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The Mathematics of Financial Risk Management

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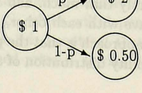
"The same equations have the same solutions"
R. Feynman

1 Financial markets

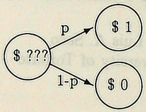
Although one can find examples of financial markets going back to the middle ages, organized financial markets became common in the second half of the 19'th century. One of the most representative cases was the bond issue that took place in 1857, to finance the construction of the railroad system in the United States. It is important to point out that the original needs that financial markets were destined to address were: the financing of enterprises, and the uncertainty of future events.

Up until recently, financial trading consisted mainly of two aspects: trading of *underlying* financial instruments (currencies, stocks and bonds), and the trading of their futures contracts. A futures contract is a transaction by which two counterparties agree on a future purchase at a pre-determined price. As will become more clear later in this article, trading of both types has a linear dependence on market events, and therefore the risk it entails is rather limited. In the seventies, new types of financial products were developed and traded in large volumes: the *derivatives.* Here is a simple example of a derivative:

Imagine a certain stock trades today at \$1 per share, and its future value (after one year, for example) can only be one of the following two values: \$2, or \$0.50. The following graph summarizes this situation.



Two individuals, A and B, agree to enter into the following transaction: A pays B \$1 if the stock increases in value, and nothing if the stock price decreases in value. The only thing left to do is for the two of them to agree on the price. This financial contract can be summarized by the following graph:



From a certain point of view, one might try to model the problem as follows: call p the probability the stock increases in value, and $q = 1 - p$ the probability it decreases in value. Therefore, it seems that the fair price for the contract would be \$p. We will see below that this is not quite correct, but for the time being the reader should realize that the price fixed in this manner is based on the participants' perception on the likelyhood of future events, and therefore it is not unlike gambling in a casino.

To understand another way to price the contract, consider the following possibility (we assume that interest rates are 0, to simplify the argument; otherwise, a small modification is needed): A sets the price at \$0.333 and B agrees; the transaction takes place, A undertakes obligations with B for a payment of \$1 if the stock value goes up, and B gives A \$0.1/3; at this point, A borrows an additional \$0.333 from a bank (interest free); with the money A charged B and borrowed from the bank, a total of \$0.666, A buys 2/3 of one unit of stock.

- If the stock increases in value, A will sell its 2/3 of stock for \$1.333; this is exactly what is needed for A to pay B what it owes (\$1), and to repay the loan (\$1/3).
- If the stock decreases in value, A pays B nothing, and selling 2/3 of stock gives A just enough money to return the value of the loan.

In other words, from a *replication* point of view, the fair value of the contract is \$0.333, because there is a trade (buying 2/3 of stock, and taking a loan of \$1/3) which costs exactly that much, and has the same cash flows as the contract that A and B have with each other, under each of the possible events. The advantage of this approach is that the price is determined without any assumption on the probability distribution of stock prices.

Although this is clearly a simple example, volumes of derivative have increased very rapidly throughout the 1990's, and they all share with our simple example the following properties:

- The contract has (or may have) a purchase price.
- Its future cash flows are linked to the behavior of the underlying (a stock, a bond, exchange rate, the price of a commodity, or many others).
- They offer the purchaser a certain type of insurance. In our example, individual B may have an interest in entering into the contract to protect himself against increasing costs linked to the increase of the value of the stock.
- They can also offer the purchaser the possibility of a speculative investment. In our example, if B simply buys the stock, and not the derivative, if the stock increases in price (by 100%), he will obtain a return of 100% also; if the stock drops by 50%, his losses wouls also be of 50%. However, by purchasing the derivative instead, an increase of the stock price by 100%, turns his investment of \$0.333 into \$1.00, that is, a 200% return on the investment. If the stock drops, he will lose his entire investment. This is what we made a reference to before as a *non-linear* investment.
- Issuing derivative contracts forces the issuer to seek replication (*hedging*) strategies, to minimize its risk.

2 Stochastic Calculus and the Black-Scholes formula

For the example presented in the previous section to be applicable to a wide variety of financial derivatives, one needs to extend the argument to take into account a continuum of future possible stock values, the possibility of trading at frequent intervals, and many different pay-off structures. These problems were tackled by a variety of authors, but the theory that has become most famous has been the one developed by Black, Merton and Sholes, which lead to the Nobel prize awarded to the two last ones (Fisher Black died a year prior to the award). In this section we present, in heuristic form, the basic arguments that lead up to the theory.

Let's begin with an example. Consider an underlying (a certain stock, for instance) with a value S, which will depend on time in some stochastic manner. A European call on S is a contract that allows the holder to purchase the stock at a predetermined price K (strike price), at a later point in time

T (expiration date). Equivalently, it will give the holder a payment (pay-off) $f_0(S)$ equal to

$$f_0(S) = (S-K)_+ = \begin{cases} S-K & \text{if} \quad S \geq K \\ 0 & \text{otherwise} \end{cases}$$

In the previous section we have computed the price for such an instrument in a simple case, when S can only take two values in the future, and $K = 1$. We also obtained a replicating (or *hedging*) strategy. We would like to extend this analysis.

The Black–Scholes analysis will allow us to create replicating portfolios. It will also allow us to price options more comlicated than the European (such as the American, which allow the hold to redeem the option at any time, before expiration).

To explain this analysis in its simplest case, consider a market in which the following three securities are available for trade:

- A riskless bond with constant interest rate r; its price at time t is given by
$$B(t) = e^{-r(T-t)}.$$
- A stock whose value evolves according to the stochastic differential equation
$$\frac{dS}{S} = \alpha\, dt + \nu\, dW, \tag{1}$$
where $\alpha > 0$ is the instantaneous rate of return on the stock per unit time, dW is the standard Wiener Process and $\nu \geq 0$ is the volatility.
- An option with payoff at maturity given by $f_0(S)$. We will denote its price by $f(S,t)$, which is determined by the price S of the stock at time t.

The derivative security can be priced by means of a replicating portfolio $\Pi(t)$ made up of $a(t)$ units of the underlying risky asset S as well as $b(t)$ pure discount riskless bonds. The composition of the replicating portfolio must be dynamically adjusted so that $a(t)$ and $b(t)$ are adapted processes with respect to $S(t)$ and the value of $\Pi(t)$ replicates that of the derivative securities whenever this one expires. One way to formulate the replication condition is to consider an investor that takes a short position in the derivative asset and implements a self-financing trading strategy by forming a portfolio $\Pi(t)$ which contains bonds and shares of the underlying risky asset.

Self-financing means that no money is put into the portfolio and no money is taken away from the portfolio after the creation of the portfolio. In other words, stock can only be purchased (or sold) by selling (or buying) bonds, and bonds can only be purchased (or sold) by selling (or buying) stock. In mathematical terms,

$$da\, S + db\, B = 0.$$

The value of the replicating portfolio $\Pi(t)$ is

$$\Pi(t) = a(t)S(t) + b(t)B(t),$$

and the replicating assumption implies that

$$\Pi(t) = f(S(t),t).$$

This equation determines the number of bonds; in fact

$$b(t) = \frac{f(S(t),t) - a(t)S(t)}{B(t)}.$$

The self-financing condition is

$$Sda + Bdb = 0.$$

Hence, we have that

$$\begin{aligned} d\Pi &= adS + Sda + bdB + Bdb \\ &= adS + bdB \\ &= dS + r(f - aS)\,dt \\ &= \Big[\alpha aS + r(f - aS)\Big]\,dt + a\sigma S\,dW. \end{aligned}$$

By using Ito's formula, we find

$$\begin{aligned} df &= \frac{\partial f}{\partial S}\,dS + \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right) dt \\ &= \frac{\partial f}{\partial S}\sigma S\,dW + \left(\frac{\partial f}{\partial t} + \alpha S\,\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right) dt. \end{aligned}$$

The hedging condition $df = d\Pi$ is therefore equivalent to

$$a(t) = \frac{\partial f}{\partial S}(S(t),t) \tag{2}$$

and

$$\frac{\partial f}{\partial t} = -\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - r\,S\,\frac{\partial f}{\partial S} + r\,f. \tag{3}$$

Equation (2) tells us the number of shares to be bought or sold at each time to replicate the price of the option. Equation (3) is a backwards difussion equation that the price of the option has to satisfy. When we couple (3) with the payoff at maturity, we end up with the backwards initial value problem

$$\begin{cases} \frac{\partial f}{\partial t} & = -\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - r\,S\,\frac{\partial f}{\partial S} + r\,f, \\ f(S,T) & = f_0(S). \end{cases} \tag{4}$$

This determines the value of the option at all times prior to maturity, for all values of the underlying stock.

One of the marvels of this theory is that the Black and Scholes equation is (in different units) the same as the heat equation, a well known partial equation that describes heat transmission in a homogenous body. Moreover, Einstein wrote a famous paper where he connected the Brownian theories with the heat equation, and Bachelier developed similar ideas in his thesis in 1900. Brownian theories borrow the name of Brown who created the concept of erratic motion when studing the movement of particles of dust, a concept refered to explicitly by Darwin in his theory of the evolution of the species. What we find there is that, the mathematical principles that lead to the Black-Scholes theory had already been developed about one hundred years ago, and they share the same mathematical foundation as the physics of heat transfer, dust motion, and share the same language as genetic biology.

3 Risk Management

It may appear, from the principles of hedging and replication explained above, that trading can always be done in a riskless manner, because one's positions can be replicated by trading in the market. However, this is only accurate taking into account a large set of assumptions; risk lurks in several places: in the validity of the gaussian assumption, in the assumption that trading will be done continuously and without transaction charges, in the assumption that the counterparty of the contract will always honor their obligations, in the assumption that securities are always available for trade, and many others. Furthermore, one might think that the previous long list of sources of risk leads makes the mathematical theory presented above nothing else than a theoretical excercise. The truth is somewhere in-between: the mathematical theory of finance is a superb tool which enables businesses to trade away their risks, but there is a residual source of risk which it does not eliminate and must be monitored and managed as a separate process.

In 1991, the Bank for International Settlements held a meeting in Basle, where it came up with a number of recommendations for the risk management of banks throughout the world. One of the recommendations is that banks should calculate a number, called Value-at-Risk, which can be roughly defined as the amount of money the bank can lose on a bad market day. This definition will be enough for our purposes below. The purpose of this least section is to show how simple mathematics can enter, in strage ways, in tackling difficult problems in banking.

4 An example

The calculation of Value-at-Risk of a portfolio can be done through the generation of future scenarios, evaluating the portfolio under each of those scenarios, and obtaining statistical conclusions about the losses inferred from those valuations. Sometimes, the bottleneck in this approach is the ability to evaluate the portfolio under a large colletion of scenarios. One simple example is a portfolio of so-called *mortgage backed securities*; these are instruments whose value are linked to the value of mortgages. Mortgages are very interesting objects, which have a very difficult dependency of market variables, as follows:

A mortgage allows the holder to borrow a certain amount of money, which is paid back in regular intervals (usually to buy a house) using an interest rate which is often fixed. Once they have been obtained, they can be considered to be an asset, which can have a positive value (the case when interest rates rise), a negative value (when interest rates drop), or stay at a constant value (when interest rates stay constant). The dependence is however more subtle, because interest rates are not given by a single number, but by a curve, representing the interest rate of all possible terms (one day, one month, several years, etc.). While the value of the mortgage clearly increases or decreases as *all* rates go up or down, it is not so easy to figure out how its value changes as the curve experiences twists and changes in shape. Moreover, mortgages give the holder the option to prepay, at least part of the mortgage, which the user does when the curve moves in certain directions which makes pre-payment increase the value of the mortgage, something which we already discussed could be difficult to determine. The end result, is that pricing a mortgage when the interest rate curve changes, can be a very difficult computational exercise, and computing this for a large portfolio of mortgages, with a fixed interest rate curve, could add up to half an hour of computater time. The implication of this is that performing a VaR calculation on the portfolio, usually requiring several thousand or millions of samples, is out of the question, as it would take years or centuries of computer time. This poses the following interesting optimization problem, in mathematical terms:

Consider a function (the portfolio value)

$$f(x), \quad x = (x_1, \cdots, x_n) \in \mathbb{R}^n.$$

Consider a multivariate probability distribution for x (we can assume it to be Gaussian). Determine a set of k points (20, for example, which allows for an overnight calculation), $x_1, \cdots, x_k$, and denote by $g(x)$ the function obtained by linearly interpolating the values

$$(x_i, f(x_i)), \quad i = 1, \ldots, k.$$

Furthermore, consider the difference $\|f - g\|$ in some appropriate norm (sup-norm, for simplicity).

The problem consists in determining the points x_i such that the norm difference between f and g is minimized.

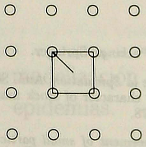
To understand what is involved in this problem, consider the simplest solution: a two-dimensional risk space, with a set of x_i which form a certain square grid.

Because we are approximating a function by its linear approximation, the value of the reduction in the number of points, by Taylor's theorem can be considered to be proportional to

$$\mu = \frac{V}{d^2},$$

where each grid point has a cell associated of volume V, and the point which is furthest from a grid point is at a distance d. The reason for this is the fact that the number of points required to fill the risk–space is inversly proportional to V, and because the largest error in the linear approximation is proportional to the square of the largest distance.

For the square grid, we have $V = 1$, and $d = 1/\sqrt{2}$. This yields

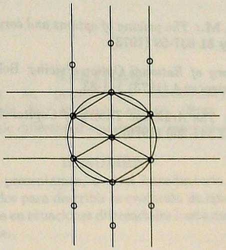


This yields

$$\mu_{\text{SquareGrid}} = 2$$

For the hexagonal grid, we have $V = \sqrt{3}/2$, and $d = 1/\sqrt{3}$. This yields

$$\mu_{\text{HexGrid}} = 3\sqrt{3}/2 \approx 2.5980.$$



Therefore, performance of the hexagonal grid will be -roughly- 30% better than the square grid.

When one tries to extend this example to the case of more dimensions (the case for sure in mortgage portfolios, as we saw, influenced by all the interest rates of all possible terms), one finds having to deal with optimal packing lattices in spaces of possibly high dimensions; this is one of Hilbert's problems and one that still remains unsolved to this date (solutions do exist in dimensions 2 and 3, and up to 8 in some form).

The previous exposition has been conducted without referencing to basic literature, to aid in the reading of the basic concepts. References that the interested reader may find useful to deepen in some of the issues presented here are:

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