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On the Dynamism of Harvesting Biological Resources

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ABSTRACT. The dynamic harvest of a biological population is examined under different growth rates and different harvesting communities. After the general model is formulated, the number of equilibria and the long term behavior of the state trajectory are discussed.

1 Introduction

In this paper, we will examine general models of harvesting biological resources. The models are based on an ordinary differential equation of the form

$$\dot{x} = F(x) - H(x) \tag{1}$$

where $x(t)$ denotes the size of the resource population, $F(x)$ is the natural growth rate, and $H(x)$ is the harvesting rate. We will examine the number of positive equilibria and the asymptotical behavior of the state trajectory $x(t)$ under different assumptions. Some special cases of this model have been earlier examined in Clark (1990), who also has discussed the optimal management of renewable resources. Special cases of model (1) have also been discussed for analyzing international fishing (Szidarovszky and Okuguchi, 1998, 2000). This paper develops as follows. The natural growth of biological resources will be first examined and then the optimal harvesting rate will be determined under various conditions. These two models will be then combined in model (1), and finally equilibrium and stability analysis will be performed.

2 Natural Growth

Assume first that in the population the birth rate b and mortality rate m are proportional to the population size x, then without harvesting, the population is driven by the differential equation

$$\dot{x} = xr \tag{2}$$

where $r = b - m$. It is easy to see that the solution of this equation is $x(t) = x(0)e^{rt}$ showing that $x(t)$ remains constant if $r = 0$, converges monotonically to zero if $r < 0$ and converges to infinity monotonically if $r > 0$. Exponential growth of a population assumes ideal conditions, however as the population increases, some environmental limitations will make the growth rate declining. In such cases, equation (2) modifies as

$$\dot{x} = r(x)x, \tag{3}$$

where $r(x)$ is a decreasing function of x. The most simple choice of $r(x)$ is linear:

$$r(x) = r\left(1 - \frac{x}{K}\right) \tag{4}$$

where r is called the intrinsic growth rate and K is the carrying capacity. In this case, $F(x) = r(x)x$ is a parabola as shown in Figure 1. Notice that if $0 < x < K$, then $\dot{x} > 0$, if $x = K$ then $\dot{x} = 0$, and if $x > K$, then $\dot{x} < 0$. Therefore the state trajectory $x(t)$ has the following simple property. If $x(0) < K$, then $x(t)$ is increasing and converges to K as $t \to \infty$, if $x(0) = K$, then $x(t)$ will remain constant for all future times, and if $x(0) > K$, then $x(t)$ decreases and also converges to K as t tends to infinity. This property is illustrated in Figure 2.

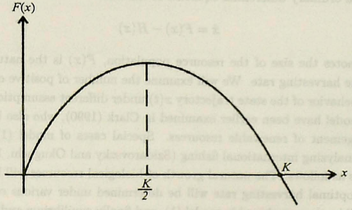

Figure 1: Linear proportional growth rate

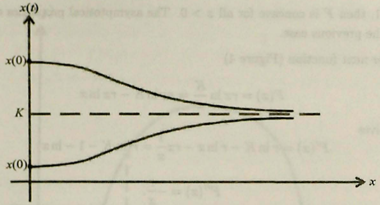

Figure 2: State trajectory

In many applications equation (4) is modified in order to obtain a nonsymmetric shape of $F(x)$. Such modifications are next introduced.

Consider first the function

$$F(x) = rx^{\alpha}\left(1 - \frac{x}{K}\right) \tag{5}$$

with some constant $\alpha > 0$ (Figure 3). Notice that the case of $\alpha = 1$ corresponds to equation (4). Since

$$F'(x) = r\alpha x^{\alpha-1}\left(1 - \frac{x}{K}\right) + rx^{\alpha}\left(-\frac{1}{K}\right) = r\alpha x^{\alpha-1} - \frac{rx^{\alpha}}{K}(\alpha+1)$$

and

$$F''(x) = r\alpha(\alpha-1)x^{\alpha-2} - \frac{r\alpha x^{\alpha-1}}{K}(\alpha+1) = r\alpha x^{\alpha-2}\left(\alpha - 1 - \frac{x(\alpha+1)}{K}\right)$$

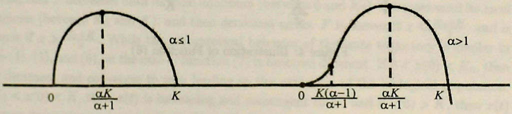

Figure 3: Illustration of Function (5)

it is easy to see the F increases if $x < \frac{\alpha K}{\alpha+1}$, decreases if $x > \frac{\alpha K}{\alpha+1}$ and it has the maximum point at $x = \frac{\alpha K}{\alpha+1}$. In addition, F is convex if $x < \frac{K(\alpha-1)}{\alpha+1}$, it is concave if $x > \frac{K(\alpha-1)}{\alpha+1}$. Notice that if $\alpha \leq 1$, then F is concave for all $x > 0$. The asymptotical properties of $x(t)$ are the same as in the previous case.

Consider next function (Figure 4)

$$F(x) = rx \ln \frac{K}{x} = rx \ln K - rx \ln x \tag{6}$$

with derivatives

$$F'(x) = r \ln K - r \ln x - rx\frac{1}{x} = r(\ln K - 1 - \ln x)$$

and

$$F''(x) = -\frac{r}{x}.$$

Therefore F is concave for all $x > 0$, it increases at $x < e^{\ln K - 1} = \frac{K}{e}$, it decreases for $x > \frac{K}{e}$, and at $x = \frac{K}{e}$ it has its unique maximum. The asymptotic behavior of $x(t)$ is the same as in the previous cases.

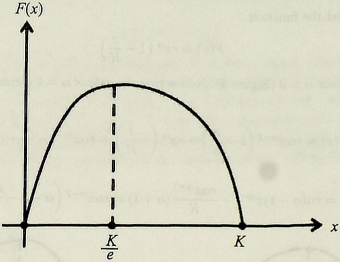

Figure 4: Illustration of Function (6)

More complicated model can be obtained by selecting

$$F(x) = rx\left(\frac{x}{K_0} - 1\right)\left(1 - \frac{x}{K}\right) \tag{7}$$

where $0 < K_0 < K$. This function is cubic with roots 0, K_0, and K. Notice that if $0 < x < K_0$ then $F(x) < 0$, for $K_0 < x < K$ function value $F(x)$ is positive, and if $x > K$,

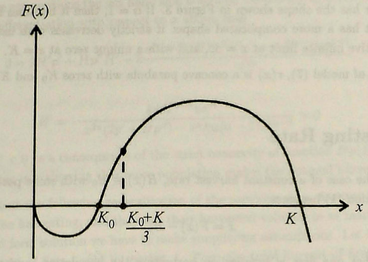

Figure 5: Cubic case of $F(x)$

then $F(x)$ becomes negative again. This case is shown in Figure 5. Notice that

$$F(x) = -\frac{rx^3}{KK_0} + rx^2\left(\frac{1}{K_0} + \frac{1}{K}\right) - rx$$

so

$$F'(x) = -\frac{3rx^2}{KK_0} + 2r\left(\frac{1}{K_0} + \frac{1}{K}\right)x - r$$

and

$$F''(x) = -\frac{6rx}{KK_0} + 2r\left(\frac{1}{K_0} + \frac{1}{K}\right)$$

showing that F decreases until its local minimum (between 0 and K_0), increases until its local maximum (between K_0 and K), and then decreases again. F is convex if $x < \frac{K_0+K}{3}$, and is concave if $x > \frac{K_0+K}{3}$. While the asymptotical behavior of the state trajectory is similar in cases (4), (5), and (6), in the case of function (7) it becomes different. If $0 < x(0) < K_0$, then $x(t)$ decreases and converges to zero leading to the extension of the biological population. If $K_0 < x(0) < K$, then $x(t)$ is increasing and convergent to K, and if $x(0) > K$, then $x(t)$ becomes decreasing and converges again to K. If $x(0) = K_0$ or $x(0) = K$, then $x(t)$ remains constant.

For further use we examine next function $r(x)$ introduced earlier in equation (3). In model (4) it is strictly decreasing and linear. In the case of model (5),

$$r(x) = rx^{\alpha-1}\left(1 - \frac{x}{K}\right).$$

If $\alpha > 1$, then it has the shape shown in Figure 3. If $\alpha = 1$, then it becomes linear. In the case of $\alpha < 1$ it has a more complicated shape: it strictly decreases with infinite limit at $x = 0$ and negative infinite limit at $x = \infty$, and with a unique zero at $x = K$.

In the case of model (7), $r(x)$ is a concave parabola with zeros K_0 and K.

3 Harvesting Rate

Consider first the case of a constant harvest rate, $H(x) \equiv H_0$ with some positive constant H_0. Then equation (1) becomes

$$\dot{x} = F(x) - H_0. \tag{8}$$

Assume next that one firm is involved in harvesting the population, and the harvested amount is sold in a market with known inverse demand function $p(H)$. Let $C(x, H)$ denote the harvesting cost. It is usually assumed that p is strictly decreasing in H, and $Hp(H)$ is strictly concave and converges to zero as $H \to \infty$. A particular form of the cost function can be derived as follows. Assume that the harvest rate is a function of the existing population and the harvesting effort. Using the form of production functions we assume that

$$H = \alpha x^{\beta} E^{\gamma}, \tag{9}$$

where E is the harvesting effort. From (9) we have

$$E = \alpha^{-\frac{1}{\gamma}} H^{\frac{1}{\gamma}} x^{-\frac{\beta}{\gamma}},$$

and if C_0 denotes the unit cost of effort, then the harvesting cost is obtained as

$$C(x, H) = C_0 \alpha^{-\frac{1}{\gamma}} \frac{H^{\frac{1}{\gamma}}}{x^{\frac{\beta}{\gamma}}} = k \frac{H^u}{x^v}$$

with k, u, and v being positive constants. It is usually assumed that $u \geq 1$. Then at each time period the profit of the firm is given as

$$\pi = Hp(H) - k\frac{H^u}{x^v} \tag{10}$$

which is strictly concave in H, so there is a unique profit maximizing harvest rate for all $x > 0$, which is denotes as $H = H(x)$. Assuming interior optimum (otherwise no harvesting takes place) simple differentiation shows that at the optimum harvest rate

$$p(H) + Hp'(H) - k\frac{uH^{u-1}}{x^v} = 0.$$

Under the above assumptions the left hand side is strictly decreasing in H, so H is a function of x. Implicit differentiation with respect to x shows that

$$0 = 2H'p' + Hp''H' - k\frac{u(u-1)H^{u-2}H'x^v - uH^{u-1}vx^{v-1}}{x^{2v}}$$

implying that

$$H' = -\frac{kuH^{u-1}vx^{v-1}}{x^{2v}(2p' + Hp'') - x^v ku(u-1)H^{u-2}} > 0$$

since $2p' + Hp'' < 0$ as a consequence of the strict concavity of function $Hp(H)$. Thus H is strictly increasing in x showing that more population makes the optimal harvest rate larger.

Consider next the following generalization of the previous case. Assume that there are n firms doing the harvesting, and they sell their harvested volumes in m markets. In order to obtain closed form solution we have to make simplifying assumptions. Let h_{ij} denote the amount of harvest of firm i sold in market j. Then the total harvest of firm i is given as $H_i = \sum_{j=1}^{m} h_{ij}$ and the supply in market j is $S_j = \sum_{i=1}^{n} h_{ij}$. Assume that the inverse demand function of market j is linear: $P_j(S_j) = a_j - b_j S_j$ where a_j and b_j are positive constants. Assume in addition that in the cost function of firm $i, u_i = 2$ and $v_i = 1$ (which is the usual assumption in examining commercial fishing). Then similarly to Szidarovszky and Okuguchi (1998, 2000) we can easily verify that the total harvest rate is given as

$$H(x) = \frac{Af(x)}{1 + f(x)} \tag{11}$$

with

$$f(x) = \sum_{i=1}^{n} \frac{1}{1 + 2B\frac{k_i}{x}}$$

where

$$A = \sum_{j=1}^{m} \frac{a_j}{b_j} \quad \text{and} \quad B = \sum_{j=1}^{m} \frac{1}{b_j}$$

if we assume that the firms are competitive and at each time period a Cournot-Nash equilibrium determines the harvest rate. If in contrary we assume that the firms are cooperative and they maximize their total profit at each time period, then

$$H(x) = \frac{ACx}{2(Cx + B)} \tag{12}$$

with

$$C = \sum_{i=1}^{n} \frac{1}{k_i}.$$

It is easy to see that in both cases (11) and (12), the harvest rate increases in x. In the case of equation (11) notice that $f(x)$ increases in x, since each term is increasing. Furthermore

$$H(x) = A - \frac{A}{1 + f(x)}$$

which shows that $H(x)$ is strictly increasing in x. In the case of equation (12) we notice that

$$H(x) = \frac{A}{2} - \frac{AB}{2(Cx + B)},$$

which is clearly strictly increasing in x. A relatively simple but lengthy calculation shows that in both cases $H(x)/x$ is strictly decreasing in x and is strictly convex, furthermore it converges to zero as $x \to \infty$.

4 Equilibrium and Stability Analysis

Consider first the case of constant harvest H_0. Let the maximal value of $F(x)$ be denoted by F_0. If $H_0 > F_0$, then there is no equilibrium in system (1), and since the right hand side is always negative, $x(t)$ always decreases and converges to zero. Thus in this case the biological population will vanish in the long run. Assume next that $H_0 = F_0 = F(x^*)$, where x^* is the maximizer of $F(x)$. Then x^* is the unique equilibrium. If $x(0) < x^*$, then $x(t)$ is decreasing and converges to zero, and if $x(0) > x^*$, then $x(t)$ also decreases and converges now to x^*. If $x(0) = x^*$, then $x(t) \equiv x^*$ for all $t \geq 0$. Assume finally that $H_0 < F_0$. Then there are two values x_1^* and x_2^* such that $0 < x_1^* < x_2^*$ and $F(x_1^*) = F(x_2^*) = H_0$. These values x_1^* and x_2^* are the equilibria of system (1). If $x(0) < x_1^*$, then $x(t)$ is decreasing and converges to zero. If $x_1^* < x(0) < x_2^*$, then $x(t)$ increases and converges to x_2^*, and if $x(0) > x_2^*$, then $x(t)$ decreases again and converges to x_2^* as $t \to \infty$. So small initial biological population cannot survive, but if it is larger than the smaller positive equilibrium, then it always converges to the larger equilibrium. In such cases the population is stable.

In more general cases equation (1) can be rewritten as

$$\dot{x} = x(r(x) - h(x)) \tag{13}$$

with $h(x) = H(x)/x$. Depending on the analytical properties of functions r and h, we might have a variable number of positive equilibria. For example, if both r and h are decreasing and convex (as in some previously discussed cases) we might have many equilibria, there is even the possibility of infinitely many equilibria. However if h is strictly decreasing and convex, and r is decreasing linear, or concave parabola or even having shape shown in Figure 3, then there are at most two equilibria, and the stability of the state trajectory is very similar to that of the constant harvest case.

Assume first that in the general case no equilibrium exists. Since $H(0) = 0$ and strictly increasing, in this case $H(x) > F(x)$ for all $x > 0$. Then $x(t)$ always strictly decreases and converges to zero as $t \to \infty$. Assume next that the set of positive equilibria is not empty. Notice that zero is always an equilibrium. $x(0)$ is larger than the largest equilibrium, then $\dot{x} < 0$, so $x(t)$ converges monotonically to the largest equilibrium (which is always below K). Assume next that $x(0)$ is between two equilibria $x_i^* < x_{i+1}^*$ and there is no further equilibrium between these points. If $F(x(0)) - H(x(0)) > 0$, then $x(t) \to x_{i+1}^*$ and if $F(x(0)) - H(x(0)) < 0$, then $x(t) \to x_i^*$ and in both cases the convergence is monotonic. If $F(x(0)) - H(x(0)) = 0$, then $x(t)$ remains constant for all future times.

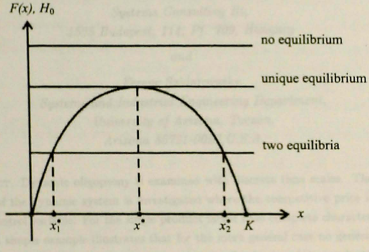

Figure 6: Cases of constant harvest rate

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[2] Szidarovszky, F. and Okuguchi, K., *An Oligopoly Model of Commercial Fishing*, Seoul Journal of Economics, **11** (3) (1998), 321–330.

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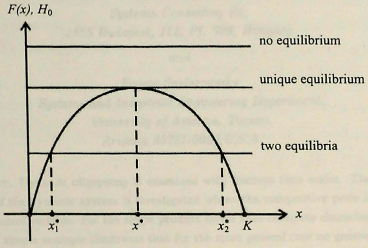

Figure 6: Cases of constant harvest rate

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