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Periodicity in Dissipative-Repulsive Systems of Functional Differential Equations*

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1 Introduction

The existence of periodic solutions has been an important subject in the qualitative theory of differential equations and dynamical systems in the last half century, and it has also been an important area of applications in nonlinear analysis and, in particular, in asymptotic fixed point and topological degree theories.

Dissipative-repulsive systems frequently arise in modeling real world problems including control systems, electrodynamics, mixing liquids, neutron transportation, and population models. Dissipativeness means that the system energy dissipates (because of friction, breaking, etc), and hence all trajectories or orbits eventually enter and remain in a bounded set of the phase space. Repulsiveness, however, is in some sense the opposite of this property: trajectories eventually go away from a given set of the phase space. A dissipative-repulsive system has a combination of the above properties: dissipative with respect to some state variables and repulsive with respect to the remaining set of state variables. It has been long conjectured in the international community of researchers in the area that such a dissipative-repulsive system does admit a periodic solution.

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The present exposition is to discuss the methods developed to solve this problem and the progress having been made. It is only introductory in nature and is not intended in any way to be a review of the area.

The problem has its roots in the theory of ordinary differential equations. Consider, for example, the linear system

$$x' = A(t)x + f(t) \tag{1.1}$$

where $A(t)$ is an $n \times n$ matrix and $p : R \to R^n$, $R = (-\infty, +\infty)$. Both A and p are continuous and T-periodic on R for some $T > 0$. If $x = 0$ is the only solution of

$$x' = A(t)x \tag{1.2}$$

which is bounded on R, then (1.1) has a T-periodic solution. The result can be broken into the following three illustrative cases. First, let us recall the Floquet theory (see Burton [3, p.52]) which states that there exists a nonsingular, T-periodic, and continuous $n \times n$ matrix $P(t)$ and an $n \times n$ constant matrix J such that $Z(t) = P(t)e^{Jt}$ being a fundamental matrix of (1.2).

Case 1. If all characteristic roots of J have negative real parts, then the periodic solution of (1.1) is

$$x(t) = \int_{-\infty}^{t} P(t)e^{J(t-s)}P^{-1}(s)f(s)ds.$$

Case 2. If all characteristic roots of J have positive real parts, then the periodic solution of (1.1) is

$$x(t) = -\int_{t}^{+\infty} P(t)e^{J(t-s)}P^{-1}(s)f(s)ds.$$

Case 3. If some roots of J have positive and some roots of J have negative real parts, then a periodic solution is of a form analogous to a combination of Case 1 and Case 2. See Burton [3, p.68] for discussion on these cases when A is a constant matrix.

This is the simplest example of a dissipative-repulsive system. However, proving that a nonlinear dissipative-repulsive system has the same periodicity property presents a significant challenge to the investigators since neither the linear property nor the technique developed for (1.1) can be applied. In the next two sections, we will discuss how to extend Case 1-3 to fully nonlinear systems.

2 Dissipative Systems

In 1944 Norman Levinson [25] initiated the study of a second order system of ordinary differential equations

$$x' = f(t, x) \tag{2.1}$$

with the property that solutions all entered a ball and remained there for all large t. Such a system is called dissipative; that is, there exists a constant $r > 0$ such that

$$\limsup_{t \to +\infty} |x(t, t_0, x_0)| < r$$

where $x = x(t, t_0, x_0)$ is any solution of (2.1) with $x(t_0, t_0, x_0) = x_0$. He posed a problem which grew into the following notion. Let $f : R \times R^n \to R^n$ be continuous and locally Lipschitz in x. Denote by $|\cdot|$ the Euclidean norm on R^n. Throughout the paper, if a function is written without its argument, we mean the argument is t.

Definition 2.1. Solutions of (2.1) are uniformly bounded (UB) if for each $B_1 > 0$ there exists $B_2 > 0$ such that

$$[t_0 \in R, |x_0| \le B_1, t \ge t_0] \text{ imply that } |x(t, t_0, x_0)| < B_2.$$

Definition 2.2. Solutions of (2.1) are uniformly ultimately bounded for bound B (UUB) if for each $B_3 > 0$ there exists $K > 0$ such that

$$[t_0 \in R, |x_0| \le B_3, t \ge t_0 + K] \text{ imply that } |x(t, t_0, x_0)| < B.$$

In search for periodic solutions of (2.1), t_0 is almost always taken as zero and one asks for UUB at $t_0 = 0$. UB and UUB are different concepts. However UUB does imply UB if $f(t, x)$ in (2.1) is periodic in t (see Yoshizawa [33, p.64]). It is obvious that UUB is a concept of uniform dissipativeness.

Levinson's problem can be formulated in the following way. For history and details, we refer the reader to Burton [3, p.142], Burton and Zhang [11].

Problem 1. Suppose that there exists a constant $T > 0$ such that

$$f(t+T,x) = f(t,x) \text{ for } (t,x) \in R \times R^n.$$

Prove that if solutions of (2.1) are UB and UUB at $t_0 = 0$, then (2.1) has a T-periodic solution.

This problem was solved for $n = 2$ independently by Cartwright [13] and Massera [29] in 1950. But it had to wait until 1959 and the asymptotic fixed point theorem of Browder [2] for a proof for general n. Such a proof can be found in Yoshizawa [34, p.158].

Attention then turned to the finite delay system

$$x'(t) = f(t, x_t) \tag{2.2}$$

where $x_t(s) = x(t+s)$ for $-h \le s \le 0$, $h > 0$ is a constant, and $f : R \times C \to R^n$ is continuous with $f(T+t,\phi) = f(t,\phi)$ and is locally Lipschitz in ϕ. $(C, \|\cdot\|)$ is the Banach space of continuous functions $\phi : [-h,0] \to R^n$ with the supremum norm. For each $(t_0,\phi) \in R \times C$, there exists a unique solution $x = x(t,t_0,\phi)$ of (2.2) with $x_{t_0} = \phi$ (see Hale [16, p.42]). Definitions of UB and UUB are extended to (2.2) by merely replacing x_0 by ϕ and $|x_0|$ by $\|\phi\|$. Note that in these new definitions, It is still the R^n norm on solutions $x(t,t_0,\phi)$. While the norm makes no difference here, it is crucial for infinite delay systems.

Problem 2. Show that problem 1 is true for (2.2).

Hale and Lope [17] used the asymptotic fixed point theorem of Horn [20] to solve Problem 2.

Fixed Point Theorem ([20]). Let $S_0 \subset S_1 \subset S_2$ be convex subsets of a Banach space X with S_0 and S_2 compact and S_1 open relative to S_2. Let $S_2 \to X$ be a continuous function such that for some integer $m > 0$,

(a) $P^j S_1 \subset S_2, \quad 1 \le j \le m-1$

(b) $P^j S_1 \subset S_0, \quad m \le j \le 2m-1.$

Then P has a fixed point in S_0.

As mentioned earlier, UUB does not in general imply UB. In fact, Kato [23] showed that even when (2.2) is autonomous and completely continuous, then UUB does not imply UB. Investigating the same problem for an infinite delay system, Burton and Zhang [9] in 1990 solved Problem 2 using UUB alone; that is without asking UB. Makay [26] continued and showed that if system (2.2) is dissipative, then it has a T-periodic solution without asking UUB.

We shall sketch the proof of the following theorem whose general version can be found in [9] to illustrate the use of Horn's theorem without UB.

Theorem 2.1.([9]) Suppose that $f(t,\phi)$ takes bounded sets of $R \times C$ into bounded sets of R^n. If solutions of (2.2) are UUB at $t_0 = 0$, then (2.2) has a T-periodic solution.

Proof. Since solutions of (2.2) are UUB for a bound B and $f(t,\phi)$ takes bounded sets into bounded sets, there exist $N > 0$ and $L_B > 0$ such that

$$[\phi \in C,\ \|\phi\| \le 2B,\ t \ge N] \text{ imply } |x(t,0,\phi)| \le B$$

and $|f(t,\phi)| \le L_B$ for all $t \ge 0$ whenever $\|\phi\| \le 2B$. Now define

$$S_0 = \{\phi \in C : \|\phi\| \le 2B,\ |\phi(u) - \phi(v)| \le L_B|u - v|,\ u, v \in [-h, 0]\}$$

and for each $t \ge 0$ define $P_t : C \to C$ by $P_t(\theta) = x(t+\theta, 0, \phi)$ for $\theta \in [-h, 0]$. Then P_t is continuous and S_0 is compact. Thus, there exists $B_1 > B$ such that

$$\phi \in S_0 \text{ implies } |x(t,0,\phi)| \le B_1 \tag{2.3}$$

for $0 \le t \le N$. Since $|x(t,0,\phi)| \le B$ for $t \ge N$, we have $|x(t,0,\phi)| \le B_1$ for all $t \ge 0$.

Next let $B_2 = B + B_1$ and find $L > L_B$ such that $|f(t,\phi)| \le L$ for all $t \ge 0$ and $\|\phi\| \le B_2$. Define

$$S_2 = \{\phi \in C : \|\phi\| \le B_2,\ |\phi(u) - \phi(v)| \le L|u - v|,\ u, v \in [-h, 0]\}.$$

By UUB, there exists $K > 0$ such that

$$[\phi \in C,\ \|\phi\| \le B_2,\ t \ge K] \text{ imply } |x(t,0,\phi)| \le B.$$

Since P_K is continuous on the compact set S_2, it is uniformly continuous; thus, there exists $\delta > 0$ ($\delta < B$) such that

$$[\phi_1, \phi_2 \in S_2, \|\phi_1 - \phi_2\| < \delta] \text{ imply } \sup_{0 \le t \le K} |x(t,0,\phi_1) - x(t,0,\phi_2)| < B/2. \tag{2.4}$$

In view of (2.3) and (2.4), $[\phi_0 \in S_0,\ \phi \in S_2,\ \|\phi - \phi_0\| < \delta]$ imply

$$\sup_{0\le t\le K} |x(t,0,\phi)| \le \sup_{0\le t\le K} |x(t,0,\phi_0)| + \sup_{0\le t\le K} |x(t,0,\phi) - x(t,0,\phi_0)| \le B_1 + B/2 < B_2.$$

Define

$$Q_1 = \bigcup_{\phi_0\in S_0} \{\phi \in C : \|\phi - \phi_0\| < \delta\}$$

and $S_1 = Q_1 \cap S_2$. Then $S_0 \subset S_1 \subset S_2$ are all convex subsets of C with S_0 and S_2 compact and S_1 open relative to S_2. Define $P : S_2 \to C$ by

$$P(\phi) = P_T(\phi) = x_T(\cdot, 0, \phi) \quad \text{for } \phi \in S_2$$

or

$$P(\phi)(\theta) = x(T+\theta, 0, \phi) \quad \text{for } \theta \in [-h, 0].$$

Then $P^j(S_1) \in S_2$ for all $j = 1, 2, \cdots$. Choose $m > 0$ such that $mT > K + T$. Then $P^j(S_1) \subset S_0$ for $j \ge m$. By Horn's theorem, $P(\phi)$ has a fixed point $\phi^* \in S_0$; that is $P(\phi^*) = x_T(\cdot, 0, \phi^*) = \phi^*$. Since $x(t, 0, \phi^*)$ and $x(t+T, 0, \phi^*)$ are both solutions of (2.2) with the same initial function, by uniqueness, they are equal. Thus, $x = x(t, 0, \phi^*)$ is a T-periodic solution of (2.2), and the proof is complete.

We turn now to the infinite delay system

$$x'(t) = F(t, x_t) \tag{2.5}$$

where $x_t(s) = x(t+s)$ for $-\infty < s \le 0$ and $F : R \times X \to R^n$ with $F(t + T, \phi) = F(t, \phi)$. Here X is the Banach space of bounded continuous function. $\phi : R^- \to R^n$, $R^- = (-\infty, 0]$, having the supremum norm $\|\cdot\|$. Define UB and UUB for (2.5) just as for (2.1) replacing x_0 by ϕ and $|x_0|$ by $\|\phi\|$.

Problem 3. Prove that if solutions of (2.5) are UUB, then there is a T-periodic solution.

When we attempt to follow the construction of S_i and P for (2.5), several things go wrong.

(a) S_0 and S_2 are not compact,

(b) $P^j(S_1)$ is never contained in S_0 because of the "tail" is never translated out of the initial function space,

(c) P_t is not necessarily continuous in $(X, \|\cdot\|)$, where $P_t(\theta) = x(t+\theta, 0, \phi)$ for $\theta \in R^-$.

Speaking for an infinite delay system, Seifert [31] showed that a necessary condition for UUB is that $F(t, \phi)$ has some kind of a fading memory. Such an idea suggested to Arino, Burton, and Haddock [1] that it may be feasible to use initial functions ϕ having the property that $|\phi(s)| \to +\infty$ as $s \to -\infty$. Let G denote the set of continuous and non-increasing functions $g : (-\infty, 0] \to [1, +\infty)$, $g(0) = 1$, $g(r) \to +\infty$ as $r \to -\infty$. Then for each $g \in G$, $(X_g, |\cdot|_g)$ is the Banach space of continuous functions $\phi : (-\infty, 0] \to R^n$ for which

$$|\phi|_g = \sup_{-\infty < s \leq 0} |\phi(s)|/g(s)$$

exists. In Burton [3, p.282], it is shown how g can always be chosen from F when (2.5) is an integrodifferential equation. To this end, we define g-UB and g-UUB by replacing x_0 by ϕ and $|x_0|$ by $|\phi|_g$ and modify S_i. Those basic sets are similar to

$$S = \{\phi \in X_g : |\phi(s)| \leq B\sqrt{g(s)}, |\phi(u) - \phi(v)| \leq L|u - v|\}.$$

Then S is compact in $(X_g, |\cdot|_g)$ and $P^j(S) \subset S$ for large j. These constructions allowed Arino-Burton-Haddock [1] to show that if solutions of (2.5) are g-UB and g-UUB, then (2.5) has a T-periodic solution. The Problem 3 is not solved completely here since g-UB, g-UUB, and the continuity of solutions with respect to the initial functions $\phi \in X_g$ are much stronger than those in terms of $(X, \|\cdot\|)$.

The key to the problem of continuity of $x(t, 0, \phi)$ with respect to ϕ and of the compactness of S when $h = +\infty$ seems to lie in the aforementioned counterexample of Seifert [31]. A fading memory is required for UUB. The fading memory, together with continuity of $F(t, \phi)$ in the supremum norm and the requirement that F takes bounded sets into bounded sets will yield the continuity of $x(t, 0, \phi)$ in ϕ with respect to a norm $|\cdot|_g$ and the compactness of S in $(X_g, |\cdot|_g)$. Burton and Feng [8] achieved this goal.

All of the work discussed here is based on translation of map P_t and is highly dependent on the uniqueness of solutions with respect to the initial functions. Unlike ordinary differential equations, uniqueness in (2.5) does not imply the continuity of $x(t, 0, \phi)$ in ϕ (see Burton and Dwiggins [5] and Kaminogo [22]).

Definition 2.3. Let $\Omega \subset X$. We say that P_t is continuous in (Ω, G) if there is a $g \in G$ and if for each $\{\phi \in \Omega,\ J > 0,\ \mu > 0\}$ there is a $\delta > 0$ such that $\{\psi \in \Omega,\ |\phi - \psi|_g < \delta\}$ imply $|P_J(\phi) - P_J(\psi)|_g < \mu$.

Definition 2.4. Equation (2.5) has a weakly fading memory in $\Omega \subset X$ if for each $[J > 0,\ D > 0,\ \mu > 0]$ there exists $K > 0$ such that

$$\{\phi_i \in \Omega,\ \|\phi_i\| \le D,\ i = 1, 2,\ \phi_1(s) = \phi_2(s) \text{ on } [-K, 0],\ 0 \le t \le J\}$$

imply that $|F(t, \phi_1) - F(t, \phi_2)| < \mu$.

Burton and Zhang [9] dropped the UB requirement and solved Problem 3 in the sense that $F(t, \phi)$ has a fading memory. Note that Burton-Feng's theorem [8] reduces condition (iii) below to conditions on the supremum norm.

Theorem 2.2. ([9]) Suppose that $F(t + T, \phi) = F(t, \phi)$ for some $T > 0$ and all $(t, \phi) \in R \times X$ and for each $\phi \in X$ there is a unique solution $x(t, 0, \phi)$ of (2.5) on $[0, +\infty)$. Suppose also that

(i) solutions of (2.5) are UUB in the supremum norm,

(ii) for each $M > 0$ there exists $L > 0$ such that

$$[\phi \in X,\ \|\phi\| \le M,\ t \ge 0] \text{ imply } |F(t, \phi)| \le L,$$

(iii) for each bounded (in the supremum norm) set $\Omega \in X$, P_t is continuous in (Ω, G).

Then (2.5) has a T-periodic solution.

A variant of Theorem 2.2 was proved by Burton, Dwiggins, and Feng [6] with the additional assumption of UB. When (2.5) is linear in ϕ then much less is required, as may be seen in the work of Hino and Murakami [19] and Makay [27]. It is also shown in Makay [28] that (2.5) has a T-periodic solution if $F(t, \phi)$ is continuous in $R \times X_g$ and locally Lipschitz in ϕ in the g-norm, and solutions of (2.5) are dissipative with respect to a compact set in X_g. By introducing a space C_h, Huang and Wang [21] showed that if solutions of (2.5) are h-UB and h-UUB, then (2.5) has a T-periodic solution.

3 Dissipative-Repulsive Systems

All of the work discussed in Section 2 is highly dependent on the dissipativeness of the system. When such a property is absent, however, the problem showing the existence of a periodic solution is extremely difficult and has been a central subject of modern research in the area.

It is obvious that the method used in the previous section will not work for systems with repulsive components. To overcome such a difficulty, investigators turned to degree-theoretic method which often relies on an a priori bound on all possible periodic solutions to a family of differential equations associated with the original one, but it does not require all solutions to be bounded. Therefore, it is possible to apply the method to non-dissipative systems for which Horn's theorem is not applicable. It is now even clear that dissipativeness is merely a sufficient condition for the existence of a periodic solution.

The degree theory many investigators have used is that of Granas [14, 15] who calls it the method of continuation of Poincaré and later refers to it as a topological transversality method. The approach of Granas explicitly avoids the calculation of Leray-Schauder degrees and thus allows investigators to study the periodicity problem in a much simpler way.

Consider system (2.5) and a companion to it

$$x'(t) = F_\lambda(t, x_t) \tag{2.5_λ}$$

where $0 \leq \lambda \leq 1$, $F_\lambda(t+T, \phi) = F_\lambda(t, \phi)$, and $F_\lambda(t, \phi) = F(t, \phi)$ if $\lambda = 1$. We construct a homotopy $h(\lambda, \phi)$ or $h_\lambda(\phi)$ on $[0,1] \times P_T$ to P_T, where P_T is the Banach space of continuous T-periodic functions $\phi : R \to R^n$. The theory of Granas is applied in the following way:

(a) An a priori bound B is found for all possible T-periodic solutions of (2.5_λ).

(b) A set $X \subset \{\phi \in P_T : \|\phi\| \leq B\}$ is constructed with the property that $h_\lambda : X \to P_T$ is compact and each fixed point of h_λ satisfies (2.5_λ). This implies that all fixed points of h_λ are bounded by B.

(c) h_0 has a fixed point $\phi_0 \in X \setminus \partial X$.

Then Granas' theorem yields that h_1 has a fixed point ϕ_1 in X. Thus, ϕ_1 is a T-periodic solution of (2.5). The idea is to deform the fixed point x_0 of h_0 to a fixed point x_1 of h_1 through h_λ since $h(\lambda, \phi)$ is fixed point free on ∂X. It turns out that constructing such a homotopy is a very difficult task as can be seen below.

Burton, Eloe, and Islam [7] studied the equation

$$x'(t) = Dx(t) + \int_{-\infty}^{t} C(t-s)x(s)ds + p(t) \tag{3.1}$$

with a view of extending Case 3 mentioned in Section 1. Here C is an $n \times n$ matrix of continuous functions which are in $L^1[0, +\infty)$, D is an $n \times n$ constant matrix, $p : R \to R^n$ is continuous and T- periodic.

Let $(P_T^0, \|\cdot\|)$ denote the subspace of P_T with $\phi \in P_T^0$ if $\phi \in P_T$ and if the mean value of ϕ is zero:

$$m(\phi) = (1/T) \int_0^T \phi(s)ds = 0.$$

To each $\phi \in P_T$ we associate the function $\bar{\phi} \in P_T^0$ defined by

$$\bar{\phi}(t) = \phi(t) - (1/T) \int_0^T \phi(s)ds \tag{3.2}$$

having the mean value zero and the consequential property that $\int_0^t \bar{\phi}(s)ds \in P_T$.

Consider the family of equations

$$x'(t) = \lambda\Big[Dx(t) + \int_{-\infty}^{t} C(t-s)x(s)ds + p(t)\Big] \tag{3.1_λ}$$

where $0 \le \lambda \le 1$.

Theorem 3.1. ([7]) Suppose that $p \in P_T^0$ and there exists a constant $B_1 > 0$ such that $\|x_\lambda\| < B_1$ for every solution $x_\lambda \in P_T$ of (3.1_λ) for every $\lambda \in (0, 1]$. Then (3.1) has a solution in P_T^0.

The proof of Theorem 3.1 relies on the construction of the following homotopy. Define $h_\lambda : [0, 1] \times P_T \to P_T$ by

$$h_\lambda(\phi)(t) = \lambda\Big[\bar{\phi}(0) + D\int_0^t \bar{\phi}(s)ds + \int_0^t \int_{-\infty}^{u} C(u-s)\bar{\phi}(s)dsdu + \int_0^t p(s)ds\Big]$$

for each $\lambda \in [0, 1]$ and $\phi \in P_T$, where $\bar{\phi}$ is defined in (3.2).

It is easy to check that h_λ is compact, $h_\lambda(\phi) \in P_T$,

$$\frac{d}{dt}h_\lambda(\phi)(t) = \lambda\Big[D\bar{\phi}(t) + \int_{-\infty}^{t} C(t-s)\bar{\phi}(s)ds + p(t)\Big]$$

and, consequently, if $\phi \in P_T$ is a fixed point of h_λ, then $\phi'(t) = \bar{\phi}'(t)$ so that $\bar{\phi}$ is a solution of (3.1_λ) and, by hypothesis, $\|\bar{\phi}\| < B_1$. This implies there exists $B > 0$ such that $\|\phi\| < B$. Since the fixed point of h_0 is $0 \in X \setminus \partial X$, $X = \{\phi \in P_T : \|\phi\| \le B\}$, Granas' theorem implies that h_1 has a fixed point ϕ_1 which is a T-periodic solution of (3.1).

It is noticed in [7] that if $p \in P_T$, if $p \notin P_T^0$, and $D + \int_0^\infty C(u)du$ is nonsingular, then a translation $y = x + d$, d is constant, can be defined so that (3.1) becomes an equation with a forcing function having mean value zero.

Example 3.1. ([7]) Consider the two dimensional system

$$x'(t) = \lambda\Big[Dx(t) + \int_{-\infty}^{t} C(t-s)x(s)ds + p(t)\Big] \tag{3.3}$$

with $D = \mathrm{diag}(1,-1)$, C is continuous with $-1 + \int_0^{+\infty} |C(u)|du < 0$, and $p \in P_T^0$. Then (3.3) has a T-periodic solution for $\lambda = 1$.

Using a Liapunov function instead of a functional here, we obtain an a priori bound for all possible periodic solutions and eliminate the condition $\int_0^\infty \int_t^\infty |C(u)|dudt < \infty$ used in [7]. Define $V(x) = x^T Ex$, where $x = (x_1, x_2)^T$ and $E = \mathrm{diag}(-1,1)$ so that $|E| = 1$. Let $x = x(t)$ be any T-periodic solution of (3.3). Then

$$\begin{aligned} V'(x(t)) &\le -2\lambda|x|^2 + 2\lambda|x| \int_{-\infty}^{t} |C(t-s)||x(s)|ds + 2\lambda|x||p(t)| \\ &\le -2\lambda|x|^2 + \lambda\int_0^{\infty} |C(u)|du\,|x|^2 + \lambda\int_{-\infty}^{t} |C(t-s)||x(s)|^2 ds + 2\lambda|x|\|p\| \end{aligned}$$

Let $\alpha = 1 - \int_0^\infty |C(u)|du$ and integrate from 0 to T to obtain

$$\begin{aligned} V(T) - V(0) &\leq -2\lambda \int_0^T |x(s)|^2 ds + \lambda \int_0^\infty |C(u)|du \int_0^T |x(s)|^2 ds \\ &\quad + \lambda \int_0^\infty |C(u)|du \int_0^T |x(t-u)|^2 dtdu + \lambda\alpha \int_0^T |x(s)|^2 ds + \lambda T\|p\|^2/\alpha \\ &= -2\lambda\Big(1 - \int_0^\infty |C(u)|du\Big) \int_0^T |x(s)|^2 ds + \lambda\alpha \int_0^T |x(s)|^2 ds + \lambda T\|p\|^2/\alpha \\ &= -\alpha\lambda \int_0^T |x(s)|^2 ds + \lambda T\|p\|^2/\alpha. \end{aligned}$$

Since $x(t)$ is T-periodic, we have $V(T) - V(0) = 0$. This in turn yields

$$\int_0^T |x(s)|ds \leq T^{1/2}\Big[\int_0^T |x(s)|^2 ds\Big]^{1/2} \leq T\|p\|/\alpha$$

for $\lambda \in (0, 1]$. An integration of the equation yields $\int_0^T |x'(s)|ds \leq \gamma$. It follows from Sobolev's inequality, there exists a constant $B_1 > 0$ such that $\|x\| \leq B_1$. By Theorem 3.1, equation (3.3) has a T-periodic solution for $\lambda = 1$.

If the equation has a linear part, for example,

$$x'(t) = Dx(t) + F(t, x_t) \tag{3.4}$$

where D is an $n \times n$ matrix, then a companion system is

$$x'(t) = Dx(t) + \lambda F(t, x_t) \tag{3.4_λ}$$

where $\lambda \in [0, 1]$. If all characteristic roots of D have negative real parts, then a homotopy for (3.4_λ) may be defined as

$$h(\lambda, \phi)(t) = \lambda \int_{-\infty}^t e^{D(t-s)} F(s, \phi_s) ds$$

for $\phi \in P_T$. Now suppose that $D = \text{diag}(D_1, D_2)$ where D_1 is a $k \times k$ matrix whose eigenvalues have positive real parts and D_2 is an $(n-k) \times (n-k)$ matrix whose eigenvalues have negative real parts, $0 \leq k \leq n$ is an integer. If $F(t, \phi_t) = \big(F_1(t, \phi_t), F_2(t, \phi_t)\big)^T$, then the homotopy is

$$h_\lambda(\phi)(t) = \lambda\Big(-\int_t^\infty e^{D_1(t-s)} F_1(s, \phi_s) ds, \int_{-\infty}^t e^{D_2(t-s)} F_2(s, \phi_s) ds\Big)^T.$$

The idea of such a construction for $h(\lambda, \phi)$ comes from expressions for periodic solutions in Case 1-3 discussed in Section 1. Many good theorems are obtained using such approach (see [10],[32],[36]). Systems discussed there are in general forms including abstract, partial, and neutral functional differential equations. The technique of adding growth conditions on F to obtain periodic solutions was also developed. We refer the reader to the work of Hatvani and Krisztin [18] and references contained therein.

Wu, Xia, and Zhang [32] summarize Granas's theory to a useful principle for topological transversality.

Theorem 3.2. ([32]) Let Y be a convex subset of a Banach space, $X \subset Y$ be closed, $p \in X \backslash \partial X$, and $N : X \to Y$ be a compact map. If $H : [0,1] \times X \to Y$ is compact such that $H(0, \phi) = p$ and $H(1, \phi) = N(\phi)$ for all $\phi \in X$, then either

(i) N has a fixed point in $X \setminus \partial X$, or

(ii) there exists $x \in \partial X$ and $\lambda \in (0, 1]$ such that $x = H(\lambda, x)$.

Burton [4] used a direct fixed point technique and extended the method of homotopy construction in Theorem 3.1 to that of nonlinear systems. He applied a fixed point theorem of Schaefer [30] which is a variant of the nonlinear alternative of Leray and Schauder degree, but much easier to use.

Theorem 3.3. ([30]) Let V be a normed space , H a continuous mapping of V to V which is compact on each bounded subset of V. Then either

(i) the equation $x = \lambda H x$ has a solution for $\lambda = 1$, or

(ii) the set of all such solutions x, for $0 < \lambda < 1$, is unbounded.

Using the direct fixed point mapping, Burton and Zhang [12] are able to link the homotopy to the right-hand side of the equations directly and avoid many difficulties encountered previously. The technique is a significant improvement of that for (3.1). Let us exam the problem we might face when writing the differential equation as an integral equation which then defines a mapping, the homotopy; if the mapping has a fixed point, then it is a solution of the differential equation. We write (2.5) as an integral equation

$$x(t) = x_0 + \int_{t_0}^{t} F(s, x_s) ds.$$

This is used to define a mapping P by

$$(P\phi)(t) = x_0 + \int_{t_0}^{t} F(s, \phi_s)ds$$

on the space $(P_T, \|\cdot\|)$. There are several things go wrong.

(a) We do not know how x_0 is to be chosen.

(b) P will not map P_T into P_T unless $F(t, \phi_t)$ has a mean value zero.

(c) If any fixed point theorem, say Schauder's, is to be used, then P must satisfy $P : K \to K$ for a closed, convex set K in P_T. In general, P does not satisfy this condition.

To overcome such difficulties, we consider a different mapping. Instead of writing the differential equation as an integral equation, write the solution as an integral equation

$$\Phi(t) = x_0 + \int_{t_0}^{t} \phi(s)ds.$$

Then define a mapping P by

$$(P\phi)(t) = F(t, \Phi_t)$$

for $\phi \in P_T$. In order for P to map P_T into P_T, we must carefully choose x_0 so that $F(t, \Phi_t)$ has a mean value zero. We state a simple version of Burton-Zhang's theorem [12].

Theorem 3.4. ([12]) Suppose the following conditions hold:

(i) for each $\phi \in P_T^0$, there is a constant $k_\phi \in R$ such that $\int_0^T F(t, \Phi_t)dt = 0$, where $\Phi(t) = k_\phi + \int_0^t \phi(s)ds$ for each $t \in R$,

(ii) $E : P_T^0 \to P_T$ defined by $E(\phi)(t) = \Phi(t)$ in (i) is continuous and for each $\alpha > 0$, there exists a constant $L_\alpha > 0$ such that $|k_\phi| \leq L_\alpha$ whenever $\|\phi\| \leq \alpha$.

(iii) $F : R \times P_T \to R^n$ is continuous and maps bounded sets into bounded sets,

(iv) there exists a constant $B > 0$ such that $\|x\| < B$ whenever $x = x(t)$ is a T- periodic solution of

$$x'(t) = \lambda F(t, x_t), \quad \lambda \in (0, 1).$$

Then Eq.(2.5) has a T-periodic solution.

Proof. Define the homotopy as $H_\lambda(\phi)(t) = \lambda F(t, \Phi_t)$. It follows from (i)-(iii) that $H_\lambda(\phi) \in P_T^0$ and is compact. By (iii), there exists a constant $L = L(B)$ such that $\|F(t, \Phi_t)\| < L$ whenever $\|\Phi\| \le B$. If ϕ is a fixed point of H_λ, then $\phi(t) = \Phi'(t) = \lambda F(t, \Phi_t)$. By (iv), we have $\|\Phi\| < B$. This implies $\|\phi\| < L$. Applying Schaefer's theorem, we conclude H_λ has a fixed point ϕ for $\lambda = 1$. Thus, Φ is a T-periodic solution of (2.5). The proof is complete.

Example 3.2. ([12]) Consider the two dimensional nonlinear system

$$x'(t) = Ag(x(t)) + \int_{-\infty}^{t} C(t-s)g(x(s))ds + p(t) \tag{3.5}$$

where $A = \mathrm{diag}(1, -1)$, $C(t) = (c_{ij}(t))_{2\times 2}$, $g(x) = (x_1^3, x_2^3)^T$, $x = (x_1, x_2)^T$, $p \in P_T^0$. If

$$\int_0^{\infty} (|c_{1j}(s)| + |c_{2j}(s)|ds)ds < 1, \ j = 1, 2 \tag{3.6}$$

then Eq.(3.5) has a T-periodic solution.

The key to apply Theorem 3.4 is to verify condition (i). Let $F(t, x_t)$ denote the right-hand of (3.5). For $\phi = (\phi_1, \phi_2)^T \in P_T^0$ and $k \in R$, we define

$$Q(k) = \int_0^T \Big(k + \int_0^t \phi_1(s)ds\Big)^3 dt.$$

Since the quadratic function

$$Q'(k) = 3\int_0^T \Big(k + \int_0^t \phi_1(s)ds\Big)^2 dt \ge 0$$

with $\lim_{k\to\pm\infty} Q(k) = \pm\infty$, there exists a unique $k_{1\phi} \in R$ such that $Q(k_{1\phi}) = 0$. Similarly, there exists $k_{2\phi} \in R$ such that

$$\int_0^T \Big(k_{2\phi} + \int_0^t \phi_2(s)ds\Big)^3 dt = 0.$$

If $k_\phi = (k_{1\phi}, k_{2\phi})^T$ and $\Phi(t) = \Big(\Phi_1(t), \Phi_2(t)\Big)^T = k_\phi + \int_0^t \phi(s)ds$ for each $\phi \in P_T^0$, then

$$\int_0^T \Phi^3(s)ds = 0 \ \text{ and } \ \int_0^T F(t, \Phi_t)dt = 0.$$

Thus, (i) is satisfied.

Finally, for the conjecture that a dissipative-repulsive system admits a periodic solution, Kupper, Li, and Zhang [24] give a positive answer for ordinary and finite delay equations. Consider a system of ordinary differential equations

$$x'(t) = f(t, x(t)) \tag{3.7}$$

where $f : R \times R^n \to R^n$ is continuous and locally Lipschitz in x with $f(t + T, x) = f(t, x)$ for all t and some $T > 0$. Denote by $x(t, x_0) = x(t, 0, x_0)$ the unique solution of (3.7) with $x(0, x_0) = x_0$. If m and l are nonnegative integers with $m + l = n$, we denote $x = (y, z)$, $y \in R^m$, $z \in R^l$.

Definition 3.1. Eq.(3.7) is said to be dissipative-repulsive if there exist possitive constants B, a_0, b_0, and a continuous T-periodic function $g : R \to R^l$ with $|g(t)| < b_0$ for all $t \in R$ such that for any $a \geq a_0$, $b \geq b_0$, There are $b_1 = b_1(a, b) \geq b_0$ and $K = K(a, b) > 0$ such that the following conditions hold for all $|y_0| \leq a$:

(i) $|y(t, x_0)| \leq B$, whenever $t \geq T$ and $|z_0| \leq b_1$;

(ii) $|z(t, x_0) - g(t)| > 0$, whenever $0 \leq t \leq T$ and $b_1 \leq |z_0| \leq b_1 + b$;
$|z(t, x_0)| > b_1$, whenever $t \geq T$ and $b_1 \leq |z_0| \leq b_1 + b$.

Theorem 3.5. ([24]) If system (3.7) is dissipative-repulsive, then it admits a T-periodic solution.

The proof is based on a modular degree theorem of Zabreiko and Krasnosel'skii [35] by constructing two homotopies and using the dissipative-repulsiveness property of the system. It is a direct generalization of the corresponding result for dissipative systems discussed in Section 2 (if $m = n$). A similar result for functional differential equations with finite delay is also obtained in [24]. Much remains to be done concerning the existence of periodic solutions to infinite delay systems of dissipative-repulsive type.

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