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On Semantic Paradoxes*

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1. Introduction

By a paradox we mean generally an argument that leads to contradiction for no clear reason. Note that in an argument to reach a contradiction is usually not a surprise, nor a disaster, but exactly what we are looking for, as in a "proof by contradiction." The difference is this: for a "proof by contradiction", there is an assumption announced explicitly in front, hence the contradiction just establishes the negation of the assumption. In contrast, for a paradox, there seems no assumption used in the argument. The contradiction occurs for no clear reason, hence is suspected to have some deep cause in the foundation of our language or logic. Of course this is very, very serious.

The most ancient and most influential paradox in history is perhaps the paradox of the Liar. Here is a well-known version of it:

The Liar paradox. $\boxed{\text{The boxed sentence is false}}$

If it is true, what it says should be the case, hence it is false. If it is false, what it says should be negated, hence it is true.

Another well known paradox of this type involves two cards:

The Jourdain's Cards paradox. Consider the following two cards.

$\boxed{\text{The sentence on the second card is true.}}$

$\boxed{\text{The sentence on the first card is false.}}$

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If the first sentence is true, what it says should be the case, hence the second sentence is true. Then what the second sentence says should be the case, hence the first sentence is false, a contradiction. Likewise, assuming the first sentence is false leads to contradiction too.

The arguments lead to contradiction. It is unclear at first glance what goes wrong. A number of theories have been proposed in the literature to resolve the Liar paradoxes, notably the hierarchy theory of language of Tarski [11], which separates sentences into different levels, and the truth-value gap theory of Kripke [5], which adopts three-valued logic. Nice accounts can be found in [1], [2], [4], [5], [6], [7], [9], [10], [11]. In [12] I presented a different solution to the Liar paradox. It is not hierarchic, and adopts the classical two-valued logic. The main observation is this:

Main Observation. *There is an assumption implicitly used in the Liar argument. With this assumption uncovered, and announced explicitly in front, the Liar argument will be found to be a usual "proof by contradiction", but not at all paradoxical.*

This is wholly supported by a "Three Cards paradox" I found recently in [12]. In this expository article I concentrate in §2 on analysis of the Three Cards paradox, and then briefly illustrate in §3 the other conclusions of [12]. Quite a part of [12] is expository already, and is simply taken here. I just feel the material is very appropriate for an expository article, which I owe Cubo.

2. The Three Cards paradox: The secret of the Lair

To present my solution to the Liar paradox, the best way is first to present a new paradox, the Three Cards paradox. Uncovering the secret of it leads directly to the solution to the Liar paradox.

The Three Cards paradox. Consider the following three cards.

The sentence on the second card is true, and the sentence on the third card is false.

Either the sentence on the first card is false, or the sentence on the third card is true.

The sentence on the first and second card are both true.

This paradox looks fancier than the Liar and Jourdain, what with the logical connectives "and" and "or". The argument is hence more complicated.

Assume the first sentence is true. Then what it says should be the case, which means the second sentence is true and the third sentence is false. Hence what the third sentence says should be negated, which means that either the first sentence is false, or the second sentence is false. Putting these together, we have that either the second sentence is true and the first sentence is false, or the second sentence is both true and false. But this "or" is impossible, because we adopt the classical two-valued logic. Thus this "either" must hold. That is, the second sentence is true and the first sentence is false. This contradicts the assumption that the first sentence is true at the beginning. Thus the first sentence must be false.

Then what the second sentence says is the case, hence the second sentence is true. Moreover, what the third sentence says is not the case, hence the third sentence is false. In summary, the first and third sentence are each false, but the second sentence is true. In particular, what the first sentence says is the case, hence the first sentence should be true, which contradicts that the first sentence is false. This way we have run out of possibilities with contradictions everywhere.

This is a new paradox I found recently. Both the statement and the argument are of the same type as the Liar and Jourdain. It is evidently a "Liar-like paradox".

But where does this paradox come from? The argument is quite complicated. How was it figured out?

To reveal the secret let me first restate the Three Cards paradox by using symbols. Denote the three sentences by A, B, and C, respectively. Then A says that "B is true and C is false", and so on. Denote by T the phrase "is true", and by F, "is false". Finally, denote by ":=" the phrase "says that", or "refers to". Then the Three Cards paradox consists of three "referential relations", written as

$$\begin{cases} A := B\mathrm{T} \wedge C\mathrm{F}, \\ B := A\mathrm{F} \vee C\mathrm{T}, \\ C := A\mathrm{T} \wedge B\mathrm{T}, \end{cases}$$

where $\wedge$ stands for "and", and $\vee$ stands for "or".

Some people take the referential relations such that $A := B\mathrm{T} \wedge C\mathrm{F}$ to be the strict equality $A = B\mathrm{T} \wedge C\mathrm{F}$. All conclusions of the present paper hold automatically under such a stronger identification, but it is clearer to keep the referential relations less special.

For the meanwhile these referential relations may be considered "presumed" ones but not "verified" ones, and the Three Cards paradox may be written as "referential equations"

$$\begin{cases} X := Y\mathrm{T} \wedge Z\mathrm{F}, \\ Y := X\mathrm{F} \vee Z\mathrm{T}, \\ Z := X\mathrm{T} \wedge Y\mathrm{T}. \end{cases}$$

Of course, to regard a relation as an equation does not exclude the possibility that the relation might be verified later, hence it is not a loss but just a precaution.

Now let me reveal the secret: In making the Three Cards paradox I had in mind the following Boolean system.

The Boolean Model for the "Three Cards." The Boolean system

$$\begin{cases} x = y\overline{z}, \\ y = \overline{x} + z, \\ z = xy \end{cases}$$

has no solution.

Proof. Assume there is a solution. We derive the following contradiction.

Assume $x = 1$. Then, by equation 1, $y = 1$ and $z = 0$. By equation 3, $z = 0$ yields either $x = 0$, or $y = 0$. Putting these together, we have either $y = 1$ and $x = 0$, or $y = 1$ and $y = 0$. But this "or" is impossible. Thus this "either" must hold. That is, $y = 1$ and $x = 0$. This contradicts the assumption $x = 1$ at the beginning. Thus $x = 0$.

Then $y = 1$ from equation 2, and $z = 0$ from equation 3. In summary, $x = z = 0$, but $y = 1$. However putting $x = z = 0$ and $y = 1$ into equation 1 yields a contradiction. This contradiction proves the system has no solution.

The reader may have noticed a clear resemblance between the Boolean problem and the Three Cards paradox. The difference is clear too: In their statements, one has a phrase "has no solution", the other does not. In their arguments, one has a standard frame of proof by contradiction, that is, the "head" "Assume there is a solution, we derive the following contradiction" and the "tail" "This contradiction proves there is no solution", while the other does not.

In fact my argument for the Three Cards was just translation of the Boolean proof into ordinary language (without the guide of the Boolean proof I would easily get lost into the complicated "paradoxical" argument), only I cut off the phrase "has no solution" from the statement, and cut off the head and the tail from the argument. As expected, the normal Boolean proof becomes a mysterious argument that leads to contradiction seemingly with no reason, that is, a "paradox".

However, removing the head does not affect the argument, because the head "Assume there is a solution" is merely an announcement for the assumption. The actual use of this assumption takes place not in the head, but in the body of the Boolean proof. Removing the tail does not affect the argument either, because the argument has finished already. Thus the solution to the Three Cards paradox must be (informally) this:

It is (the translation of) the assumption of existence of a solution that causes contradiction in the Three Cards paradox. The assumption is tacit and goes unnoticed.

It is believed traditionally that Liar-like paradoxes are logically different from Boolean problems. It is believed that in Boolean "proofs by contradiction" one assumes existence of solution and hence derives contradiction, but in Liar-like paradoxes one does not assume anything, except some basic rules of language and logic, hence contradictions must have some deep, yet unknown cause in our language or logic. The Three Cards paradox shows this is not the case.

3. Conclusions drawn from the Three Cards paradox

Up to this point I have presented up the most important idea of [12]. I believe anyone who agrees with the above analysis for the Three Cards paradox will reach by himself all the conclusions of [12], which I briefly go over now.

1. An informal solution to the Liar paradox.

   The Liar and Jourdain have the same secret. Denoting the term "The boxed sentence" by X, the Liar paradox is written as a "sentence equation"

   $$X := XF,$$

   with Boolean model

   $$x = \overline{x}.$$

   Likewise, the Jourdain's Cards paradox is written as two "sentence equations"

   $$\begin{cases} X := YT, \\ Y := XF, \end{cases}$$

   with Boolean model

   $$\begin{cases} x = y, \\ y = \overline{x}. \end{cases}$$

   One can check that, for both the Liar and Jourdain, the "paradoxical" argument is just the translation of the corresponding Boolean proof, with the head and tail removed. (Here by translation I mean logically, but not historically. Historically, the Liar paradox is perhaps 2500 years older than Mr. Boole.) Thus the solution to the Liar or Jourdain paradox is (informally) this:

Solution to the Liar and Jourdain (Informal version). It is (the translation of) the assumption of existence of a solution that causes contradiction in the Liar or Jourdain paradox.

2. The formal solution to the Liar paradox.

The above solution to the Liar or Jourdain paradox is informal. What it needs is how to translate the term "existence of solution" from Boolean algebra into our ordinary language. The term "existence" needs no translation, which is a universal term used in many disciplines. The term "solution" reduces to other two terms "given" and "equation" (in algebra a solution is just a given that satisfies an equation), which need some preparation. There are not yet corresponding notions for sentences in our ordinary language. We need first to formally establish these notions before we can do the translation.

Assuming this formal work done [12], hence the notions of "sentence given", "sentence equation", "sentence solution", and so on are all available, I can state the formal solution as follows. I take the Liar paradox. For Jourdain it is similar.

Solution to the Liar paradox (Formal version). It is the assumption of existence of a sentence solution to the Liar sentence equation that causes contradiction in the Liar argument. In other words, there can be no sentence *given* that says, of itself that it is false.

Thus the solution to the ancient Liar paradox is simply the negation of the original Liar relation, with only one word "given" put in! This sounds like cheating. But actually this is the right conclusion, as right as to say "There can be no *given* that equals to itself plus one" to which, besides the negation of the original equation $x = x + 1$, only one word "given" is put in! (What else we can say?)

3. The huge class of Liar-like paradoxes.

The reader can create a huge class of "Liar-like paradoxes," corresponding in this way to inconsistent Boolean systems. The Liar paradox, Jourdain's Cards paradox, and the Three Cards paradox are just the three simplest examples in the class. The number of sentences or cards involved can be arbitrarily large, and the argument could be arbitrarily complicated.

In fact, without help of Boolean theory, we would not suspect there is such a huge class of "paradoxes" in ordinary language. All Liar-like paradoxes have the same secret, and can be solved the same way. In particular, criteria in Boolean algebra that determine inconsistent Boolean systems become automatically criteria in ordinary language that determine paradoxical referential systems of sentences.

4. The Truth-teller.

Thus a system of sentences is paradoxical if the corresponding Boolean system has no solution. But what if the Boolean system does have solution? Here is such a problem, known as the *Truth-teller*.

The Truth-teller. | The boxed sentence is true. |

This is expressed as a sentence equation $X := XT$. The corresponding Boolean equation is hence $x = x$, which certainly has solutions. Thus Boolean diagnosis reveals nothing wrong.

But in some sense something is wrong with the Truth-teller. A diagnosis for Truth-teller is given in [12]. According to the diagnosis, the Truth-teller equation has solutions respecting some interpretations of "refer to", but no solution respecting some other interpretations. This fact is not perceivable by Boolean diagnosis. Boolean diagnosis is coarse. If it says "fine", things may not be really fine, as the Truth-teller shows. (But if it says "ill", things must be serious. Contradictions that appear in Liar-like arguments are of serious Boolean nature.)

5. The Formal work.

The main issue is to establish formally, or axiomatically, the notions of "refer to" and "sentence given". Then the notions of "sentence unknown", "sentence equation", and "sentence solution" will follow. Another issue is to make precise the correspondence between Liar-like paradoxes and inconsistent Boolean systems. While the ideas are very natural, the formal work is delicate [12], and omitted here. There is an application to the Löb's paradox in [12], which is omitted here too.

4. Given vs unknown: The main lesson

The main lesson we learned from the Liar paradox is that to distinguish between "given" objects and "unknown" objects is extremely important. In algebra this has been so standard that we never even thought about it seriously.

For instance we often say "There is no x that satisfies $x = x + 1$." (We even do not bother saying "There is no *given* x that satisfies $x = x + 1$.") But imagine a serious thinker who never learned the distinction of given and unknown might object like this: "What? No such x satisfies the expression? But you have written $x = x + 1$ in front of my eyes. I see x, and see the whole expression. Why do you say no such x exists?" This sounds strange, but if one objects like this, what shall we say? Clearly, we would say "Well, the equality $x = x + 1$ is merely a presumed one, but not a verified one. Or, the letter x in front of your eyes is merely an 'unknown', but not a 'given'. In fact by 'no x satisfies the equation' we really mean 'no *given* x satisfies the equation'". Thus the word "given" put in is really not cheating, but the key.

Indeed, without the distinction of "given" and "unknown", we would be seriously confused by such a "strange" equality $x = x + 1$ in front of our eyes. The contradiction would not be understandable, and $x = x + 1$ would become a "paradox"! Moreover, not only equalities, but also inequalities such as $x > x + 1$ would become a "paradox". In fact the problem raised by the Liar paradox is highly philosophical. For any relation, say the above semantic referential relation ":=", if there is a lack of distinction between "given" and "unknown" for sentences, hence a lack of distinction between presumed relations and verified relations, we would have many, many "paradoxes" of Liar type, and this was exactly the situation we had before. Note that it is commonly believed that the truth predicate is responsible to the paradoxical feature of the Liar paradox. Our analysis shows this is not really relevant. As long as there is a lack of distinction between "given" and "unknown", the same paradoxical story would happen anywhere, even in algebra, which involves no truth predicate. It is also commonly believed that self-reference is responsible to the paradoxical feature. This is not really relevant either. Algebra contains many self-references like $x = x + 1$ or $x = 2x + 1$, which are never regarded as "paradox".

The distinction between "given" and "unknown" is much subtler than we thought. Even we learned this distinction from algebra, we still can hardly notice that some assumption of "givenness" for sentences is implicitly used in the Liar argument. This leads our best salutation to a man who lived in more than 2000 years ago, which was about 1000 years before algebra was born:

—— "One ancient logician, Philetas of Cos, supposedly died prematurely from frustration caused by his inability to solve the problem (of the Liar paradox)." (quoted from [2])

The present paper is dedicated to him, the legendary hero of human's logical thinking.

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