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BROUWER'S FIXED POINT THEOREM

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1 General overview

Let A and B be sets with $A \subseteq B$, and $f : A \to B$. Fixed point theory identifies conditions on A and/or on f which assure that f has at least one *fixed point*; that is a point $x \in A$ exists for which $f(x) = x$.

With the theory couched in such basic terms the variety of potential approaches to the study seems almost endless. This alone has attracted interest in the subject and led to elegant discoveries. Much of the theory, however, is firmly grounded in its potential usefulness, either directly to the study of other problems, most notably differential equations, or indirectly through its impact on the study of functional analysis.

Before turning to more serious considerations we mention two abstract facts about fixed point theory, each of which is remarkable in its simplicity and yet, perhaps surprisingly, both have applications in deeper settings. The proofs are left as exercises.

Proposition 1.1 *Let S be a set and $f : S \to S$ a mapping. Then f has a fixed point if and only if there exists an integer n such that both f^n and f^{n+1} have a common fixed point. In fact the fixed point set of f coincides with the intersection of the fixed point sets of f^n and f^{n+1} for any integer $n \in \mathbb{N}$.*

Proposition 1.2 *Let S be a set and $f : S \to S$ a mapping. Then f has a fixed point if and only if there is a constant map $h : S \to S$ which commutes with f (that is, $h(f(x)) = f(h(x))$ for all $x \in S$).*

There are three major (over-lapping) branches of fixed point theory in functional analysis. The *topological* branch was inspired by the 1909 theorem of Brouwer; the *order theoretic* branch has its origins in the early set-theoretic work of Zermelo which dates back to 1908; and the *metric* branch inspired by the method of successive approximations and Banach's contraction mapping principle put forth in 1922. These branches are identified largely by the methods they employ. In most instances the underlying framework is functional analytic, although the set-theoretic branch finds wide application in the metric branch of the theory and, for example, in the study mathematical economics and logic programming languages.

In this article we discuss only the theorem credited to the celebrated Dutch mathematician L. E. J. Brouwer, along with some ideas either related to or motivated by its discovery.

Theorem 1.3 *(Brouwer) Let B denote the unit ball in euclidean space $\mathbb{R}^n$. Then any continuous mapping $f : B \to B$ has at least one fixed point.*

We begin with the case $n = 1$, which significantly predates the general case.

2 Mappings in $\mathbb{R}^1$

The Czech mathematician and philosopher Bernard Bolzano[1] (1781-1848), published the following result in 1817.

[1]It appears that historically the mathematical community has been slow to recognize many of Bolzano's contributions. In addition to the Intermediate Value Theorem and the Bolzano-Weierstrass Theorem (from which Weierstrass's name probably should be dropped), it appears that Bolzano discovered the modern definitions of convergent sequences and even the notion of a Cauchy sequence. See Russ [24]; also [7] (pp. 48 - 49) for a brief discussion.

Theorem 2.1 *Suppose f and g are continuous mappings of the closed interval $[a,b] \to [a,b]$, and suppose $f(a) < g(a)$ and $f(b) > g(b)$. Then there exists a number x_0 in the interval (a,b) for which $f(x_0) = g(x_0)$.*

Upon taking $g(x) \equiv 0$ one obtains the usual Intermediate Value Theorem in calculus, which is the version of the theorem most widely known today. However there are two fixed point theorems implicit in Bolzano's theorem. First, suppose $f : [a,b] \to [a,b]$ is continuous and define the mapping $F : [a,b] \to \mathbb{R}$ be setting $F(x) = x - f(x)$. Then, since $f(a) \geq a$, $F(a) = a - f(a) \leq 0$; similarly $f(b) \leq b \Rightarrow F(b) \geq 0$. By the Intermediate Value Theorem there exists a point x_0 in $[a,b]$ for which $F(x_0) = 0$; hence $f(x_0) = x_0$. This gives we have Brouwer's Theorem in the case $n = 1$.

Theorem 2.2 *Any continuous mapping $f : [a,b] \to [a,b]$ has a fixed point.*

On the other hand, the case $g(x) \equiv x$ in Theorem 2.1, produces another fixed point theorem.

Theorem 2.3 *Any continuous mapping $f : [a,b] \to \mathbb{R}$ for which $f(a) \leq a$ and $f(b) \geq b$ has a fixed point.*

It is apparent that any generalization the Intermediate Value Theorem should also lead to a generalized fixed point theorem. Here is a recent example (see [10]). A mapping f is said to be *upper semicontinuous* on $[a,b]$ if the set

$$S_t = \{x \in [a,b] : f(x) \geq t\}$$

is closed for each $t \in \mathbb{R}$. Therefore, if $\{x_n\}$ is any sequence in S_t which converges to $x_0 \in [a,b]$ then it must be the case that $f(x_0) \geq t$. In particular, $f(x_0) \geq \limsup_n f(x_n)$. This leads to the following definition. A mapping $f : [a,b] \to \mathbb{R}$ is said to be *upper semicontinuous from the right* if $f(x_0) \geq \limsup_n f(x_n)$ for each sequence $\{x_n\}$ in $[a,b]$ such that $x_n \downarrow x_0 \in (a,b]$. Similarly, f is *lower semicontinuous from the left* if $f(x_0) \leq \liminf_n f(x_n)$ for each sequence $\{x_n\}$ in $[a,b]$ for which $x_n \uparrow x_0 \in [a,b)$.

Theorem 2.4 *Let $f : [a,b] \to \mathbb{R}$ be a mapping which is upper semicontinuous on the right and lower semicontinuous on the left. If $f(a) \leq a$ and $f(b) \geq b$, then f has a fixed point.*

3 Brouwer's Theorem: $n = 2$

Because of the absence of an appropriate analog of the Intermediate Value Theorem in higher dimensions, one immediately encounters difficulties extending Brouwer's Theorem even to the case $n = 2$; indeed none of the many proofs that have been given for Brouwer's Theorem is significantly simpler in the case $n = 2$ than in the general case. However there is one approach to Brouwer's Theorem that is surprisingly elementary from a conceptual point of view. We describe this approach in the case $n = 1$, show how it extends to the case $n = 2$, and then indicate how it extends to the general case. Our point of departure is the following simple fact about the real line.

Proposition 3.1 *Let* $\mathcal{P} : (0 = P_0 < P_1 < \cdots < P_k = 1)$ *be a partition of the interval* $[0, 1]$ *with each of the points* P_i *'labeled' either* 0 *or* 1. *Then the cardinality of the set* $N_{\mathcal{P}} = \{i \in \{1, \cdots, k\} : P_{i-1}$ *and* P_i *are labeled differently*$\}$ *is an odd integer.*

This proposition has a straightforward induction proof the reader might want to try. However we motivate the approach to Brouwer's theorem given below with a 'counting' argument. Begin with the ordinary unit interval $[0, 1]$ with $P_0 = 0$ and $P_1 = 1$, and label points $P_i \in (0, 1)$ arbitrarily with 0's and 1's. We now count the number of 'endpoints' of intervals defined by the partition which are labeled 0 in two ways. First, if an endpoint labeled 0 is in $(0, 1)$ then it must be counted twice since it is a 0 endpoint of two different intervals. By adding P_0 to these endpoints we see that there are an odd number of such endpoints, say k. Now divide the subintervals defined by the partition into two subclasses: (i) Those which have only 0 as endpoints and (ii) those which have a 0 and a 1 as an endpoint. If there are m intervals in class (i) then these m intervals contribute $2m$ endpoints to the counting. If there are n intervals in class (ii) then, since each such interval contributes only one endpoint to the total, we conclude that $2m + n = k$. This proves that n must be an odd number and establishes Proposition 3.1. Note in particular that the conclusion of Proposition 3.1 implies $N_{\mathcal{P}} \neq \emptyset$.

We now show how to use Proposition 3.1 to give proof of Brouwer's Theorem in $\mathbb{R}^1$. While this proof is no simpler than the one usually given for the Intermediate

Value Theorem, we shall see that this same line of argument leads to a simple proof of Brouwer's Theorem in $\mathbb{R}^2$, and in turn it points the way to a proof of Brouwer's theorem in its full generality. In contrast to this, the Intermediate Value Theorem fails to *suggest* a higher dimensional proof. (However this does not mean that no such proof exists. Indeed the Intermediate Value Theorem can be exploited to give a proof of Brouwer's Theorem in the case $n = 2$; see, for example, [7], p. 71.)

To proceed with the proof in $\mathbb{R}^1$ we prove the following. The reader might think of quick proofs of this proposition, but it is the *method* of proof that we are interested in here.

Proposition 3.2 *Let* C_0 *and* C_1 *be two closed sets in* $\mathbb{R}^1$ *with* $0 \in C_0$ *and* $1 \in C_1$, *and suppose* $[0,1] \subset C_0 \cup C_1$. *Then* $C_0 \cap C_1 \neq \emptyset$.

Proof. For each $k \in \mathbb{N}$ let $\mathcal{P}_k$ be a partition of $[0,1]$ into subintervals of equal length. Assign the label 0 to P_0 and 1 to P_k. For $i \in \{2, \cdots, k-1\}$ assign labels as follows. If $P_i \in C_0$ assign the label 0 to P_i; otherwise assign the label 1 to P_i. Since $[0,1] \subset C_0 \cup C_1$ this results in a labeling of $\mathcal{P}_k$ as in Proposition 3.1. Consequently for each k there exists an interval of length $1/k$ whose endpoints $\{P_k, Q_k\}$ are labeled differently, say P_k is labeled 0 and Q_k is labeled 1. Thus for each k, $P_k \in C_0$ and $Q_k \in C_1$. It is possible to choose convergent subsequences $\{P_{k_j}\}$ and $\{Q_{k_j}\}$ with, say, $\lim_{j\to\infty} P_{k_j} = P$ and $\lim_{j\to\infty} Q_{k_j} = Q$. Since $\lim_{j\to\infty} |P_{k_j} - Q_{k_j}| = \lim_{j\to\infty} 1/k_j = 0$, $P = Q$, and since both C_0 and C_1 are closed this point lies in $C_0 \cap C_1$. ■

It is somewhat remarkable that the above ideas provided the basis for the general proof of Brouwer's theorem. Proposition 3.1 is a special case of a celebrated result known as Sperner's Lemma while Proposition 3.2 is a special case of a famous result due to Knaster, Kuratowski and Mazurkiewicz, known as the KKM Theorem ([18]). We shall discuss more general cases below. First, however, we show how the above results lead another proof of the fact that any continuous mapping $f : [0,1] \to [0,1]$ has a fixed point. Notice that every point in $P \in [0,1]$ can be written in the form

$$P = \lambda_0(P) \cdot 0 + \lambda_1(P) \cdot 1$$

where $\lambda_0(P) + \lambda_1(P) = 1$ by simply taking $\lambda_1(P) = P$ and $\lambda_0(P) = 1 - P$. Set

$$C_0 = \{P \in [0,1] : \lambda_0(f(P)) \leq \lambda_0(P)\};$$
$$C_1 = \{P \in [0,1] : \lambda_1(f(P)) \leq \lambda_1(P)\}.$$

Thus

$$C_0 = \{P \in [0,1] : P \leq f(P)\};$$
$$C_1 = \{P \in [0,1] : f(P) \leq P\}.$$

Also $0 \in C_0$ and $1 \in C_1$. The assumption $P \notin C_0 \cup C_1$ for some $P \in [0,1]$ implies both $f(P) > P$ and $f(P) < P$ which is absurd, so it must be the case that $[0,1] \subset C_0 \cup C_1$. We now invoke Proposition 3.2 to conclude $C_0 \cap C_1 \neq \emptyset$. But if $P \in C_0 \cap C_1$ then clearly $f(P) = P$.

We now give a detailed proof of Sperner's Lemma in the case $n = 2$ and we also indicate how to prove the theorem for arbitrary n. Our approach is inspired by a discussion found in Zeidler [30] (p. 798).

Let M be a closed triangle in $\mathbb{R}^2$ with vertices P_0, P_1, P_2. In order to understand the general case it is important to be precise about some definitions. The *r-dimensional sides* of M are:

The vertices P_0, P_1, P_2 for $r = 0$.

The sides $\overline{P_0P_1}, \overline{P_1P_2}, \overline{P_2P_0}$ for $r = 1$.

The triangle itself for $r = 2$.

We define the *base of a point P in M* to be the side of M of smallest dimension which contains P. An integer i ($i = 0, 1, 2$) is said to be a *Sperner label* for P if P_i belongs to the base of P in M. It is important to understand exactly what this means. The points P_0, P_1, P_2 are labeled $0, 1, 2$ respectively. Any *other* point on the side $\overline{P_0P_1}$ may be labeled either 0 or 1 regardless of how any other such points are labeled. The same holds for the remaining sides. Any point *strictly* inside M may be labeled 0, 1 or 2.

The following is Sperner's Lemma [28] for the case $n = 2$.

Lemma 3.3 *Suppose the triangle M of $\mathbb{R}^2$ with vertices P_0, P_1, P_2 is divided into subtriangles, and suppose to each vertex of the subtriangles is assigned a Sperner* *label. Then the cardinality of the set of subtriangles whose vertices have distinct labels is odd.*

Proof. Call a subtriangle of the triangulation of M a *Sperner simplex* if its vertices have distinct labels, and call a side of a subtriangle *distinguished* if its vertices carry the labels $\{0, 1\}$.

Notice that if a distinguished side of T has a vertex which lies in the interior of M then this side is also a distinguished side of another subtriangle in the triangulation of M. Thus every distinguished side which has a vertex lying strictly inside the triangle M must be counted two times, and so there are an even number of such distinguished sides. On the other hand, by Proposition 3.1, there are an odd number of distinguished sides lying on $\overline{P_0P_1}$, because each such side is side of only one triangle. Therefore the total number of distinguished sides is an odd integer, say k. At the same time, the vertices of any subtriangle which has distinguished sides and which is *not* a Sperner triangle must be labeled either $\{0, 1, 1\}$ or $\{0, 0, 1\}$. Each such triangle has precisely two distinguished sides. Suppose there are m subtriangles of this type and suppose there are n Sperner triangles. Then $2m + n = k$ and we conclude that n is odd. ■

We now prove the KKM Theorem for the case $n = 2$. Note that here we use M inclusively, to denote the vertices of the triangle along with its sides and interior.

Proposition 3.4 *Suppose M is a triangle in $\mathbb{R}^2$ with vertices P_0, P_1, P_2, and suppose C_0, C_1, C_2 are closed subsets of $\mathbb{R}^2$ which satisfy*
(i) $P_i \in C_i$, $i = 0, 1, 2$;
(ii) $P_iP_j \subset C_i \cup C_j$, $i, j = 0, 1, 2$;
(iii) $M \subset C_0 \cup C_1 \cup C_2$.
Then $\cap_{i=0}^{2} C_i \neq \emptyset$.

The proof Proposition 3.4 closely follows that of Proposition 3.2. For $k = 1, 2, \cdots$ consider a sequence of triangulations $\mathcal{P}_k$ of M where the diameter of the largest triangle in $\mathcal{P}_k$ is ϵ_k where $\epsilon_k \to 0$ as $k \to \infty$. (Intuitively it is clear that such a triangulation can be always be constructed; describing a precise algorithm for doing so is a little more difficult). Assign a Sperner labeling to each triangulation with the added provision that each vertex Q is assigned a label i consistent with $Q_i \in C_i$. This restriction is important, and (ii) and (iii) insure that it can be done. Then by Lemma 3.3 each triangulation $\mathcal{P}_k$ contains a Sperner simplex with vertices $\{P_0^{(k)}, P_1^{(k)}, P_2^{(k)}\}$ where $P_i^{(k)} \in C_i$, $i = 0,1,2$. Now choose respective convergent subsequences $\{P_i^{(k_j)}\}_{j=1}^{\infty}$ $(i = 0,1,2)$ with

$$\lim_{j\to\infty} P_i^{(k_j)} = Q_i \ (i = 0,1,2).$$

Since the diameters of the subtriangles tend to zero as $j \to \infty$ it must be the case that $Q_0 = Q_1 = Q_2 := Q$, and since each of the sets C_i is closed, $Q \in \cap_{i=0}^{2} C_i$.

We are now in a position to prove Brouwer's Theorem for the case $n = 2$.

Theorem 3.5 *Suppose M is a triangle in $\mathbb{R}^2$ with vertices P_0, P_1, P_2, and suppose $f : M \to M$ is continuous. Then f has a fixed point.*

Outline of Proof. Every point P in M can be written in the form

$$P = \lambda_0(P)P_0 + \lambda_1(P)P_1 + \lambda_2(P)P_2$$

where $0 \le \lambda_i(P) \le 1$ $(i = 0,1,2)$ and $\lambda_0(P) + \lambda_1(P) + \lambda_2(P) = 1$.

Set $C_i := \{P \in M : \lambda_i(f(P)) \le \lambda_i(P)\}$, $i = 0,1,2$. Since f is continuous each of the sets C_i is closed. Also, since $\lambda_i(P_i) = 1$ for each i, $P_i \in C_i$ for $i = 0,1,2$. It is easy to check that conditions (ii) and (iii) of Proposition 3.4 hold for the sets C_i, so we conclude that there exists $P \in \cap_{i=0}^{2} C_i$. This implies

$$\lambda_i(f(P)) \le \lambda_i(P), \ i = 0,1,2$$

which in turn implies $1 = \sum_{i=0}^{2} \lambda_i(f(P)) \le \sum_{i=0}^{2} \lambda_i(P) = 1$. Therefore $\lambda_i(f(P)) = \lambda_i(P)$, $i = 0,1,2$, and thus $f(P) = P$. ■

The difficulties in extending Brouwer's Theorem to higher dimensions largely involve terminology. An *n-dimensional closed simplex* in $\mathbb{R}^n$ is the convex hull of $n+1$ points $P_0, \cdots, P_n$ in $\mathbb{R}^n$ which do not lie in an $n-1$- dimensional subspace of $\mathbb{R}^n$. As before, the *base* of a point in M is the subsimplex of lowest dimension of which the point lies. To see how the induction goes we illustrate how to pass from $n = 2$ to $n = 3$.

Let M be a closed 3-dimensional simplex in $\mathbb{R}^3$, and partition M into 3-dimensional subsimplexes, and assign to each vertex of the partition a number which belongs to its base in M. Now call a triangular face of a subsimplex *distinguished* if it carries the labels $\{0, 1, 2\}$, and observe that if a vertex of a distinguished triangle lies in interior of M then it is a face of two subsimplexes of the partition. By the case $n = 2$ there are an odd number of distinguished triangles on the boundary of M. (Here it is important to observe that *only the face initially labeled* $\{0, 1, 2\}$ can have distinguished triangles.) Therefore the total number of distinguished triangles is odd. On the other hand the vertices of a simplex which is *not* a Sperner simplex (i.e., one which does not have distinguished labels) and which *does have a distinguished triangle* must have one of the following labellings: $\{0, 1, 2, 0\}$, $\{0, 1, 2, 1\}$, or $\{0, 1, 2, 2\}$. Each such simplex has precisely two distinguished triangular faces. If there are m such simplexes and if n are the number of Sperner simplexes, then $2m + n$ must equal an odd number; whence n is odd.

A statement of Sperner's Lemma in its full generality requires a little more explanation. Let $M^n = [P_0, \cdots, P_n]$ be a closed n-dimensional simplex in $\mathbb{R}^n$. By a subdivision of M^n we mean a decomposition of M^n into finitely many non-overlapping n-simplexes $s_1, \cdots, s_k$ such that (1) the intersection of any two simplexes in the subdivision is either empty or a common face of each, and (2) each $n-1$-simplex in the subdivision that is not on the boundary of M^n is a common face of exactly two n-simplexes of the subdivision.

Lemma 3.6 [*Sperner*] *Let S be a subdivision of M^n and label each vertex $P \in S$ with one of the numbers $\{i_0, \cdots, i_s\}$ whenever $[P_{i_0}, \cdots, P_{i_s}]$ is the base of v in M^n. Then the number of simplexes in S which are labeled $\{0, \cdots, n\}$ is odd.*

The proof is by induction with the inductive step applied on the boundary of M^n.

Sperner's Lemma can be used to extend the KKM theorem to higher dimensions, and this in turn leads to Brouwer's Theorem. The general version of the KKM theorem needed to carry this out is the following. The path to the general case of Brouwer's Theorem closely follows that given in the case $n = 2$.

Theorem 3.7 *Suppose $\{P_0, \cdots, P_n\}$ is a finite set in $\mathbb{R}^n$ and suppose $C_0, \cdots, C_n$ are closed subsets of $\mathbb{R}^n$ for which*

$$conv\{P_{i_0}, \cdots, P_{i_k}\} \subset \cup_{j=0}^{k} C_{i_j}$$

for every subset $\{P_{i_0}, \cdots, P_{i_k}\}$ of $\{P_0, \cdots, P_n\}$. Then

$$\cap_{i=0}^{n} C_i \neq \emptyset.$$

4 Some consequences Brouwer's Theorem

We stated Brouwer's Theorem for the closed unit ball, yet for $n = 2$ we discussed only the case when the domain is a triangle. The fact that this restriction is inconsequential follows from the observations below.

If X is a topological space and $M \subseteq X$, then a continuous mapping $r : X \to M$ is called a *retraction* if $r(x) = x$ for all $x \in M$. When this occurs, M is said to be a *retract* of X.

We begin with the following simple fact.

Proposition 4.1 *Every closed convex subset K in $\mathbb{R}^n$ is a retract of $\mathbb{R}^n$.*

Proof. . Let $p \in \mathbb{R}^n$ and set

$$r = \inf\{\|p - x\| : x \in K\}.$$

Then it is possible to choose a sequence $\{x_n\} \subseteq K$ such that

$$\lim_{n\to\infty} \|x_n - p\| = r.$$

Since the sequence $\{x_n\}$ is bounded it has a convergent subsequence. (An extension of the Bolzano-Weierstrass Theorem assures that K is compact). So assume that $\lim_{k\to\infty} x_{n_k} = x$. Then $x \in K$ because K is closed, and clearly $\|x - p\| = r$. ■

Now suppose there are two points $x, y \in K$, such that

$$\|x - p\| = \|y - p\| = r.$$

Suppose $x \neq y$ and consider the point $m = \frac{1}{2}(x + y)$. Then by the Pythagorean Theorem

$$\|p - m\|^2 + \|m - x\|^2 = r^2.$$

If $m \neq x$ this would imply $\|p - m\| < r$ – a contradiction since convexity of K implies $m \in K$. Therefore $m = x = y$. This assures that for each point $p \in \mathbb{R}^n$ there is a unique point $\rho(p) \in K$ which is nearest p.

We assert that ρ is the desired retraction. To do this we need to show that ρ is continuous. However (and we omit the details) it is possible to show even more; namely that ρ is nonexpansive, that is, for each $u, v \in \mathbb{R}^n$

$$\|\rho(u) - \rho(v)\| \leq \|u - v\|.$$

The above fact permits an extension of Brouwer's Theorem to arbitrary bounded closed convex subsets of $\mathbb{R}^n$. For such a set K select a simplex S sufficiently large that $K \subseteq S$ (remember, K is bounded). By Proposition 4.1 there exists a retraction $r : S \to K$. The composition map $f \circ r$ is a continuous mapping of S into itself and therefore must have a fixed point x. Moreover, x must lie in K. Since $r(x) = x$ for points in K, we have

$$x = f \circ r(x) = f(x).$$

Therefore we have the following.

Theorem 4.2 *Let K be a bounded closed convex subset of $\mathbb{R}^n$ and let $f : K \to K$ be continuous. Then f has a fixed point.*

One very interesting consequence of Brouwer's theorem is the following fact, which seems intuitively clear.

Proposition 4.3 *The boundary, ∂B, of a nontrivial closed ball B in $\mathbb{R}^n$ is* ***not*** *a retract of B.*

Proof. Suppose r is a retraction of B onto ∂B. Then $-r$ would be a continuous mapping of B into B which is fixed point free, and this contradicts Brouwer's Theorem. ∎

In fact Proposition 4.3 is equivalent to Brouwer's Theorem. To see this, suppose $f : B \to B$ is continuous. If $f(x) \neq x$ for each $x \in B$ then one could construct a retraction $r : B \to \partial B$ as follows. For each x follow the directed line segment from $f(x)$ through x and find its intersection with $S(x; r)$. Call this point $r(x)$. The existence of such a retraction contradicts Proposition 4.3. Therefore f must have a fixed point.

In 1935 S. Ulam posed the following problem (Problem 36) in the famous collection of mathematical problems known as *The Scottish Book* (named after the Scottish Cafe in what was then Lwøw, Poland - a cafe in which Stefan Banach and his collaborators frequently gathered to discuss mathematics): "Can one transform continuously the solid sphere of a Hilbert space into its boundary such that the transformation should be the identity on the boundary of the ball?" In other words, Ulam is asking whether Proposition 4.3 holds in infinite dimensional Hilbert space. An addendum indicates that Tychonoff provided an affirmative answer to Ulam's question; thus showing that Proposition 4.3, hence Brouwer's Theorem, fails in infinite dimensional.

Another nice solution to Ulam's problem, and one that holds in an arbitrary Banach, space was published by Victor Klee in 1955. In [16] Klee proved that any infinite dimensional Banach space X is homeomorphic with the punctured space $X\backslash\{0\}$. Let $h : X \to X\backslash\{0\}$ be such a homeomorphism. (Thus h is continuous, one-to-one, and surjective (onto), and h^{-1} is also continuous). Now assume (as one may) that $h(x) = x$ if $x \in X$ and $\|x\| \geq 1$. The required retraction is now given by the mapping $R : X \to h(x)/\|h(x)\|$.

It is also important to realize that Brouwer's Theorem extends to $\mathbb{R}^n$ with any other norm. This follows from an elementary fact in functional analysis: Any linear mapping $T : E \to F$ where E and F are Banach spaces with E finite dimensional is necessarily continuous. In particular the identity mapping $I : (\mathbb{R}^n, \|\cdot\|_2) \to (\mathbb{R}^n, \|\cdot\|)$

is continuous. Thus there exist constants $\alpha, \beta \in R$ such that for any $x \in \mathbb{R}^n$

$$\alpha \|x\| \leq \|x\|_2 \leq \beta \|x\|$$

where, of course, $\|\cdot\|_2$ denotes the usual euclidean norm in $\mathbb{R}^n$. Consequently any closed convex set K_1 in $(\mathbb{R}^n, \|\cdot\|)$ is homeomorphic to a closed convex set K_2 in $(\mathbb{R}^n, \|\cdot\|_2)$ via the identity mapping $I : K_2 :\to K_1$.

Since any finite dimensional Banach space can be viewed as $\mathbb{R}^n$ with an equivalent norm simply by identifying the respective basis vectors, Brouwer's Theorem can be restated as follows.

Theorem 4.4 *Suppose K is a nonempty bounded closed convex subset of a finite dimensional Banach space X, and suppose $f : K \to K$ is continuous. Then f has at least one fixed point.*

REMARKS. We mention some further historical comments. Brouwer announced his result in 1908 and proved the case $n = 3$ in 1909. In 1910 Hadamard gave a proof using Kronecker's index theory. Brouwer gave another proof (for arbitrary n) in 1912 using the theory of topological degree. In devising this proof the fundamental properties of classical topological degree were explicitly formulated by Brouwer for the first time, although these ideas were implicitly used by Poincaré and others much earlier (see [6]). The underlying idea is the continuation method. Suppose $f : B \to \mathbb{R}^n$ is continuous, and let $B_t := (I - tf)(B)$ for $t \in [0, 1]$. Clearly $0 \in B_0$. The idea is to show that the condition $x - tf(x) \neq 0$ for all x in the boundary of B and $t \in (0, 1)$ implies $0 \in B_1$. The approach we have just outlined, based on Sperner's Lemma appeared in 1929 in a joint paper of Knaster, Kuratowski, and Mazurkiewicz [18].

Finally, it is interesting, indeed curious, that Brouwer's later embrace of intuitionist logic caused him later to reject the Bolzano-Weierstrass Theorem, the result which forms the basis of the proof of Intermediate Value Theorem (see [5]). Intuitionists reject "arguments by contradiction", an approach readily accepted by most mathematicians. Thus Brouwer's fixed point theorem has had a tremendous impact on mathematics, while his approach to logic (about which he held strong views) has been largely ingored by mathematicians.

The history of mathematics is replete with instances in which ideas have been mis-attributed (recall our comment about Bolzano). This question also arises in the case of Brouwer's Theorem. In 1904 the German-Baltic mathematician Piers Bohl published a proof of the following theorem (stated here in the style of today's writing).

Theorem 4.5 *(Bohl) Suppose $T : B \to \mathbb{R}^n$ is continuous (B is the unit ball in $\mathbb{R}^n$) and suppose $T(x) \neq 0$ for all $x \in B$. Then there exists a point x in the boundary (surface) of B and a number $\mu < 0$ such that*

$$T(x) = \mu x.$$

In fact the above result was even known to Henri Poincaré as early as 1883, who announced it without proof in [22] but later, in 1886, published an argument ([23]) which forms the basis of a proof. This latter result has come to be known as the theorem of Miranda who, in 1940 [21], showed it actually to be equivalent to Brouwer's Theorem.

It is easy to see that Bohl's Theorem implies Brouwer's Theorem. Suppose $f : B \to B$ is continuous. Notice that since $f : B \to B$ it is necessarily the case that $f(x) \neq \lambda x$ for all x in the boundary of B and $\lambda > 1$. Let $f = I - T$. Then

$$x - T(x) \neq \lambda x$$

for all x in the boundary of B and $\lambda > 1$. This in turn implies that

$$T(x) \neq (1 - \lambda)x = \mu x$$

for all x in the boundary of B and $\mu < 0$. By Bohl's Theorem there exists $x \in B$ such that $T(x) = 0$. But

$$T(x) = 0 \Leftrightarrow x - f(x) = 0 \Leftrightarrow f(x) = x.$$

The first major extension of the Bohl-Brouwer Theorem was obtained by Juliusz Schauder in 1930, although Birkhoff and Kellogg had earlier extended it to compact convex subsets of certain function spaces. This theorem can be derived from Brouwer's Theorem in a relatively straightforward way using finite dimensional approximations.

Theorem 4.6 *(Schauder) Suppose K is a compact and convex subset of an arbitrary Banach space. Then every continuous mapping $f : K \to K$ always has at least one fixed point.*

This was followed in 1934 by a theorem known as the Schauder-Tychonoff Theorem.

Theorem 4.7 *(Schauder-Tychonoff) Suppose K is a compact and convex subset of a locally convex Hausdorff linear topological vector space. Then every continuous mapping $f : K \to K$ has at least one fixed point.*

The following is a stability result which can also be derived from Brouwer's Theorem. A mapping f from a topological X space into a metric space is said to be *ε-continuous* if given any $x \in X$ there is a neighborhood U_x of x for which $diam(f(U_x)) \leq \varepsilon$.

Theorem 4.8 *(Klee) Suppose K is a compact convex subset of a normed linear space, and suppose $f : K \to K$ is ϵ-continuous. Then there exists a point $x_0 \in K$ such that $\|f(x_0) - x_0\| \leq \epsilon$.*

5 Another look at $\mathbb{R}^1$

Simple questions are sometimes difficult (or impossible!) to answer. In 1954 Eldon Dyer posed the following question: If two continuous mappings $f, g : [0,1] \to [0,1]$ commute, do they necessarily have a common fixed point? In fact, A. L. Shields posed the same question in 1955, as did Lester Dubins in 1956, but the problem first appeared in the literature as part of a more general question posed by J. R. Isbell in 1957 [13]. This question immediately attracted a lot of attention, in part because it had long been known that the answer is yes for polynomial functions. In 1964 R. DeMarr [8] gave a partial positive answer by showing that if f and g satisfy

$$|f(x) - f(y)| \leq \alpha\,|x - y| \text{ and } |g(x) - g(y)| \leq \beta\,|x - y|$$

for all $x, y \in [0,1]$, where $\beta < (\alpha+1)/(\alpha-1)$, then f and g must have a common fixed point. Another partial answer to the question was established by Schwartz in 1965, who showed in [26] that if f has a continuous derivative, then there is a common fixed point of f and some iterate of g.

However the general question resisted a complete solution for over ten years. In 1967 both W. M. Boyce [3] and H. Huneke [12] gave counterexamples to the problem. In fact, Boyce describes a counterexample in which f and g are both lipschitzian.

We conclude by looking back to our opening remarks. Suppose $f : [0,1] \to [0,1]$ has the property that for some $n \in \mathbb{N}$ both f^n and f^{n+1} are continuous. Does f have a fixed point? Notice that f^n and f^{n+1} commute, so in view of Proposition 1.1 the answer is 'yes' if f^n and f^{n+1} have a common fixed point.

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