Cubo A Mathematical Journal
Vol.05/N$^{\underline{o}}$03 - OCTOBER 2003

Sliding Mode Control

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1 Control as a Basic Human Activity

Most human activities are based on the control exerted by the brain on the body. For example, when we walk around the street we employ several control actions on our legs in order to move step by step in the correct direction, to remain reasonable stable during the motion, to avoid obstacles we encounter, to maintain or to change if needed the desired speed, etc.

These activities are directed to let the physical system (i.e. our body) operate in such a way to achieve some preassigned aims, taking into account the external world which of course influences the behavior of the system (not necessarily cooperating with the prassigned goals). Moreover the system must be able to react in a reasonable way, if some unpredictable event happens suddenly during the control actions.

For example, if we are peacefully walking on the street and a crazy driver crosses suddenly our street, we are (sometimes !) able to react quickly in order to avoid a crash. Our sensors (eyes, ears etc.) are such that when receiving a message of danger, we are able to process it in a very short time and to provide a corresponding behavior. The environment which sorrounds the system (i.e. our body) we want to control is, after all, full of uncertain events and we should be able to handle most of them successfully.

During centuries men tried to imitate such natural behaviors in order to build control systems, to obtain prespecified results from controlled machines. A quantitative step in this direction, from which the modern history of control systems and control technology started, was taken by J. Maxwell around 1868.

2 Feedback Control

Driving a car is one of the most common control activities. The basic point is the following: the decisions of the driver, at each time instant, are dictated not only by his or her explicit will (e.g. to reach some point of the town in the shortest possible time), but also by the instantaneous state of the system (i.e. the controlled car).

When the car comes close to a red light, the driver's action will of course take this into account. At the time when the car approaches several crossing roads, the driver will perhaps need to turn right say. Of course the corresponding time instant cannot be exactly planned in advance, due to unavoidable small delays during the driving, or to more or less unpredictable circumstances. Hence the controller (the driver in this example) acts on the physical system (the car) by driving in the appropriate way on the basis not only of the time instant at which some action must be exerted, but also of the actual state of the system.

The instantaneous state, if mathematically modeled, will consist of several functions of the time, like the coordinates of the position (with respect to some fixed reference frame), those of its velocity, the available fuel etc.

The control action, if mathematically modeled, will be represented by a vector depending at least on time and the instantaneous state. The usual control terminology is *feedback control*, meaning that the control action, as explained before, will be some function of time and state. The basic feature is that the control action depends in a crucial way upon the measured state vector.

If an automated pilot system is built, to control the motion of a vehicle by an automatic procedure in the form of some suitably feedback control law, then, based on the instantaneous system state, the controller can react in the appropriate way to changes of the state in order to follow the desired control aims, taking into account such disturbances.

3 Variable Structure and Sliding Mode Control

A basic control problem is to stabilize a controlled dynamical system. To simplify, suppose that the ideal state we want to reach at least asymptotically is the identically zero vector. Let the controlled system be described by a linear second order ordinary differential equation with scalar input u and scalar unknown state x. We write

$$\dot{x}(t) = \frac{d}{dt}x(t)$$

for the time-derivative of the function $x(t)$. The state variables are then $x(t), \dot{x}(t)$ which give rise to a two-dimensional vector. In a mechanical system we interpret the pair $(x(t), \dot{x}(t))$ as comprising all available information about the instantaneous position and velocity at time t. As discussed in the previous section 2, the feedback control $u(x_1, x_2)$ is a mathematical model for the control device we can use corresponding to an available state variable x_1, x_2. Hence if the pair $(x(t), \dot{x}(t))$ is available for measures (by an ideally exact measurement equipment) for each time instant t, then the corresponding control action will be represented by $u[x(t), \dot{x}(t)]$.

We want to select the input u in such a way that every solution x to

$$\ddot{x} + a\dot{x} + bx = u \tag{2}$$

fulfils

$$x(t) \to 0 \text{ and } \dot{x}(t) \to 0 \text{ as } t \to +\infty. \tag{3}$$

Here the time derivative d/dt is denoted by a dot and similarly $d^2x/dt^2 = \ddot{x}$, moreover a, b are given real numbers. The state variables (x_1, x_2) are related to x by the phase plane coordinates

$$x_1 = x, \; x_2 = \dot{x}.$$

Hence an equivalent rewriting of the control system (2) in terms of (x_1, x_2) is given by the following system of two first order differential equations

$$\dot{x}_1 = x_2, \; \dot{x}_2 = -a\, x_2 - b\, x_1 + u.$$

We want to choose the feedback control law as a linear function of the state, namely

$$u = hx + k\dot{x} \tag{4}$$

where the constant parameters h, k are at our disposal. By injecting (4) into (2) we obtain

$$\ddot{x} + (a - k)\dot{x} + (b - h)x = 0$$

a linear homogeneous differential equation of second order with constant coefficients.

We know that (3) is fulfilled if and only if every complex characteristic root, i.e. every complex solution λ to the algebraic equation of second degree

$$\lambda^2 + (a - k)\lambda + b - h = 0$$

fulfils the condition

$$\text{real part of } \lambda < 0.$$

This is true (by explicit computation) if and only if

$$a > k \text{ and } b > h. \tag{5}$$

Now suppose that only $x(t)$ is available for feedback: for example, maybe we are able to measure $x(t)$ but not $\dot{x}(t)$. Then we have the parameter $k = 0$ in (4). Hence if , for example, $a < 0$, no choice of k in (4) allows the controller to stabilize the system, because (5) fails; some solution to (2) will not satisfy (3).

Consider as an example

$$\ddot{x} - 2\dot{x} + 4x = u \tag{6}$$

$$u = 5x \text{ or } u = x, \tag{7}$$

which are particular cases of (2), (4). If $u = 5x$ then we get

$$\ddot{x} - 2\dot{x} - x = 0 \tag{8}$$

hence every solution x is a linear combination of

$$y_1(t) = e^{t(1+\sqrt{2})},\ y_2(t) = e^{t(1-\sqrt{2})}.$$

The corresponding phase portrait in the plane $(x, \dot{x})$ is shown in fig. 1.

If $u = x$ then we get instead

$$\ddot{x} - 2\dot{x} + 3x = 0 \tag{9}$$

hence every solution x is now a superposition of

$$y_3(t) = e^t \ \cos(t\sqrt{2}),\ y_4(t) = e^t \ \sin(t\sqrt{2}).$$

The corresponding phase portrait in the phase plane $(x, \dot{x})$ is shown in fig. 2.

Both control laws $u = 5x$ and $u = x$ induce unstable solutions, hence we have not solved this way the stabilization problem with the choice of the control law (7).

However a basic discovery (by Russian control theorists around the late fifties) allows us to obtain an asymptotically stable behavior from the two unstable structures (8) and (9), as follows.

Fix any $c > 0$ such that, in the phase plane (x_1, x_2) the line

$$s(x_1, x_2) = c\,x_1 + x_2 = 0 \tag{10}$$

lies between the axis x_1 and the line of slope $1-\sqrt{2}$ through the origin, which is an asymptote of the trajectories of the system (8). The a closer look to the phase portrait of fig. 3 shows that, if we follow in a suitable way, starting from any initial point, selected portions of the trajectories of (8) and (9), we reach the straight line (10) in finite time. The trajectories of (8) and (9) are directed towards the line (10) if we are sufficiently close to it. Therefore, as soon as some trajectory $(x, \dot{x})$ reaches the line (10), it remains on it forever, hence fulfills the constraint

$$s = c\,x_1(t) + x_2(t) = 0 \tag{11}$$

whence the differential equation

$$c\,x_1(t) + \dot{x}_1(t) = 0 \tag{12}$$

because $\dot{x}_1 = x_2$. From (12) we get

$$x_1(t) = \text{(constant)}\; e^{-ct} \to 0 \text{ as } t \to +\infty$$

because $c > 0$, and $x_2(t) \to 0$ as well. Then we see that asymptotically stable behavior is obtained, as desired.

Therefore, by varying in a suitable manner the two unstable structures obtained from (6) by the feedbacks (7), we obtain an asymptotically stable system, thereby solving the stabilization problem. Moreover we see that, starting from a second order dynamics (6) we need only to consider (asymptotically) only a first order dynamics (12), hence a simplified model has been obtained.

The control system (6), (10) employs a feedback control law

$$u^*(x_1, x_2) = \begin{cases} x_1 & \text{if } x_1 s(x_1, x_2) > 0, \\ 5x_1 & \text{if } x_1 s(x_1, x_2) < 0 \end{cases} \tag{13}$$

which turns out to be discontinuous along the line $s = 0$ given by (10), and the line $x_1 = 0$. This discontinuous feedback forces the state variable to slide on the line (10), hence the name *sliding mode control*. In a sense, only states (x_1, x_2) fulfilling (10) are of interest, and using (13) we satisfy the *viability constraint* (10).

4 A Generalization of Differential Equations

How to give a mathematically correct meaning to sliding mode control we described in the previous section 3 ? The differential equation we obtain injecting the feedback (13) in (6) can be written, as we did before, in the standard format of a first order differential system making use of the phase coordinates

$$x_1 = x, \quad x_2 = \dot{x}$$

thus obtaining

$$\dot{x}_1 = x_2, \dot{x}_2 = 2x_2 - 4x_1 + u^*(x_1, x_2). \tag{14}$$

The phase portrait of the control system, as we have seen, shows that using the discontinuous feedback control (13) we obtain, from a two - dimensional system, a one - dimensional dynamics on the *sliding manifold* (10) which is asymptotically stable, starting from two unstable dynamics. This is of great interest to control engineers, and can be generalized to handle successfully several control problems for non linear systems as well. However (14) does not fulfill the standard assumptions which guarantee existence and uniqueness of the solution to a system of ordinary differential equations with a given initial data. At least continuity, and something more (like Lipschitz continuity) is required, which clearly fails in (14).

In general, an initial value problem for an ordinary differential equation with discontinuous right-hand side has no (usual) solution at all.

For example, consider the opposite of the signum function

$$f(x) = \begin{cases} -x/|x| \text{ if } x \neq 0, \\ 1 \text{ if } x = 0. \end{cases}$$

Claim : there is no solution to

$$\dot{x} = f(x), \quad x(0) = 0. \tag{15}$$

Here, solution means the usual concept, namely x is required to be (absolutely) continuous, thus almost everywhere differentiable and

$$\dot{x}(t) = -\text{ sgn } x(t) = -x(t)/|x(t)|$$

for almost every t in an interval containing 0. Indeed, if $x(t^*) > 0$ at some $t^* > 0$, then near t^* we must have, from the differential equation, $\dot{x}(t) = -1$ hence

$$x(t) = \text{ constant } - t,$$

and similarly if $x(t^*) < 0$ then near t^*

$$x(t) = \text{ constant } + t,$$

thereby forbidding existence.

This means that the sliding mode control of the previous section cannot be interpreted in the standard sense. A different concept of solution to discontinuous differential equations was developed (nearly forty years ago by the Russian mathematician A. F. Filippov) in order to establish a mathematically rigorous theory of sliding mode control and discontinuous ordinary differential equations. In order to modify the concept of solution to (15) we enlarge the right-hand side f at its unique discontinuity point 0, by taking into account the behavior of f at $x \neq 0$. We want to obtain more room at 0 for the admissible values of $\dot{x}(t)$ by replacing the single - valued function $f : R \to R$ by a set - valued function F, in the following way.

If $x = 0$ we consider the set of all values of $f(y)$ for y in a neighborhood of 0, hence obtaining the set $\{-1, 1\}$. Then we take its closed convex hull, namely the smallest closed convex set containing $\{-1, 1\}$, hence obtaining the whole interval $[-1, 1]$, and finally we take the intersection of such sets over all neighborhoods of 0 (which in this case has no effect). We repeat the same construction at every x. The final result will be $F(x)$. Hence

$$F(0) = [-1, 1], \; F(x) = \{f(x)\} \text{ if } x \neq 0$$

since f is continuous at every $x \neq 0$.

Then we modify the solution concept by defining a (Filippov) solution y to (15) as a function which satisfies the initial value problem

$$\dot{y}(t) \in F[y(t)], \; y(0) = 0$$

for the differential inclusion

$$\dot{y}(t) \in F[y(t)]$$

which replaces the original differential equation we started from. In this way we see that the (constant) solution $y(t) = 0$ for every t has been found.

The analogous procedure is applied to control systems

$$\dot{x} = f(t, x, u^*)$$

with a discontinuous feedback u^*, and with this definition we get a coherent theory, which gives a mathematically rigorous way to properly handle sliding mode control problems.

This is an interesting case, where a rather simple concept (sliding mode control) invented by control engineers in order to solve problems posed by technology, required developing a new (forty years ago) mathematical theory, namely differential inclusions, to treat in a mathematically sound way such problems: in a sense, a nonstandard mathematical concept for a (rather simple) physically motivated idea.

However, a discontinuous control law can often be avoided as far as the mathematical description of the sliding mode control is concerned. Indeed, if the sliding line is reached, then by (11)

$$0 = c\dot{x}_1 + \dot{x}_2.$$

By (6)

$$\dot{x}_1 = x_2, \dot{x}_2 = 2x_2 - 4x_1 + u,$$

hence

$$0 = cx_2 + 2x_2 - 4x_1 + u$$

whence

$$u = \bar{u} = 4x_1 - (c+2)x_2$$

is the control law we obtain from the dynamics of our control system, namely the so called equivalent control law: clearly a continuous function of x_1, x_2.

It can be proved that $\bar{u}$ induces the same motion as the discontinuous feedback u^* on the sliding line.

Both theories of differential inclusions and of sliding mode control have been greatly developed, sometimes independently to each other. It is of further interest to know that some more recent problems, again related to stabilization via discontinuous feedback, cannot be handled satisfactorily within this theory, and new concepts of solution to discontinuous feedback control systems, have been proposed quite recently.

Differential equations are used as a basic mathematical model of classical mechanics and physics. New technological issues raised by engineers interested in controlling dynamical systems led mathematicians to introduce new concepts and generalizations. In a sense, cross fertilization of theory and applications can be considered, not surprisingly, quite effective, as we know from several points of the history of the development of science and mathematics.

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