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About Models of Ferromagnetic Hysteresis

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ABSTRACT

The evolution of ferromagnetic systems can be described by coupling the Maxwell equations with suitable constitutive relations. These relations can be established for nondistributed systems, but the formulation of a P.D.E. model raises nontrivial questions. Macroscopic and a mesoscopic models of ferromagnetism are here reviewed.

1 Maxwell Equations

This paper discusses the mathematical modelling of electromagnetic evolution of either ferromagnetic or ferrimagnetic materials which exhibit hysteresis. It intends to address a composite audience which might include physicists, mathematical-physicists, and electric engineers. Accordingly, an effort is made to use a broadly accessible mathematical language.

We say that a system is *distributed in space* whenever the relevant state variables depend on the space point. This is the domain of partial differential equations, for instance. On the other hand, we say that a system is *nondistributed in space* if all the relevant variables do not depend on space.

Let us consider an either ferromagnetic or ferrimagnetic material which occupies a region Ω in a time interval $]0,T[$. In Gauss units the magnetic field, $\vec{H}$, the magnetization, $\vec{M}$, and the magnetic induction, $\vec{B}$, are related by the condition

$$\vec{B} = \vec{H} + 4\pi\vec{M} \qquad \text{in } Q_\infty := \mathbb{R}^3\times]0,T[, \tag{1.1}$$

that is, in the whole physical space, for a time interval $]0,T[$ $(T > 0)$. We denote the electric field by $\vec{E}$, the electric displacement by $\vec{D}$, the electric current density by $\vec{J}$, the electric charge density by $\hat{\rho}$, and the speed of light in vacuum by c. With this notation, the Maxwell equations read

$$c\nabla \times \vec{H} = 4\pi\vec{J} + \frac{\partial\vec{D}}{\partial t} \qquad \text{in } Q_\infty \ (\nabla\times := \text{curl}), \tag{1.2}$$

$$c\nabla \times \vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \text{in } Q_\infty, \tag{1.3}$$

$$\nabla\cdot\vec{B} = 0 \qquad \text{in } Q_\infty \ (\nabla\cdot := \text{div}), \tag{1.4}$$

$$\nabla\cdot\vec{D} = 4\pi\hat{\rho} \qquad \text{in } Q_\infty. \tag{1.5}$$

These laws must be coupled with appropriate constitutive relations.

Let us denote the electric conductivity by σ and the dielectric permeability by ϵ. We assume that the material is homogeneous and isotropic; σ and ϵ are then constant scalars. We assume Ohm's law in Ω and insulation conditions outside, namely,

$$\begin{array}{rcll} \vec{J} & = & \sigma(\vec{E}+\vec{g}) & \text{in } Q := \Omega\times]0,T[, \\ \vec{J} & = & \vec{0} & \text{in } (\mathbb{R}^3 \setminus \Omega)\times]0,T[. \end{array} \tag{1.6}$$

Here $\vec{g}$ represents an applied electromotive force. We also assume that

$$\vec{D} = \epsilon\vec{E} \qquad \text{in } Q_\infty. \tag{1.7}$$

Appropriate initial and boundary conditions must then be appended to the above equations.

By applying the operator $c\nabla\times$ to (1.2) and $\epsilon\frac{\partial}{\partial t}$ to (1.3), using $(1.6)_1$ and (1.7), we get the following equation in the magnetic variables $\vec{H}$ and $\vec{B}$:

$$\epsilon\frac{\partial^2\vec{B}}{\partial t^2} + 4\pi\sigma\frac{\partial\vec{B}}{\partial t} + c^2\nabla\times\nabla\times\vec{H} = 4\pi c\sigma\nabla\times\vec{g} \qquad \text{in } Q. \tag{1.8}$$

If our material is a metal (as all ferromagnetic materials are), then $\epsilon = 1$ (in Gauss units); hence $c^2 \gg \epsilon$ and $4\pi\sigma \gg \sqrt{\epsilon}$. Therefore, whenever the field $\vec{B}$ does not vary too rapidly, $\epsilon\frac{\partial^2\vec{B}}{\partial t^2}$ is negligible in comparison with $4\pi\sigma\frac{\partial\vec{B}}{\partial t}$. This yields the so-called *eddy current approximation:*

$$4\pi\sigma\frac{\partial\vec{B}}{\partial t} + c^2\nabla\times\nabla\times\vec{H} = 4\pi c\sigma\nabla\times\vec{g} \qquad \text{in } Q. \tag{1.9}$$

Obviously, this is tantamount to dropping the displacement current $\frac{\partial \vec{D}}{\partial t}$ in (1.2). (For the nonlinear problem, a rigorous analytic justification of this reduction does not look obvious.)

On the other hand, if our material is an insulator (as several ferrimagnetic materials are), then $\sigma = 0$ and (1.8) reads

$$\epsilon \frac{\partial^2 \vec{B}}{\partial t^2} + c^2 \nabla \times \nabla \times \vec{H} = \vec{0} \qquad \text{in } Q. \tag{1.10}$$

We then need a further constitutive law, in order to represent hysteresis in the M vs. H behaviour.

2 Non-Space-Distributed Constitutive Law

Let us review a classic measurement procedure. Let us consider a homogeneous isotropic toroidal specimen of our magnetic material, wound it uniformly with an electrically conducting coil, and let a current flow through the latter by connecting it to a battery. By Ampère's law (1.2), this induces a magnetic field having uniform intensity, H, along the axis of the torus. Because of symmetry, this determines a parallel magnetization field having uniform intensity, M, which can be measured by means of a probe.

We represent the M vs. H relation in the form $\vec{M} = \mathcal{F}(\vec{H})$. On account of the symmetry, here we have no space dependence (in other terms, we deal with a space-non-distributed system). We assume $\mathcal{F}$ to be a *hysteresis operator;* by this we mean that

(i) $\mathcal{F}$ contains *memory.* This means that at each instant t, $\vec{M}(t)$ depends not only on $\vec{H}(t)$ but also on $\vec{H}(\cdot)$ (i.e., on the history of $\vec{H}$ in the time interval $[0, t]$), and on the initial value $\vec{M}(0)$. (In the formula $\vec{M} = \mathcal{F}(\vec{H})$ the latter dependence is not displayed.) In more refined models the initial value includes internal variables, too.

(ii) $\mathcal{F}$ is *rate-independent.* This means that the path of the pair $(\vec{H}(t), \vec{M}(t))$ is invariant w.r.t. any increasing homeomorphism $\varphi : [0, T] \to [0, T]$, that is,

$$\mathcal{F}(\vec{H} \circ \varphi) = \mathcal{F}(\vec{H}) \circ \varphi \qquad \text{in } [0, T]. \tag{2.1}$$

In other terms, if $\mathcal{F}$ maps $\vec{H}$ to $\vec{M}$, then it maps $\vec{H} \circ \varphi$ to $\vec{M} \circ \varphi$. In particular, if the function $\vec{H}$ is periodic, then the $\vec{M}$ vs. $\vec{H}$ relation does not depend on the frequency.

The latter property is regarded as characteristic of hysteresis, and is fulfilled within a good degree of approximation whenever the rate of $\vec{H}$ is not too large. For high frequencies the relaxation dynamics should be included.

Scalar hysteresis operators have been extensively studied in the last thirty years; see e.g. [1,3,6,7,9,11,12]. Several results are known for continuous hysteresis operators. Discontinuous operators have also been studied, and coupled with P.D.E.s.; in several cases this corresponds to so-called *free boundary problems*.

For one-dimensional systems, $\mathcal{F}$ may represent either the *Preisach model* [10], or one of it many generalizations, or another hysteresis model. In the scalar setting, the operator $\mathcal{F}$ must also account for the dynamics in the interior of the region bounded by the main hysteresis loop, cf. Fig. 1.

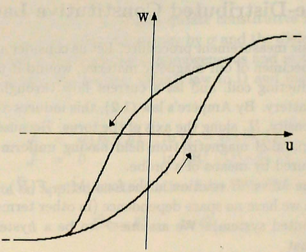

Figure 1. Hysteresis dynamics, for a univariate system.

For univariate systems, the operator $\mathcal{F}$ can be identified by means of a series of tests, in which a suitable set of *input* functions $H(\cdot)$ is applied, and the corresponding *output* functions $M(\cdot)$ are measured; see e.g. [2,5]. In the vector setting the number of measurements to be performed for direct identification would be very large. It seems then convenient to devise a vector model which is strictly related to a scalar one, and to identify the latter.

Analogous situations are met in ferroelectricity, in elasto-plasticity, in pseudo-elasticity, and in a number of other phenomena which also exhibit hysteresis.

3 Space-Distributed Constitutive Law

Now we go back to our *space-distributed* system. For the sake of simplicity, here we just deal with equations in Ω, assuming that the relevant boundary conditions are known. However, we must point out that it is not easy to formulate physically appropriate boundary conditions. A more sound model would then be obtained by prescribing $\vec{M} = \vec{0}$ outside Ω, and studying the problem in the whole space. In any case, our discussion also applies to the latter setting.

As a first guess, we insert the space variable, x, as a parameter in the constitutive law, and write

$$\vec{M}(x,t) = [\mathcal{F}(\vec{H}(x,\cdot))](t) \qquad \text{in } Q, \tag{3.1}$$

We then couple this constitutive relation with the P.D.E. (1.9),

$$4\pi\sigma\frac{\partial}{\partial t}(\vec{H} + 4\pi\vec{M}) + c^2\nabla \times \nabla \times \vec{H} = 4\pi c\sigma\nabla \times \vec{g} \qquad \text{in } Q, \tag{3.2}$$

and with suitable initial and boundary conditions.

In univariate systems, this problem is mathematically *well-behaved.* For several choices of the operator $\mathcal{F}$, including the Preisach model, there exists a unique solution; this depends continuously on the data, and can efficiently be approximated; see e.g. [12; Chap. XI]. A natural vector extension of the Preisach model has been proposed in [4]; see also [8.9]. This is based on a rather simple idea: for any unit vector $\vec{\theta}$, the input $\vec{H}$ is projected along $\vec{\theta}$; a prescribed scalar hysteresis operator ($\mathcal{F}_{\vec{\theta}}$) is then applied to this projection, and is multiplied by the unit vector $\vec{\theta}$. Finally, as $\vec{\theta}$ varies in the unit sphere (S^2), all these vectors are averaged w.r.t. a prescribed finite Borel measure, μ. In formula, we have

$$\vec{H} \to \vec{H}\cdot\vec{\theta} \to \mathcal{F}_{\vec{\theta}}(\vec{H}\cdot\vec{\theta})\vec{\theta} \to \int_{S^2} \mathcal{F}_{\vec{\theta}}(\vec{H}\cdot\vec{\theta})\vec{\theta}\, d\mu(\vec{\theta}).$$

Although this simple model does not account for all of the complexity of magnetic hysteresis, its qualitative agreement with experiments seems satisfactory, and several engineers regard it as useful, cf. [8,9].

Monotonicity. At this point, a further difficulty arises: the structure of the P.D.E. (1.9) is such that we are not able to exclude the onset of rapid space-oscillations in the field $\vec{H}$, as we have no control on $\nabla\cdot\vec{H}$. (On the other hand, under natural assumptions, the energy estimate can be derived, and this provides an estimate for $\int_0^T dt \int_{\mathbb{R}^3} |\nabla\times\vec{H}|^2\, dx$).

In order to give the reader an idea of the difficulties that arise in the analysis of (1.9), we compare the latter with the analogous equation in which the hysteresis operator is replaced by a superposition operator. In this case the latter difficulty is overcome; some mathematical technicalities are needed to illustrate this issue. Let us assume that

$$\vec{M}(x,t) = \vec{G}(\vec{H}(x,t)) \qquad \text{in } Q,$$

with $\vec{G}$ a prescribed *monotone* vector function, i.e.,

$$[\vec{G}(\vec{v}_1) - \vec{G}(\vec{v}_2)] \cdot [\vec{v}_1 - \vec{v}_2] \geq 0 \qquad \forall \vec{v}_1, \vec{v}_2 \in \mathbb{R}^3. \tag{3.3}$$

For reasons we are not going to explain here, in this case the main issue is the stability of the product $\iint_Q \vec{M} \cdot \vec{H}\, dxdt$ with respect to weak L^2-convergence of both factors. By (1.1), we then have

$$\iint_Q \vec{M} \cdot \vec{H}\, dxdt = \iint_Q \vec{B} \cdot \vec{H}\, dxdt - 4\pi \iint_Q |\vec{H}|^2 dxdt.$$

As $\nabla \cdot \vec{B} = 0$ in Q, a *compensated compactness* argument yields the stability of the first integral; the second one is obviously weakly lower semicontinuous in $L^2(\Omega; \mathbb{R}^3)$. This yields the desired stability property.

This argument however does not take over to hysteresis constitutive laws, for only trivial hysteresis operators are monotone in the sense of (3.3). This is illustrated by the following scalar counterexample, which can easily be extended to nondegenerate hysteresis operators. Let $\mathcal{F} : W^{1,1}(0,T) \times \mathbb{R} : (u, w^0) \mapsto w$ be defined by the following Cauchy problem

$$\frac{dw}{dt} = \left(\frac{du}{dt}\right)^+ \qquad \text{in }]0,T[, \qquad w(0) = w^0.$$

This is a hysteresis operator: causality and rate-independence are straightforward. (In this case, the pair (u, w) cannot move along any closed hysteresis loop; but this is immaterial.) Let us fix any $T > 3\pi/2$, and take

$$\begin{array}{llll} u_1(t) := \sin t & \text{in } [0, 3\pi/2], & u_1(t) := -1 & \text{in } [3\pi/2, T], \\ u_2 := 0 & \text{in } [0,T]. & & \end{array}$$

Setting $w_i := \mathcal{F}(u_i, 0)$ for $i = 1, 2$, we have $(w_1 - w_2)(u_1 - u_2) = -1$ for $t \geq \frac{3}{2}\pi$. Hence

$$\int_0^T (w_1 - w_2)(u_1 - u_2) dt < 0 \qquad \text{if } T \text{ is large enough},$$

that is, the monotonicity fails.

The folllowing weaker monotonicity-type property applies to a large class of *scalar* hysteresis operators, and is especially convenient for the analysis of several P.D.E.s:

$$\begin{aligned}&\forall(u, w^0) \in \mathrm{Dom}(\mathcal{F}), \forall[t_1, t_2] \subset [0, T],\\ &\text{if } u \text{ is nondecreasing (nonincreasing, resp.) in } [t_1, t_2],\\ &\text{then } \mathcal{F}(u, w^0) \text{is also nondecreasing (nonincreasing, resp.) in } [t_1, t_2].\end{aligned} \tag{3.4}$$

This means that *hysteresis branches* are nondecreasing, and entails that

$$\forall(u, w^0) \in \mathrm{Dom}(\mathcal{F}) \text{ such that } \; u, w := \mathcal{F}(u, w^0) \in W^{1,1}(0, T),$$

$$\frac{dw}{dt}\frac{du}{dt} \geq 0 \quad \text{a.e. in }]0, T[.$$

Despite of these difficulties, some progresses have been achieved in the analysis of the system (3.1) and (3.2). Existence of a solution has been proved for a corresponding initial- and boundary-value problem, through a suitable weak formulation of the vector Preisach hysteresis operator, and by using a homogenization technique known as *two-scale convergence;* see [16].

4 A Mesoscopic Model

A possible reason of the above difficulties stays in the fact that the hysteresis relation (3.1) does not account for space interaction in the $\vec{M}$ vs. $\vec{H}$ constitutive law. On a mesoscopic length-scale, the theory known as *micromagnetics* accounts for this interaction. In particular, the initial- and boundary-value problem obtained by coupling a classic equation due to Landau and Lifshitz with the system of the Maxwell equations and with the Ohm law has a solution, which can also be approximated; see e.g. [13]. The Landau-Lifshitz equation reads

$$\begin{cases} \dfrac{\partial \vec{M}}{\partial t} = \lambda_1 \vec{M} \times \vec{H}^e - \lambda_2 \vec{M} \times (\vec{M} \times \vec{H}^e), \\ \vec{H}^e := \Delta \vec{M} - A \cdot \vec{M} + \vec{H}; \end{cases} \tag{4.1}$$

λ_1 and λ_2 are constants, $\lambda_2 > 0$ (typically $\lambda_1 > \lambda_2$, in some cases $\lambda_1 \gg \lambda_2$), A is a positive-definite symmetric tensor. Equation $(4.1)_1$ represents a natural relaxation dynamics for a magnetic moment of constant modulus, which is under the action of the *effective magnetic field* $\vec{H}^e$. The vector $-\vec{M} \times (\vec{M} \times \vec{H}^e)$ is the projection of the *effective magnetic field* $\vec{H}^e$ onto the tangent plane at $\vec{M}$ to the sphere with center $\vec{0}$ and radius $\mathcal{M}$. By (4.1) this term drives $\vec{M}$ to move towards $\vec{H}^e$ and is dissipative. The vector $\vec{M} \times \vec{H}^e$ stays in the same tangent plane, and is orthogonal to $\vec{M} \times (\vec{M} \times \vec{H}^e)$. It drives $\vec{M}$ to rotate around $\vec{H}^e$ by forming a constant angle *(precession motion)* with angular velocity proportional to $|\vec{H}^e|$, and is not dissipative. As a result of the composition of these two forces, $\vec{M}$ asymptotically converges to $\vec{H}^e$ along a nonplanar spiral on the sphere of radius $\mathcal{M}$. The relaxation time is proportional to λ_2^{-1}.

The dynamics (4.1) can equivalently be expressed by the *Gilbert equation*

$$\frac{\partial \vec{M}}{\partial t} = \mu_1 \vec{M} \times \left(\vec{H}^e - \frac{\mu_2}{\mu_1} \frac{\partial \vec{M}}{\partial t} \right) \quad \text{in } Q. \tag{4.2}$$

Indeed, multiplying (4.1) vectorially by $\vec{M}$ and eliminating $\vec{M} \times (\vec{M} \times \vec{H}^e)$, we get (4.2). Conversely, multiplying (4.2) vectorially by $\vec{M}$ and eliminating $\vec{M} \times \frac{\partial \vec{M}}{\partial t}$, we get (4.1). This calculation shows that the two pairs of constants (λ_1, λ_2) and (μ_1, μ_2) are related by the following transformation formulae:

$$\begin{cases} \lambda_1 = \dfrac{\mu_1}{1 + \mu_2^2 \mathcal{M}^2}, \\ \lambda_2 = \dfrac{\mu_1 \mu_2}{1 + \mu_2^2 \mathcal{M}^2} \end{cases} \quad \text{or equivalently} \quad \begin{cases} \mu_1 = \dfrac{\lambda_1^2 + \lambda_2^2 \mathcal{M}^2}{\lambda_1}, \\ \mu_2 = \dfrac{\lambda_2}{\lambda_1} \end{cases} \tag{4.3}$$

This model does not account for rate-independence in the $\vec{M}$ vs. $\vec{H}$ constitutive law. However, a suitable modification fulfils that property, cf. [14].

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