CUBO *a Mathematical Journal*
Vol. 6, № 1, (1-13). March 2004.

The Geometric Algebra of Leonardo da Pisa, a.k.a. Fibonacci

An Octo-Centennial Tribute 1202 -2002.[0]

Barnabas Hughes, O.F.M.
California State University, Northridge

Eight hundred years ago Fibonacci, also known as Leonardo da Pisa, finished the initial copy of his book, *Liber Abbaci.*[1] Medieval Europe was ready for such a compendious volume on calculations of whatever kind in the fields of arithmetic, geometry, and algebra-as these subjects were becoming better known in his day. Roman numerals were yielding to Hindu-Arabic numerals to such an extent that, although advocated by Leonardo at the beginning of his book with complete (tedious!) instructions on how to use them in every arithmetic operation, their use in Florence was forbidden in 1299.[2] Syntheses of Euclidean geometry were being replaced by translations of Euclid's *Elements*.[3] Three translations of al-Khwārizmī's *al-jabr* were circulating in many copies.[4] The complement to the translations of Arabic and Greek texts into Latin was Fibonacci's foundational text, *Liber Abbaci.*

Precious little is known about Leonardo, apart from an autobiographical précis in the introduction to Liber Abbaci. Born in Pisa, Italy, about 1175 he was reared as a teenager in Bougie on the Algerian coast where his father was a customs agent for Pisan merchants. He learned basic computational arithmetic from a local teacher, an experience that motivated him to study mathematics in greater detail as he traveled about the Mediterranean on business. Returning to Italy he completed the writing of *Liber Abbaci* in 1202. His *Practica geometria*, a collection of useful theorems from geometry and (what would eventually be named) trigonometry, appeared in 1202, to be followed in about five years by *Liber quadratorum*,[5] a work on indeterminate analysis. His prowess in mathematics came to the attention of the Emperor, Frederick II, who invited Leonardo to his court, to engage in mathematical tournaments. The date of his death is usually put at 1250.[6]

[0]A different version of this paper was delivered at the annual meeting of the Canadian Society for the History and Philosophy of Mathematics, Toronto, Canada, May 26, 2002.

The bulk of the *Liber Abbaci* is devoted to solving a spectrum of problems, despite Fibonacci's remark in the dedicatory and initial paragraph, "This book focuses more on theory than on applications."[7] This statement, from my viewpoint as a student of Leonardo's text, correctly describes only the third section of Chapter 15. Here he develops algebra as a tool for solving number problems, the unraveling of relations between given and desired numbers, the soft of thing his contemporary, Jordanus de Nemore, would craft in greater generality in his influential *De numeris datis.*[8] The algebraic section on *Liber Abbaci*, the focus of this paper, has few practical problems: six coin problems [9] and a seventh problem involving interest.[10] The remaining problems ask for a number given certain conditions together with the numbers involved in the conditions. We shall see that he beckons geometry from the wings to step forward and take center stage in order to make the process of the solution clear. Fibonacci continues to align measure and number, an Arabic conjunction of geometry and arithmetic that leads to an algebraic way of thinking. In short, his algebra is interwoven with geometry.[11] This essay on one part of medieval mathematics has several objectives: to offer an understanding of Leonardo's concept of algebra; to illustrate his use of geometric concepts, figures, and methods to solve problems involving number relations, each of which has been reduced to an algebraic equation;[12] and to challenge interested readers to flesh out solutions to several problems from *Liber Abbaci.*

At this point we would consider what Fibonacci understood by algebra from reviewing his sources and what he apparently gleaned from them. He certainly used three treatises on algebra, by al-Khwārizmī (*ca.* 775 -*ca.* 850), abū Ka mil (*ca.* 850 -*ca.* 930) perhaps in its Latin version known as *Liber mahameleth*[13], and al-Karaĵi (953 -*ca.* 1029). His science of algebra (or theory of equations) is obviously based on *Liber algebre et almuchabala* of al-Khwārizmī. Leonardo summarized the six kinds of equations as they appear in this treatise: three simple equations and three compound quadratics with widely accepted terminology, *numerus = dragma = denarius, res, radix, quadratus, census, equari, restaurare, proportio.* He perhaps first of all writers used the noun *equatio* in exactly the same way as we use the word *equation* today.[14] Further, he adopted 22 problems apparently from the al-Khwārizmī text. Since these are 12 more than the 10 problems with which I found correspondence in the Latin translations,[15] I conclude that Fibonacci used an Arabíc text, or Latin version of one, closely related to that of al-Khwārizmī. Levey identified thirteen problems in *Liber Abbaci* identical with problems in the algebra of abū Kāmil, another four which kept the form but changed the numbers, and a final twelve that are similar.[16] Woepke found twenty-two problems in *Liber Abbaci* that are either identical with problems in al-Karaĵi's text or are quite similar, constants and coefficients alone differing.[17] There is an overlap of some six problems among the three basic resources, apart from the introductory sections with its many examples of the six classical equations. In short, Leonardo was well acquainted with Arabic algebra.

Our appreciation of Leonardo's understanding of algebra is greatly assisted by the title he wrote for the algebraic section of Chapter 15: *Now begins the third part on the solution of certain problems by means of algebra and almuchabala, namely proportion and restoration.*[18] The title clearly implies that Leonardo understood algebra to be a method for solving problems. Two tools stand out, proportion and restoration. A third imbedded in his explanation on solving quadratic equations is completing the square, more on this shortly. He had prepared his readers well for proportion. Apart from its use in Chapters 8, 9, and 12 of *Liber Abbaci*, the first two sections of Chapter 15, which precede the section on algebra, are devoted exclusively to proportion.[19] Leonardo would use it to solve several problems in the algebraic section of Chapter 15, particularly Problem 86 below. Restoration is for him an algebraic rule whereby terms are moved from one side of an equation to the other; the verb, *restaura*, has no other meaning in *Liber Abbaci.* The technique of completing the square is used time and again for solving the bulk of the problems. In summary then, for Leonardo Pisano algebra was the science of equations for solving problems reducible to standardized forms, employing common terminology, and solved by rues unique to the science or adapted from geometry. Only algebraic symbols are missing for which European algebra[20] would have to wait until the renaissance, although I shall use them freely in what follows.

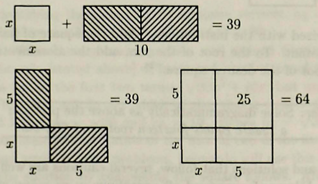
The tool he uses more than any other for solving quadratic equations is the method of *completing the square.* Now considered an algebraic method, it is derived from Euclid's *Elements* II.6.

> If a straight line is bisected and a straight line be added to it on a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.[21]

The phrase, *completing the square*, describes exactly what the method does, is exemplified in the solution of the example for the problem *square and roots equal to numbers*. Leonardo borrowed the example from al-Khwārizmī, to become the classical quadratic equation in modern symbols,

$$x^2 + 10x = 39$$

(The reader need be aware that the diagram and explanations which follow are based solely on al-Khwārizmī's instructions.[22] The terms of the equation are represented by geometric figures, a square and a rectangle. The sum of their afeas is given. The rectangle is cut into two parts of equal areas which are attached to adjacent sides of the square. The **L**-shaped result is then filled in with a second square which completes the square. The addition increases the given size of the figure. The solution follows easily.



$$(x+5)(x+5) = 64$$
$$x + 5 = 8$$
$$x = 3$$

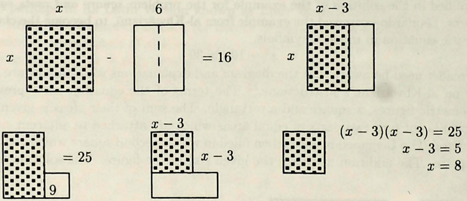
Challenge to the reader: Solve diagrammatically as above the problem *a square and fourteen roots* equals *fifty-one.*

There is no better introduction to solving quadratic equations for secondary students than beginning with this geometric routine; any discussion about the second root of 64 is better left for later.

Although the algebraic technique is nearly the same for solving the problem, *roots and numbers equal to square*, say, $6x + 16 = x^2$, not quite so obvious is the geometric diagrammatic procedure. To make this viable, the equation is transformed to

$$x^2 - 6x = 16.$$

As can be seen, the square and rectangle have a side in commnon, with the halves of the rectangle being subtracted from the square. The first subtraction is easy; the second not so. Half the rectangle, $3x$, is easily removed from the original square, but not enough remains to remove the other half and leave a square. Here a new square the dimensions of which are half the number of roots is added to the base of the diminished square, thereby increasing the area so that the second half of the rectangle can be subtracted from it. Finally, the side of the remaining square is easily found and the problem is solved.[23]



All of this Leonardo summarized with the instructions, "Add the square of half the [number of] roots to the constant. To the root of the sum add the aforementioned half, and you will have the root of the desired square."[24]

Challenge to the reader: Solve diagrammatically as above the problem *a square equals fourteen roots and fifty-one.*

In studying the problems and solutions that follow, several cautions are well kept in mind. As implied above, Fibonacci expressed all operations in so many words;

he did not have the convenience of our Cartesian convention ($a, b, c, \cdots, x, y, z$ for known and unknown numbers), and symbols for operations and relations. The letters he does employ refer to geometric points, line-segments, and area. All operations are performed on known numbers and given line-segments. While he did express equations and their solutions in so many words, he could not symbolize them as I have done below. With these remarks in mind, let us reflect on three problems.

More complex than the foregoing example is Problem 59:

> I multiply the root of six times a quantity by the root of five times the same quantity. Then 1 add to the product 20 dinars and 10 times the quantity. The whole is equal to square of the quantity.[25]

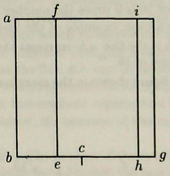
The three statements are easily transfonned into modem symbols,

$$(\sqrt{6x})(\sqrt{5x}) + 10x + 20 = x^2$$

which in so many words Leonardo reduces to the equation

$$\sqrt{30x^2} + 10x + 20 = x^2.$$

That is, a square of unknown dimensions is composed of three identifiable areas; thus,



Identifying the square by its opposite vertices, $ag = x^2$, he recognizes the included areas $ae = \sqrt{30x^2}$, $fh = 10x$, and $ig = 20$.

With the algebraic equation well defined Leonardo continues, "This falls under the rule [stated above] of roots and number equal to a square."[26] His initial step is to factor the first two terms, $\sqrt{30x^2} + l0x$; but unlike us he does it geometrically. Fibonacci sets $bg = x$ which implies that $x = ab = fe = ih$. It follows from Elements II.1 that $be = \sqrt{30}$ and $eh = 10$. Hence $bh = \sqrt{30} + 10$; and he has factored the algebraic representation of the sum of two of the inner areas into $(\sqrt{30} + 10)x$. Time and again in other problems, Leonardo will use this devise to factor some $\sqrt{ax^2} + bx$. The solution by completing the square follows. After observing that $\sqrt{30} + 10$ is a fourth binomial,[27] he divides the line bh in two equal parts at point c. Then

$$[1] \qquad bc = ch = 5 + \sqrt{7\tfrac{1}{2}}$$

$$[2] \qquad ig = 20 = (ih)(hg)$$

$$[3] \quad \text{from}[1]: \qquad (ch)^2 = (5 + \sqrt{7\tfrac{1}{2}})^2 = 32\tfrac{1}{2} + \sqrt{750}$$

$$[4] \quad [2]+[3]: \qquad ig + (ch)^2 = (cg)^2 = 52\tfrac{1}{2} + \sqrt{750}$$

$$[5] \quad \sqrt{[4]} \qquad cg = \sqrt{52\tfrac{1}{2} + \sqrt{750}}$$

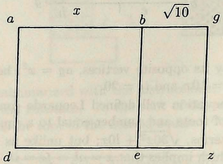
$$[6] \quad \therefore \qquad x = bc + cg = (5 + \sqrt{7\tfrac{1}{2}}\ ) + \sqrt{52\tfrac{1}{2} + \sqrt{750}}$$

The essence of the technique is an application of *Elements* II.6: join an area [2] to the square of half a line-segment [3] to create a new square [4]. The square root [5] ofthis new area sets up the solution [6]. In practice, the technique of completing the square becomes a bit more complicated than the simple examples in the introductory section suggest.

Problem 54, which may be Leonardo's own creation, is interesting because it leads to a most unusual statement about what today is called an irrational number. The problem is "Find a number such that the sum of its square and the product of the number and the square root of 10 is 9 times the number itself."[28] Leonardo begins the solution with words that are a clear example of the fusing of measure and number,

> Let the thing (x) which is the line $.a.b.$ represent the number. Add to it the line $.b.g.$ which is the root of 10. [29]

Thereupon he constructs the figure shown in the margin of the manuscript but without the modern symbols:



to conclude that "the entire area of rectangle $.d.g.$ is 9 times the number $.a.b.$"[30] Hence,

$$\begin{array}{lllll}
[1] & & (.a.b.)(.a.b.) & = & .b.d. \\
[2] & & (.e.b.)(.b.g.) & = & (.a.b.)(.b.g.) = .e.g. \\
[3] & [1]+[2]: & .b.d. + .e.g. & = & 9.a.b. \\
[4] & \therefore & .a.b. + \sqrt{10} & = & 9 \\
[5] & \therefore & .a.b. & = & 9 - \sqrt{10}
\end{array}$$

Having established step [3], Leonardo moves to [4] without instructions to the reader. Note however in the figure: since the width of the area is $.a.d. = .a.b.$, [3] implies that the length is $9 = a.b. + \sqrt{10}$. Hence, [4] follows immediately, a consequence of *Elements* II.1.

In a variation on this problem, namely that the area of the rectangle is 20,[31] Fibonacci remarks, "you will find that the square and as many of its roots *as there are units in the roots of* 10 equal 20" [32] (italics mine). How extraordinary! To state that there are so many units in an irrational number! Nevertheless, if we set the problem up and complete the square on the equation

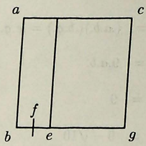
$$x^2 + x\sqrt{10} = 20$$

we shall find that $x = \sqrt{10}$. We are so accustomed to representing the measure of the diagonal of a square by an irrational number, say $\sqrt{2}$, because there is no unit measuring the side of the square which measures the diagonal, that we overlook the converse: what measures the diagonal does not measure the side. (I can almost imagine Fibonacci remarking that the line $.a.b.$ can be bisected and its halves bisected as long as one chooses; and that when the bisection has stopped, there is a very small, unnamed unit that measures the line-segment represented by $\sqrt{10}$.)

The third is my favorite problem, 86, because of the expected contradiction that leads to the solution:

> I have divided 10 into 3 parts. The product of the smallest and largest parts equals the square of the middle part. Further, the sum of the squares of the smallest and middle parts equals the square of the largest part.[33]

Fibonacci devoted nearly four pages of text to the problem for which he discusses several approaches to its solution, to find the three parts of ten.[34] He begins by noting that the first condition, the product $1(x^2) = (x)^2$ requires the three parts of ten to be *one*, *root*, and *square*, respectively the smallest, middle, and largest parts of 10. Since the largest part is already a square, the second condition creates an unknown of the fourth degree, *census census* as the text words it specifically, "When the square of the square is equal to a square and a drachma (squared), it is as though a square is equal to a thing and a drachma."[35] This is clearly equivalent to $(x^2)^2 = x^2 + 12$. Regardless of the approach to solving the problem, and we shall see that there are three approaches, the end result will be the three parts of ten, differing according to the respective approach. The firsts approach or beginning envisions $10 = x^2 + x + 1$. With this in mind, let us begin with a figure drawn after Leonard's instructions:



The solution begins by recognizing which parts of the first condition are represented in the figure.

[1] Assignments $.a.g. = x^4 \quad .a.e. = x^2 \quad .b.e. = 1 \quad (\frac{1}{2}).b.e. = \frac{1}{2} = .b.f. = .f.e.$

[2] $.a.e. = x^2 = (.a.b.)(.b.e.) = x^2(1)$

[3] $.c.e. = 1 = (.e.g.)(.g.c.)$

[4] *Elements* II.6 $(.e.g.)(.g.c.) + (.f.e.)^2 = (.f.g.)^2$

[5] That is $1 + (\frac{1}{2})^2 = \frac{5}{4} = (.f.g.)^2$

[6] From [1]&[5] $.b.f + .f.g. = .b.g. = \frac{1}{2} + \sqrt{\frac{5}{4}} = x^2$

[7] $[6]^2$ $\frac{3}{2} + \sqrt{\frac{5}{4}} = x^4$

[8] $\therefore\ x^4 = x^2 + 1$ $\frac{3}{2} + \sqrt{\frac{5}{4}} = \frac{1}{2} + \sqrt{\frac{5}{4}} + 1$ Q.E.D.

From [1] to [8] the mathematics flows perfectly; the logic is impeccable. Regardless, the conclusion is wanting. If the square roots of the three terms in [8] are added, their sum is 5.2360..., far short of the 10 required by the statement of the problem. What are we to think? A study of the problem in his putative resource, the *al-jabr* of abū Kāmil, only informs us that Leonardo added the geometric explanation to an otherwise algebraic presentation. Both begin with the same assumption: *Set the smallest part equal to one.* The statement begins the procedure of single false position. After the discovery of the false value resulting from the assumption, a further step is required to reach the true value (here) of the smallest part of 10. The further step is a proportion the false sum is to the true sum as the false value is to the true value. Fibonacci takes this step: "(A)s the sum of the three parts which were found is to 10, so is the drachma [our 1] to the thing."[36] For lack of adequate vocabulary (symbols ?) he has shifted the term *res* to the smallest part; I show the change, using y for the smallest term. Hence,

[1] $$\frac{1+\sqrt{\sqrt{\frac{5}{4}}+\frac{1}{2}}+(\sqrt{\frac{5}{4}}+\frac{1}{2})}{10}=\frac{1}{y}$$

[2] $XM[1]$ $$y+\sqrt{\sqrt{\frac{5}{4}y^4}+\frac{1}{2}y^2}+(\sqrt{\frac{5}{4}y^2}+\frac{1}{2}y^2)=10$$

[3] $([2])^2$ $$\sqrt{\frac{5}{4}y^4}+\frac{1}{2}y^2=100-\frac{5}{4}y^2-\sqrt{\frac{11}{4}y^4}-30y-\sqrt{500y^2}$$

[4] $$10y=25(3-\sqrt{5})+y^2$$

[5] CTS [4] for the smallest part $y=5-\sqrt{\sqrt{3125}-50}$ [=2.5706. modern equivalent]

To find the largest part, Leonardo sets it equal to *one* drachma, the smallest term becomes the thing (*res*), and the middle term the square root of the thing. Consequently, $1>\sqrt{x}>x$. Without referring to the geometric figure, he proceeds immediately to

[1] Given $x^2+x=l$

[2] CTS $x=\sqrt{\frac{5}{4}}-\frac{1}{2}$

[3] $\sqrt{[2]}$ $\sqrt{x}=\sqrt{\sqrt{\frac{5}{4}}-\frac{1}{2}}$

[4] But $1+(\sqrt{\frac{5}{4}}-\frac{1}{2})+\sqrt{\sqrt{\frac{5}{4}}-\frac{1}{2}}\neq 10$

[5] Accordingly $\frac{1+(\sqrt{\frac{5}{4}}-\frac{1}{2})+\sqrt{\sqrt{\frac{5}{4}}-\frac{1}{2}}}{10}=\frac{1}{y}$

[6] XM[5] $y^2+50=(5+\sqrt{125})y$

[7] CTS [6] $y=2\frac{1}{2}+\sqrt{31\frac{1}{4}}-\sqrt{\sqrt{728\frac{1}{4}}-12\frac{1}{2}}$ the largest part.

Adding this to the smallest part above and subtracting the sum from 10, the middle part becomes

$$2\frac{1}{2}+\sqrt{\sqrt{3125}-50}+\sqrt{\sqrt{728\frac{1}{4}}-12\frac{1}{2}}-\sqrt{31\frac{1}{4}}$$

These three sum to 10, as required.

Fibonacci did not let his readers move on after completing the solution. Perhaps he felt that the technique required reinforcement for ease of future use. So he begins anew by seeking the largest of the three numbers: "If you wish to find the largest part, set it equal to one and the smallest part to the thing."[37] He has exchanged an instruction from the beginning of the first solution.

Challenge to the reader: Find the three numbers as above only with the change stated here.

To round out the exposition and offer a new challenge to hisreaders (and mine!), Leonardo suggests another way (*per hanc aliam viam*), the third approach to begin with: "Set the middle part equal to 2 dragmas and the smallest part equal to the root of the thing."[38] The reason for the selection of the two values is that they are quite suitable for finding the largest part by the "sum" condition.

Challenge to the reader: Find the three numbers with the information given here.

Now I trust that my readers appreciate why this is my favor problem.

A further indication of Fibonacci's geometric thinking appears in the Chapter 14 which focuses on finding and computing with square and cube roots and conjugate binomial surds. In the introduction to the chapter he lists six propositions from Book II of Euclid's *Elements* which he identifies as necessary keys (*quedam necessaria que claues dicuntur*) for learning about the various roots. After the last two propositions he writes: "All the problems of algebra and almuchabala can be reduced to these last two propositions,"[39] namely II.5 and II.6. This statement alone prompts the question, Did Fibonacci think he was creating a "geometric algebra"? While I describe his work with this expression, I leave it to others to argue their positions.4O Geometry was certainly a guiding light.

In the body of the section on algebra in Chapter 15 after the introduction, Leonardo posed 97 problems. You have seen how three of them are solved by geometric tools, by completing the square and proportion. Of the remaining 94 problems, some thirty-five are solved either immediately by geometric tools or by geometric methods acting as aids. Clearly, Leonardo used geometric figures, concepts, and skills to assist in the solution of algebraic equations. Such is the geometric algebra of Leonardo Fibonacci, Pisano.

+ ++ +++ ++++ +++ ++ +

Note My thanks to the three referees who helped me improve the clarity and exposition of this essay.

---

1. B. Boncompagni, *Il Liber Abbaci di Leonardo Pisano*, Vol. 1 of *Scritti di Leonardo Pisano Matematico del secolo decimoterzio.* Rome: Tipografia delle Scienze Matematiche e Fisiche, 1857 [*hereafter cited as* BLA, the whole number being the page number and the decimal part the line numbers]. Despite the date 1202 in the title, the initial paragraph makes it clear that this is the revised and corrected edition of 1228. G. Libri, *Histoire des Sciences Mathématique en ltalie depuis la renaissance des lettres, jusqu' à la fin du 17^e siècle*, Vol.II (Paris, 1838; Halle: H.W. Schmidt, 1865 -2nd ed.), 307-479, the transcription available to me contains only Ch. 15 [*hereafter cited as* LLA]. For a highly detailed analysis of the text and commentar y, consult H. Lüneburg, *Leonardi Pisani Liber Abbaci oder Lesevergnügen eines Mathematikers* Mannheim: Wissenschaftsverlag,1993. For readers who have access to L.E. Sigler's monumental translation, *Fibonacci's Liber Abaci A Translation into Modem English*

of Leonardo Pisano's Book o/Calculation (New York: Springer, 2002), please remember that Professor Sigler incorporated into his translation page notation, "[p193] for example, (which) refers to the approximate beginning of each new page in the Latin (Boncompagni) editíon." Hence, there is no need to reference the English translation here. Finally, my translations are for the most part less *a verbo ad verbum* than those of Professor Sigler.

2. D. Struik, "The Prohibition of the Use of Arabic Numerals in Florence,"*Archives Internationales d'Histoires des Sciences*, 1968 21:291-94.

3. H.L.L. Busard, *The Medieval Latin Translation of Euclid's Elements Made Directly from the Greek.* Stuttgart: Franz Steiner Verlag, 1984,

4. B. Hughes, "Gerard of Cremona's Translation of al-Khwārizmī's *al-Jabr*: A critical edition,"*Mediaeval Studies* 48, 1986,211-63. See algo B. Hughes, *Robert of Chester's Latin Translation of al-Khwārizmī's al-jabr*, Stuttgart: Franz Steiner Verlag, 1989, and W. Kaunzner, *Die Lateinische Algebra in MS Lyell52 der Bodleian Library, Oxford, früher MS Admont 612*, Vienna: Österreichische Akademie der Wissenschaften Phil.-Hist. Klasse, Sitz. 1986,475. Band, 47-89. No other Latin translations of al-Khwārizmī's *al-Jabr* have been found.

5. L.E. Sigler, *Leonardo Pisano Fibonacci's Book ofSquares.* Bastan: Academic Press, 1987.

6. Easily accessible material on his life and works may be found at the MacTutor web site, *http://www-history.mcs.st-and.ac.uk/ history* The most recent work on the life and works of Leonardo is by R. Franci, "*Il Liber Abaci* di Leonardo Fibonacci 1202-2002," in La Matematica nella Societa e nella Cultura, *Bollettino della Unione Matematica Italiana* (8) 5-A, Agosto 2002, 293-328. A comprehensive description of mathematics in medieval Europe is Ch. 8 of V. Katz' *A History of Mathematics An Introduction* (New York: HarperCollins, 1993),266-301, in which Leonardo's other works on practical geometry and number theory are discussed in detail, along with *Liber Abbaci.*

7. "Sane hic liber magis ad theoricam spectat quam ad practicam." BLA 1.14-15.

8. B. Hughes, *Jordanus de Nemore De numeris datis A Critical Edition and Translation.* Berkeley: University of California Press, 1981.

9. Questiones 12-14; BLA 413.6 -415.5. The Libri text is corrupt for these problems. The problem numbers are mine, which I developed for a research project, "Fibonacci, Teacher of Algebra. "

10. Questio 53; BLA 495.8 -426.4; LLA 401.15 -403.20.

11. After preparing this paper I remembered the carefully researched article by R. Franci and L. Toti Rigatelli (1985), "Toward a History of Algebra from Leonardo of Pisa to Luca Pacioli, *Janus* 72(1-3): 17-82. Their evaluation, Leonardo prefers to justify the greatest part of the algebraic calculations geometrically" (p. 22), is clase to mine, though without the same detail K.H. Parshall's remark, "Then following his above named ancestors, [Leonardo] gave specific examples written out rhetorically, solved algebraically, and justified geometrically," needs some adjustment. Many problems were not solve algebraically nor were all justified geometrically. See her, "The Art of Algebra from al-Khwārizmī to Viète: A Study in the Natural Selection of Ideas," *History of Science* (1988) 26: 138, unchanged at her web site *http://www.lib.virginia.edu/science/ parshall/algebrall.htm.*

12. Leonardo was the first to use the noun, equation, to describe the verb, equals, which had been current since at least the Latin translations of al-Khwārizmī's al-jabr; see BLA 407.4 and LLA 357.2.

13. J. Sesiano, "La version latine médiévale de l'Algèbre d'Abu Kamil," in M. Folkerts & J.P. Hogendijk, *Vestigia Mathematica: Studies in medieval and early modern mathematics in honor of R.L.L. Busard* (Amsterdam: Rodopi, 1993),314-452. See also his, *Une introduction à l'histoire de l'algèbre Résolution des equations des Mésopotoamiens à la Renaissance* (Lausanne: Presses Polytechniques et universitaires Romandes, 1999), 100-20.

14. "Vnde curo in aliqua questione inuenientur census uel partes unius census equari radicibus uel numero, debent reddigi ad equationem." BLA 407.2-4, LLA 356.29-357.2

15. See note 4 above. The following are also pertinent here. L.C. Karpinski's *Robert of Chester's Latin Translation of the Algebra of al-Khowarizmi with an Introduction, Critical Notes and an English Version* (New York: Macmillan, 1915) should be used with care, as I explain in my edition, despite the value of its English translation. Finally, R. Rosen, *The Algebra of Mohammed ben Musa* (London, 1831) which contains the additional 12 problems.

16. M. Levey, *The Algebra of Abū Kāmil Kitāb fi al-jábr wa'l-muqūbala in a commentary by Mordecai Finzi* (Madison, WI: University of Wisconsin Press, 1966),217-20.

17. F. Woepcke, *Extrait du Fakrî, Traité d'Algèbe par Aboù Bekr Mohammed ben al Hacan al Karkhî* (Paris: l'Imprimerie Impériale, 1853), 24 -32.

18. "Incipit pars tertia de solutione quarundam questionum secundum Modum algebre et almuchabale, scilicet ad proportionem et restaurationem. " BLA 406.334-35, LLA 356.11-13.

19. For more information on Leonardo's use of proportion apart from the text, see M. Bartolozzi and R. Franci, "La teoría delle proporzioni nella matematica dell'abaco da Leonardo Pisano a Luca Pacioli," *Bollettino di Storia delle Scienze Matematiche* (1990), 10(1):3-9.

20. In the Maghréb, an area encompassing the Algerian coast, Arabic algebrists of the thirteenth century had developed symbols much like those of Diophantus, which unfortunately did not transfer across the seas. See M. Abdeljaouad, "Le manuscrit mathématique de Jerba: Une pratique des symboles algébriques maghrébins en pleine maturité," *(Presentation at) Septième Colloque Maghrébin sur l'histoire des mathéma tiques arabes* (Marrakech, 30-31 mai et 1er juin 2002), 1-20 et seq. J. H/oyrup raised the real possibility "of a borrowing from the Islamic world" across the Mediterranean in "Artificial Language in Ancient Mesopotamia -A Dubious and a Less Dubious Case," *(Contribution to the Workshop) Asian Contributions to the Formation of Modern Science: The Emergence of Artificial Languages* (Leiden, 20-21 September 2002), 10.

21. T .L. Heath, *The Thirteen Books of Euclid's Elements I* (New York: Dover Publ. Inc, 1956), 385

22. BLA 407.37 -408.26 and LLA 358.24 -361.2. Fibonacci, *pace* al-Khwārizmī, explains a second technique in which the rectangle is divided into four equal parts each being attached to a side of the square, leaving open corners that need filling to complete the square.

23. The diagram is neither Leonardo's nor al-Khwārizmī's. I have no recollection of taking it from any source.

24. "... quadratum medietatis radicum addes super numerum; et super radicem eius quod prouenerit, addes numerum medicetatis (*sic*) radicum, et habebis radicem quesiti census." BLA 408.37-39, LLA 361.18-21

25. "Rvrsus multiplicaui radicem sexupli cuiusdam aueris in radicem quincupli eius, et addidi decuplum ipsius aueris et denarios 20. Et fuerunt hec omnia sicut multiplicatio ipsius aueris in se." BLA 427.35-37, LLA407.19-22. See mention of this problem in F. Woepcke, *op. cit.* 25.

26. "In hac cadit regula radicum et numeri que equantur censui." BLA 427.43, LLA 408.1-3

27. BLA 356.24 -358.25 where Leonardo summarized from Euclid X the doctrine on the six kinds of binomia.

28. "Inueniat quis numerum, quo multiplicato in se et in radicem de 10, faciat nonuplum ipsius numeri." BLA 426.4-5, LLA 403.21-22.

29. "... ponam pro ipso numero rem, que sit linea .a.b., et addam ei lineam .b.g. que sit radix de 10." BLA 426.5-6, LLA 403.22-24.

30. "... superficies .d.g. que est nonuplum numeri .b.a. " BLA 426.8-9, LLA 403.28-29.

31. "Item est numerus, qua multiplicato in se et in radicem de 10, proueniunt 20." BLA 426.32, LLA 405.12-13

32. " ... census et tot radices eius quot unitates sunt in radicibus de 10 equantur 20." BLA 426.35, LLA 405.16-17

33. "... diuisi 10 in 3 partes. Et fuit multiplicatio minoris per maiorem sicut multiplicatio medie partis in se. Et multiplicationes minoris in se et medie partis in se sunt sicut multiplicatio maioris partis in se." BLA 448.9-: , LLA 453.3-7. A logical reading of the received texts demanded the corrections that appear in my development of the demonstration. The problems and its solution are closely parallel to what may be read in J. Sesiano, *op.cit.*, 405.3ff. See algo M. Levey, *op.cit.*, 186 [No. 61], in which the editor offers a complete symbolic solution for each of the three approaches to the problem, based on quite plausible assumptions involving 10 and its three parts. From much the same viewoint, see J.L. Berggren, *Episodes in the Mathematics of Medieval Islam* (New York: Springer-Verlag, 1986), 110-11

34. BLA 448.9-451.39, LLA 453.3.-461.13

35. " ... quando census census equatur censui et dragme est sicut quando equatur census rei et dragme." BLA 448.17-18, LLA 453.16-18

36. "(E)rit sicut coniunctum ex predictis partibus inuentis ad 10, ita dragma ad rem." BLA 448.33- 34, LLA 454.18-19.

37. "Et si uolumus maiorem partem inuenire, pones pro ipsa dragmam, et pro media radicem rei, et pro minori parte rem." BLA 449.28-29; LLA 456.15-17.

38. "... ut ponamus pro ipsa media parte duas dragmas et pro prima radicem rei ... " BLA 450.27-28; LLA 458.19-20.

39. "Ad has quidam ultimas duas diffinitiones reducunt omnes questionestiones (*sic*)." BLA 353.2-4.

40. A good place to obtain background on the topic is in Heath, *op. cit.*, I, 372-74.