(1/(T^*-t))}} = 1, \quad \text{a.s.}$$

Since h is in $C^1((0,\infty);\mathbb{R})$ and $\tilde{X}$ has continuous sample paths, we have

$$\lim_{t\to T^*-} \frac{h(\tilde{X}(T^*)) - h(\tilde{X}(t))}{\tilde{X}(T^*) - \tilde{X}(t)} = h'(\tilde{X}(T^*)) = \frac{1}{g(\tilde{X}(T^*))}, \quad \text{a.s.}$$

Hence

$$\begin{aligned}
&\limsup_{t\to T^*-} \frac{\tilde{X}(T^*) - \tilde{X}(t)}{\sqrt{2(T^*-t)\log\log(1/(T^*-t))}} \\
&= \limsup_{t\to T^*-} \frac{\tilde{X}(T^*) - \tilde{X}(t)}{h(\tilde{X}(T^*)) - h(\tilde{X}(t))} \frac{h(\tilde{X}(T^*)) - h(\tilde{X}(t))}{\sqrt{2(T^*-t)\log\log(1/(T^*-t))}} \\
&= g(\tilde{X}(T^*)), \quad \text{a.s.}
\end{aligned}$$

Therefore, as $T(s) \to \infty$ as $s \uparrow T^*$, and $\langle M\rangle(T(s)) = s$ for $s \in [0, T^*]$, we have

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{L(\xi) - X(t)}{\sqrt{2\int_t^\infty \sigma^2(s)\,ds \log\log(1/\int_t^\infty \sigma^2(s)\,ds)}} \\
&= \limsup_{t\to\infty} \frac{X(\infty) - X(t)}{\sqrt{2(\langle M\rangle(\infty) - \langle M\rangle(t))\log\log(1/\langle M\rangle(\infty) - \langle M\rangle(t))}} \\
&= \limsup_{s\uparrow T^*} \frac{X(T(T^*)) - X(T(s))}{\sqrt{2(\langle M\rangle(T(T^*)) - \langle M\rangle(T(s)))\log\log(1/\langle M\rangle(T(T^*)) - \langle M\rangle(T(s)))}} \\
&= \limsup_{s\uparrow T^*} \frac{\tilde{X}(T^*) - \tilde{X}(s)}{\sqrt{2(T^*-s)\log\log(1/(T^*-s))}} \\
&= g(\tilde{X}(T^*)) \\
&= g(L(\xi)), \quad \text{a.s.}
\end{aligned}$$

An analogous argument gives

$$\liminf_{t\to T^*-} \frac{\tilde{X}(T^*) - \tilde{X}(t)}{\sqrt{2(T^*-t)\log\log(1/(T^*-t))}} = -g(\tilde{X}(T^*)), \quad \text{a.s.},$$

from which we can infer that

$$\liminf_{t\to\infty} \frac{L(\xi) - X(t)}{\sqrt{2\int_t^\infty \sigma^2(s)\,ds \log\log(1/\int_t^\infty \sigma^2(s)\,ds)}} = -g(L(\xi)), \quad \text{a.s.},$$

as required.

4.3 Proof of Theorem 3.6

By Proposition 3.2, it follows that $\lim_{x\to 0^+} v(x) < +\infty$ implies $\tilde{S}_0(\xi) < +\infty$, a.s.

In case (b) when $\sigma \notin L^2([0,\infty);\mathbb{R})$, $T(s) \to \infty$ as $s \to \infty$. Hence, as $\tilde{X}(s) = X(T(s))$ for $s \in [0,\infty)$, we have $S_0(\xi) = T(\tilde{S}_0(\xi)) < +\infty$, a.s., and

$$\lim_{s\to S_0(\xi)^-} X(s) = \lim_{t\to\tilde{S}_0(\xi)^-} X(T(t)) = \lim_{t\to\tilde{S}_0(\xi)^-} \tilde{X}(t) = 0, \quad \text{a.s.},$$

because $\tilde{S}_0(\xi) < +\infty$, a.s.

In case (a) when $\sigma \in L^2([0,\infty);\mathbb{R})$, $T(s) \to \infty$ as $s \to T^{*-}$. Define the event

$$A = \{\omega : \tilde{S}_0(\xi) \geq T^*\}.$$

Then because $\tilde{X}(s) = X(T(s))$ for $s \in [0,T^*]$, we have $S_0(\xi) = T(\tilde{S}_0(\xi))$, and so

$$A = \{\omega : \tilde{S}_0(\xi) \geq T^*\} = \{\omega : S_0(\xi) = +\infty\}.$$

Clearly, as $\tilde{S}_0(\xi) \geq T^*$ on A, we have

$$\lim_{t \to T^{*-}} \tilde{X}(t) = \tilde{X}(T^*) > 0, \quad \text{a.s. on } A.$$

Thus, as $\tilde{X}(s) = X(T(s))$ for $s \in [0,T^*]$,

$$\lim_{t\to\infty} X(t) = \lim_{s \to T^{*-}} X(T(s)) = \lim_{s\to T^{*-}} \tilde{X}(s) = \tilde{X}(T^*) > 0, \quad \text{a.s. on } A.$$

Hence

$$\left\{\lim_{t\to\infty} X(t,\xi) = L(\xi) > 0,\ S_0(\xi) = +\infty\right\} = \{\tilde{S}_0(\xi) \geq T^*\}, \quad \text{a.s.}$$

On the other hand, consider the event $\bar{A}$, where

$$\bar{A} = \{\omega : \tilde{S}_0(\xi) < T^*\} = \{\omega : S_0(\xi) < +\infty\},$$

by virtue of the fact that $S_0(\xi) = T(\tilde{S}_0(\xi)) < T(T^*) = +\infty$. Then

$$\lim_{t\to \tilde{S}_0(\xi)^-} \tilde{X}(t) = 0, \quad \text{a.s. on } \bar{A}.$$

Thus, as $\tilde{X}(s) = X(T(s))$ for $s \in [0,T^*]$,

$$\lim_{t\to S_0(\xi)^-} X(t) = \lim_{s\to \tilde{S}_0(\xi)^-} X(T(s)) = \lim_{s\to \tilde{S}_0(\xi)^-} \tilde{X}(s) = 0, \quad \text{a.s. on } \bar{A}.$$

Hence

$$\left\{\lim_{t\to S_0(\xi)^-} X(t,\xi) = 0, S_0(\xi) < +\infty\right\} = \{\tilde{S}_0(\xi) < T^*\}, \quad \text{a.s.}$$

Received: July 2008. Revised: August 2008.

References

[1] J.A.D. Appleby *Almost sure subexponential decay rates of scalar Itô-Volterra equations*, Electron. J. Qual. Theory Differ. Equ., Proc. 7th Coll. QTDE, Paper no.1, 1–32, 2004.

[2] J.A.D. Appleby and D. Mackey, *Almost sure polynomial asymptotic stability of scalar stochastic differential equations with damped stochastic perturbations*, Electron. J. Qual. Theory Differ. Equ., Proc. 7th Coll. QTDE, Paper no.2, 1–33, 2004.

[3] J.A.D. Appleby, X. Mao and A. Rodkina, *Pathwise superexponential decay rates of solutions of autonomous stochastic differential equations*, Stochastics, **77**(3), 241–270, 2005.

[4] J.A.D. Appleby, A. Rodkina and H. Schurz, *Pathwise non-exponential decay rates of solutions of scalar nonlinear stochastic differential equation*, Disc. Con. Dynam. Sys. Ser. B., **6**(4), 667–696, 2006.

[5] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, North Holland, New York, 2nd edition, 1981.

[6] K. Itô and H.P. McKean, *Diffusion processes and their sample paths*, Springer, Berlin, 2nd edition, 1974.

[7] I. Karatzas and S.E. Shreve, *Brownian Motion and Stochastic Calculus*, volume 113 of Graduate Texts in Mathematics, Springer, New York, 1991.

[8] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*. Third edition, Springer, New York, 1999.