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Lattice Points

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1 The unit square lattice.

A **lattice point** in the plane is a point with integer coordinates. The set of all lattice points is a configuration arranged in equally spaced rows and columns called the **unit square lattice**. It is remarkable that this simple pattern has attracted the attention of many celebrated mathematicians for nearly two centuries. Ever since the time of Gauss, many interesting and deep properties of the integers have been discovered by studying the unit square lattice. In fact, a branch of mathematics known as **geometric number theory**, or the **geometry of numbers**, was created as a result of investigations concerning lattice points.

The concept of lattice point can be extended in an obvious way to 3-space, and more generally to n-space for any integer $n \geq 2$. We will discuss primarily the unit square lattice in the plane, with occasional remarks about extensions to higher dimensions.

A monograph by J. Hammer [6] gives a splendid compendium of known results and describes many unsolved problems. This article treats a small sample of theorems concerning lattice points and outlines some of the methods used to prove them. We begin with the simplest types of theorems.

2 Pick's Theorem.

When the plane is covered by unit squares, like those on graph paper, the vertices of the squares are the lattice points. The same lattice can be generated by covering the plane with parallelograms having lattice points as vertices, provided there are no further lattice points inside or on the boundary of each parallelogram. Figure 1 shows some examples of parallelograms that could be used to cover the plane and whose vertices generate the unit square lattice.

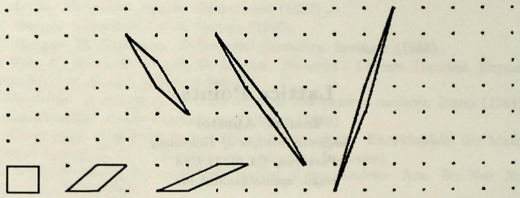

Figure 1. Parallelograms generating the unit square lattice.

The first observation we make is that all these parallelograms have equal area. That is, *any covering parallelogram whose vertices generate the unit square lattice must have area equal to 1.*

We will deduce this result as a corollary of a more general theorem concerning **lattice polygons**, that is, polygons all of whose vertices are lattice points. Figure 2 shows an example.

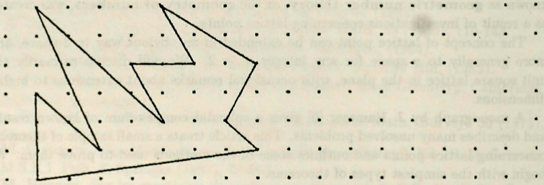

Figure 2. A lattice polygon. All its vertices are lattice points.

There is a remarkable formula, discovered around 1900 by G. Pick [10], which states that the area of any lattice polygon can be calculated by counting lattice points.

PICK'S THEOREM. *Given a lattice polygon K, let $A(K)$ denote the area of the region enclosed by K. Let $I(K)$ denote the number of lattice points inside K, and let*

B (K) denote the number lattice points on the boundary of K. Then we have

$$A(K) = I(K) + \frac{1}{2}B(K) - 1. \tag{1}$$

In the example shown in Figure 2 we have 15 lattice points inside and 18 on the boundary, so $A(K) = 15 + 9 - 1 = 23$.

A proof of Pick's formula can be given by induction on the number of edges of the polygon, after making the observation that the expression on the right of (1) is an additive function of the polygon K.

3 Applications of Pick's formula.

We mention two simple applications of Pick's formula. If K is a parallelogram whose vertices generate the unit square lattice, then $I(K) = 0, B(K) = 4$, and the formula gives $A(K) = 1$. This proves the result stated earlier, that any parallelogram (such as those in Figure 1) whose vertices generate the unit square lattice must have area equal to 1.

Another application of Pick's formula shows that there is no **regular** lattice polygon except for the square. Thus, for example, there is no equilateral lattice triangle.

To see why, take a regular lattice polygon K with n sides, each of length a, say, and calculate the area $A(K)$ in two ways. Pick's formula tells us that the area is rational because it is either an integer or an integer plus $\frac{1}{2}$. On the other hand, the diagram in Figure 3 shows that K is the union of n isosceles triangles, each with area $\frac{1}{4}a^2 \cot\frac{\pi}{n}$.

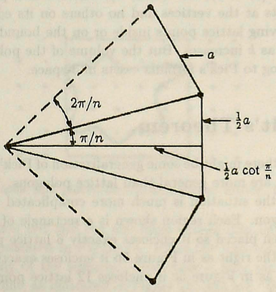

Figure 3. Area of a regular polygon with n sides is n times that of an isosceles triangle.

Therefore $A(K) = \frac{1}{4}na^2 cot\ \frac{\pi}{n}$. But a^2 is an integer, as is easily seen by the Pythagorean Theorem, so the last formula shows that $cot\ \frac{\pi}{n}$ is rational. But it is known ([8], p. 41) that if $n \geq 3$, $cot\ \frac{\pi}{n}$ is rational only for $n = 4$. Therefore K must be a square.

Incidentally, the square need not have its edges parallel to the lines of the lattice. Figure 4 shows some examples of lattice squares whose edges are not parallel to the lines of the lattice.

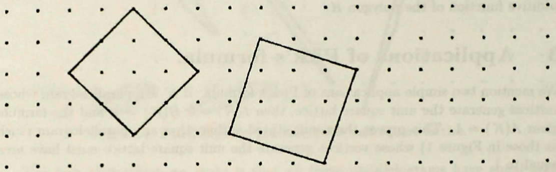

Figure 4. Lattice squares with edges not horizontal or vertical.

Perhaps it is not too surprising that there is a simple formula like Pick's Theorem for determining areas by counting lattice points. After all, lattice polygons are very special figures. But there is no 3-dimensional analog of Pick's formula for finding the volume of a lattice polyhedron by counting lattice points inside, on the faces, and on the edges. To see why, consider the tetrahedron whose four vertices are at (0, 0, 0), (1, 0, 0), (0, 1, 0), and (1, 1, k), where k is a positive integer. There are four lattice points at the vertices and no others on its edges, faces, or interior, so any formula involving lattice points inside or on the boundary of this polyhedron would stay constant as k increases. But the volume of the polyhedron increases as k increases, so no analog to Pick's formula exists in 3-space.

4 Blichfeldt's Theorem.

It is natural to ask if there might be some generalization of Pick's Theorem that applies to plane figures that are more general than lattice polygons. The three examples in Figure 5 show that the situation is much more complicated when the region is no longer a lattice polygon. Each region shown is a rectangle of area 10. In Figure 5a the rectangle has been placed so it encloses exactly 6 lattice points. By shifting the rectangle slightly to the right as in Figure 5b it encloses exactly 8 lattice points, and by raising it slightly as in Figure 5c it encloses 12 lattice points. Thus, the number of lattice points inside a region of given area can vary considerably, depending on where the region is located. Notice that in Figure 5c the number of lattice points enclosed, 12, is greater than the area of the rectangle. This is no accident. There is a remarkable theorem of H. F. Blichfeldt [3], discovered in 1914, that explains this.

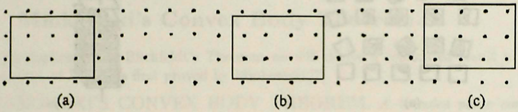

Figure 5. The number of lattice points enclosed depends on the location of the region.

BLICHFELDT'S THEOREM. *Let K denote a bounded plane region with positive area $A(K)$. Then there exists a translation of K that contains at least $A(K)$ lattice points in its interior.*

This theorem is remarkable because nothing is assumed about the shape of the region. We assume only that it is bounded, which means it can be enclosed in a large square. The other remarkable feature of this theorem is that a proof can be given by a simple intuitive geometric argument.

To be specific, suppose a region of area 11.7 has the shape shown in Figure 6. Place it anywhere on the lattice. In the position shown in Figure 6 it contains 7 lattice points inside. Place a large lattice square around the region, in this case, the 5×5 square shown.

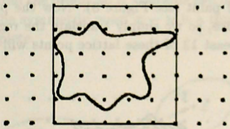

Figure 6. A region of area 11.7 placed on the unit square lattice. It contains 7 lattice points inside.

Next, paint the region red and cut the large square into unit squares along the lines of the lattice as indicated in Figure 7a. Place the unit squares in a pile, being careful not to rotate any of the squares. Then take a long pin and pierce the pile through an arbitrary point, as indicated in Figure 7b, and ask "How many times does the pin strike red paint?" Call this number N. *What are the possible values of N?*

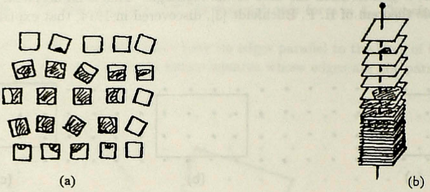

Figure 7.

For some regions it might happen that the pin is placed in such a position that it never strikes red paint, in which case $N = 0$. It's also conceivable that we could have a region such that the pin could strike red paint on each of the 25 squares. In other words, we have $0 \leq N \leq 25$. Of course, this is not very profound. What *is* profound, is that *there is at least one position of the pin for which* $N \geq 12$. If not, we would have $N \leq 11$ for all positions of the pin, and this would tell us that each point of the bottom square is covered by at most 11 red points on or directly above it, contradicting the fact that the total amount of red paint is 11.7. Therefore, $N \geq 12$ for some position of the pin.

We place the pin in this position, then remove the pin and replace the squares in their original positions on the lattice. The original figure is now restored, except for a collection of equally spaced pin holes where we placed the pin. At least 12 of these pin holes go through red paint (see Figure 8). Now the pin holes have exactly the same spacing as the lattice, so we simply translate the region so these holes fall on lattice points. Then at least 12 of these lattice points will lie inside the region, and the proof is complete. ∎

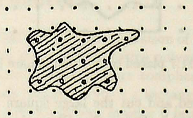

Figure 8. Proof of Blichfeldt's Theorem.

For those who do not think the foregoing argument is rigorous, rest assured that this argument can be converted into a rigorous proof using double integrals. In fact, if we use the same argument with n-fold integrals instead of double integrals we can establish the following generalization of Blichfeldt's Theorem in n-space.

If a bounded region K in n-space has n-dimensional volume $V(K) > 0$, then some translation of K contains at least $V(K)$ lattice points in its interior.

5 Minkowski's Convex Body Theorem.

As an application of Blichfeldt's Theorem we will derive a celebrated result in the Geometry of Numbers first proved by Minkowski [7].

MINKOWSKI'S CONVEX BODY THEOREM. *A bounded plane convex region K that is symmetric about a lattice point in its interior and has area $A(K) > 4$ contains at least 3 lattice points in its interior.*

Proof. Without loss of generality we can assume that K has its center of symmetry at the origin. Consider the region $K' = \frac{1}{2}K$ obtained by multiplying the coordinates of all points of K by $\frac{1}{2}$. The new region K' is also convex and symmetric about the origin, and its area, being $\frac{1}{4}$ that of K, is greater than 1. By Blichfeldt's Theorem, some translation K'' of K' contains at least 2 lattice points in its interior. Denote these points by $\boldsymbol{x}$ and $\boldsymbol{y}$ in vector notation, where $\boldsymbol{x} \neq \boldsymbol{y}$. (See Figure 9.)

Suppose K'' is obtained from K' by translation by a vector $\boldsymbol{a}$. Then the two points $\boldsymbol{x} - \boldsymbol{a}$, and $\boldsymbol{y} - \boldsymbol{a}$ are interior to K'. These are not necessarily lattice points, however, because the translation vector $\boldsymbol{a}$ need not have integer coordinates. But K' is symmetric about the origin, so the point $\boldsymbol{a} - \boldsymbol{x}$ is also interior to K'. And because K' is convex, the point midway between $\boldsymbol{y} - \boldsymbol{a}$ and $\boldsymbol{a} - \boldsymbol{x}$ is also interior to K'. This is the point $\frac{1}{2}(\boldsymbol{y} - \boldsymbol{x})$. Since $K' = \frac{1}{2}K$, the point $\boldsymbol{y} - \boldsymbol{x}$ is interior to K. But $\boldsymbol{y} - \boldsymbol{x}$ is a lattice point different from the origin, so K contains two lattice points in its interior, the origin and $\boldsymbol{y} - \boldsymbol{x}$. The symmetric point $\boldsymbol{x} - \boldsymbol{y}$ is a third lattice point interior to K, and the proof is complete. ■

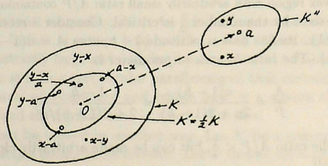

Figure 9. Proof of Minkowski's convex body theorem.

Minkowski's theorem can also be extended to n-space. *If K is a bounded convex region in n-space symmetric about a lattice point in its interior and with n-dimensional volume $V(K) > 2^n$, then K contains at least 3 lattice points in its interior.*

The proof is along the same lines as that given above, except we use the n-dimensional version of Blichfeldt's theorem.

6 Other existence theorems.

The theorems of Blichfeldt and Minkowski can be classified as **existence theorems.** We are given information about the size and shape of a region, and from this we deduce the existence of one or more lattice points inside the region. Pick's Theorem is of the same type because it tells us something about the number of lattice points inside the region if we know the area of the region and the number of lattice points on the boundary.

We turn next to another group of existence theorems discovered in the latter half of the 20th century. This body of research began in 1962 when Caltech undergraduate Edward Bender was working on some homework problems on lattice points in my number theory course and discovered an interesting existence theorem about lattice points derived from an area-perimeter relation.

Let's start with a simple example. Suppose we have a plane convex region with area A and perimeter P. Figure 10 shows a long thin convex region that contains no lattice points in its interior. It has a large perimeter P and a very small area A, so the ratio A/P is small. By extending the length of the region we can make the ratio A/P arbitrarily small and no lattice points will be captured. Now we ask: *How large can the ratio A/P be and still capture no lattice points?* Bender proved that it can't be greater than $\frac{1}{2}$. In other words, if $A/P > \frac{1}{2}$, then the region must contain a lattice point inside.

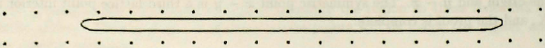

Figure 10. A convex region with arbitrarily small ratio A/P containing no lattice points.

We can easily see why the number $\frac{1}{2}$ is critical. Consider a rectangular region like the one in Figure 11. Its has base n, altitude $1 - h$, area $A = n(1 - h)$, and perimeter $P = 2n + 2(1 - h)$. The ratio of area to perimeter is

$$\frac{A}{P} = \frac{n(1-h)}{2n + 2(1-h)} = \frac{1-h}{2 + 2(1-h)/n}.$$

This shows that the ratio $A/P < \frac{1}{2}$, but can be made arbitrarily close to $\frac{1}{2}$ by making n large and h small.

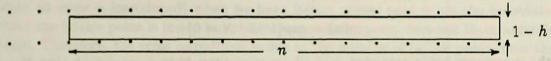

Figure 11. A region with no lattice points inside and with A/P arbitrarily close to $\frac{1}{2}$.

Bender's result was published in [2]. Later it was noted that Bender's theorem is an easy corollary of an inequality found some 15 years earlier by Nosarzewska [9]. Her inequality states that

$$A - \frac{1}{2}P < N < A + \frac{1}{2}P + 1,$$

where N is the number of lattice points inside a convex plane region with area A and perimeter P. Clearly, if $A > \frac{1}{2}P$, the leftmost inequality implies $N > 0$, so $N \geq 1$, and we get Bender's theorem. As might be expected, the proofs of Nosarzewska's inequalities are rather complicated.

Nosarzewska's leftmost inequality was extended to 3-space by Wolfgang Schmidt [13] in 1972 and (independently and almost simultaneously) by J. Bokowski, H. Hadwiger and J. M. Wills [4]. In 3-space the inequality states that

$$V - \frac{1}{2}S < N,$$

where V is the volume and S is the surface area of the convex region. Bokowski, Hadwiger and Wills, in a joint paper [4] proved in 1972 that this inequality holds in n-space as well, where now V is the n-dimensional volume of the region and S is its $(n-1)$-dimensional surface area. Of course , this inequality also gives us an existence theorem for lattice points in n-space:

$$If \quad V > \frac{1}{2}S, \text{ then } N \geq 1.$$

Nosarzewska's rightmost inequality, $N < A + \frac{1}{2}P + 1$ has not been extended to higher dimensional space. In fact, it cannot hold even for $n = 3$. There exist convex bodies in 3-space with N arbitrarily large and with both volume V and surface area S arbitrarily small. An example is a long rectangular parallelepiped that contains n lattice points along its longest axis of symmetry and whose base is a square of edge $1/n^2$. The volume $V = 1/n^3$ and the surface area is $S = 2/n^4 + 4/n$.

These results can be viewed in another way. Let K be a convex body in n-space that contains no lattice points in its interior. Let V denote its n-dimensional volume and S its $(n-1)$-dimensional surface area. Form the ratio V/S and ask: *How large can this ratio be without the body capturing a lattice point?* This is the exact analog of the question considered by Bender in the plane. The question suggests that we consider the number $\rho(n)$ defined by

$$\rho(n) = sup\{V/S : N = 0\},$$

the supremum being extended over all convex bodies K in n-space such that $N = 0$. It seems reasonable to call this number the **capture ratio**, because if V/S exceeds $\rho(n)$ then a lattice point is sure to be captured. In other words,

$$V/S > \rho(n) \text{ implies } N \geq 1.$$

Bender's theorem states that the capture ratio is $\frac{1}{2}$ in 2-space: $\rho(2) = \frac{1}{2}$. The extended Nosarzewska inequality $N > V - \frac{1}{2}S$ implies that $\rho(n) \geq \frac{1}{2}$ for all n.

In 1968 Wills [15] proved that $\rho(3) = \frac{1}{2}$, and in 1970 he also proved [16] that $\rho(4) = \frac{1}{2}$. Later that year Hadwiger [5] made a major breakthrough and proved that $\rho(n) = \frac{1}{2}$ for every dimension n, and in doing so, he completely demolished this particular problem. The problem has been generalized and there are many further results concerning area-perimeter relations that will not be described here. Instead, we turn to some other types of lattice point problems.

7 Counting problems.

The problems discussed above deal with existence theorems. Certain facts are given about the size and shape of a region and we deduce the existence of one or more lattice points inside. We consider next a class of problems called *counting problems* in which we know at the outset that there are lattice points inside, but we want to find how many. The prototype of such problems is the so-called *Gauss circle problem*.

Suppose you have a large circle of radius r with center at the origin. *How many lattice points are there inside or on the boundary of the circle?* The number will depend on the radius r and we denote it by $N(r)$. It's not hard to see that $N(r)$ is approximately πr^2, the area of the region. This is because every lattice point in the plane can be regarded as belonging to exactly one square of the unit square lattice, namely that square having the lattice point as its lower left hand corner. So, counting lattice points in a region is the same as counting squares in the region, and this is the area of the region. The approximate relation $N(r) \approx \pi r^2$ can be written as an exact equation:

$$N(r) = \pi r^2 + E(r),$$

where $E(r)$ is the error, $E(r) = N(r) - \pi r^2$. Of course, this equation merely defines $E(r)$ and tells us nothing interesting. What we really want is an estimate for the size of the error in terms of the radius r. We can get a rough idea of the size of the error by referring once more to the correspondence between lattice points and squares of the lattice. When the squares of the lattice lie inside the region, the one-to-one correspondence between lattice points and squares is exact. It fails to be exact (and hence an error is introduced) when we have lattice points near or on the boundary, where the lattice point is inside but the corresponding square does not lie completely inside. Therefore, the error introduced is due to the number of lattice points near the boundary, and this is roughly equal to the perimeter of the boundary, $2\pi r$. In other words, it is reasonable to expect that

$$N(r) = \pi r^2 + O(r).$$

This was discovered by Gauss in 1834, and is very easy to prove by drawing two concentric circles, one of radius $r + \sqrt{2}$, and one of radius $r - \sqrt{2}$. If a lattice point is on the boundary of the circle of radius r, the corresponding square that goes with it lies inside the larger circle of radius $r + \sqrt{2}$, so $N(r)$ is certainly no more than the area of this circle, $\pi(r + \sqrt{2})^2$. Similarly, $N(r)$ is greater than or equal to the area of the smaller circle of radius $r - \sqrt{2}$, so we have the upper and lower bounds

$$\pi(r - \sqrt{2})^2 \leq N(r) \leq \pi(r + \sqrt{2})^2.$$

Squaring the binomials and transposing terms we find

$$-2\pi r\sqrt{2} + 2\pi \leq N(r) - \pi r^2 \leq 2\pi r\sqrt{2} + 2\pi,$$

which implies that $|E(r)| \leq 2\pi r\sqrt{2} + 2\pi$, so $E(r) = O(r)$, as asserted.

Gauss's result is now regarded as trivial because the actual size of the error is, in fact, smaller than $O(r)$. In 1904 Sierpinski showed that $E(r) = O(r^{2/3})$, and later, in 1923, van der Corput showed that $E(r) = O(r^{\theta})$, for some $\theta < 2/3$. In 1915, Hardy and Landau proved that the error is not $O(r^{1/2})$. The smallest exponent θ for which the error is $O(r^{\theta})$ is still not known. Here is a list of values of θ that have been obtained since Gauss's time:

$\theta \leq 1$	(Gauss, 1834)
$\theta \leq \frac{2}{3} = 0.66666$	(Sierpinski, 1904)
$\theta \geq \frac{1}{2}$	(Hardy and Landau, 1915)
$\theta < \frac{2}{3}$	(van der Corput, 1923)
$\theta \leq \frac{37}{56} = 0.66071$	(Littlewood and Walfisz, 1924)
$\theta \leq \frac{163}{247} = 0.65995$	(Walfisz, 1925)
$\theta \leq \frac{17}{26} = 0.65384$	(Vinogradov, 1932)

$\theta \le \frac{15}{23} = 0.65217$ (Titchmarch, 1934) $\theta \le \frac{13}{20} = 0.65000$ (Hua, 1942)

$\theta \le \frac{24}{37} = 0.64864$ (Chen-Jing-Run, 1969) $\theta \le \frac{70}{108} = 0.648148$ (Kolesnik, 1982)

The determination of the smallest θ for which the error is $O(r^\theta)$ is now known as the Gauss circle problem. Incidentally, there is an interesting exact formula that expresses the error as an infinite series. In 1915 Hardy proved that

$$N(r) = \pi r^2 + r \sum_{n=1}^{\infty} r_2(n) J_1(2\pi r\sqrt{n})/\sqrt{n},$$

where J_1 is a Bessel function and $r_2(n)$ is the number of ways that n can be expressed as a sum of two squares. The Bessel function oscillates, and sign cancellation takes place in the series to lower the size of the error term.

The Gauss circle problem has been extended from plane regions bounded by circles to those bounded by ellipses, hyperbolas, and other curves. It has also been generalized to higher dimensions. There is a vast literature on this subject. In 1957, Walfisz published a 500 page book [14] dealing entirely with lattice points in n-dimensional spheres.

8 Visibility problems.

We turn next to lattice point problems having to do with visibility. We say that a lattice point P is **visible** from another lattice point Q if the line segment joining them contains no further lattice points. Imagine a forest of trees arranged in a unit square lattice; if you stand at one tree, you will be able to see the trees at the visible lattice points but not at the invisible lattice points.

Figure 12 shows examples of lattice points visible from the origin, indicated by small circles $\odot$. The points (1, 0), (1, 1), (0, 1) are visible from the origin, as are the points (2, 1), (1, 2), (3, 1), (3, 2), (2, 3) ,and (1, 3). The lattice points (2, 2), (3, 3), (4, 2), and (2, 4), shown by crosses $\times$ in the figure, are not visible from the origin.

7	$\times$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\times$	$\odot$
6	$\times$	$\odot$	$\times$	$\times$	$\times$	$\odot$	$\times$	$\odot$	$\times$
5	$\times$	$\odot$	$\odot$	$\times$	$\odot$	$\times$	$\odot$	$\odot$	$\odot$
4	$\times$	$\odot$	$\times$	$\odot$	$\times$	$\odot$	$\times$	$\odot$	$\times$
3	$\times$	$\odot$	$\odot$	$\times$	$\odot$	$\odot$	$\times$	$\odot$	$\odot$
2	$\times$	$\odot$	$\times$	$\odot$	$\times$	$\odot$	$\times$	$\odot$	$\times$
1	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$
0	$\odot$	$\odot$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**

Figure 12. Lattice points visible from the origin are marked by small circles $\odot$, invisible points by crosses $\times$.

It is easy to prove that a lattice point (x, y) is visible from the origin if, and only if, its coordinates x and y are relatively prime, that is, have no prime factor in common. Also, two lattice points (x, y) and (x', y') are mutually visible if, and only if, the differences $x - x'$ and $y - y'$ are relatively prime.

Let's call those lattice points visible from the origin **visible points**. There are infinitely many visible points and it is natural to ask how they are distributed among all the lattice points. For example, how "dense" are the visible points among all the lattice points?

There is a well known method for answering this question. We count the total number N' of visible points in a large square centered at the origin, and divide this number by the total number N of lattice points in the square, then find the limit of the ratio N'/N, as the edge of the square increases to infinity. Because of symmetry, it suffices to consider an isosceles right triangle of base X as shown in Figure 13.

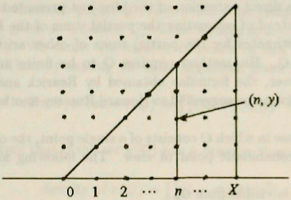

Figure 13. Points visible from the origin.

For a fixed $n \le X$, the number of visible points on the vertical segment $x = n, 0 < y \le n$ is the number of lattice points (n, y) with y relatively prime to n and $y \le n$, and this, of course, is Euler's famous totient function $\varphi(n)$. Therefore, the number of visible points in the right triangle shown in Figure 13 is equal to

$$N' = \sum_{n \le X} \varphi(n).$$

It is known ([1], p. 62) that for large X the value of this sum is asymptotic to $3X^2/\pi^2$.

The total number N of lattice points in the triangle is equal to the area of the triangle plus an error of the order of magnitude of its perimeter, so $N = \frac{1}{2}X^2 + O(X)$. Dividing N' by N and letting $X \to \infty$ we find that the ratio N'/N tends to the limit $6/\pi^2$ as $X \to \infty$. In other words, the density of visible points among all lattice points in the plane is $6/\pi^2$, or about $2/3$.

This result can also be interpreted probabilistically. If you pick a lattice point at random, the probability that it is visible from the origin is $6/\pi^2$. Or, if you pick two integers at random, the probability that they are relatively prime is $6/\pi^2$.

Incidentally, the number $6/\pi^2$ is the reciprocal of $\zeta(2)$, the sum of the infinite series of reciprocals of all the squares. (See [1], p. 266.) For the same problem in n-space, the answer is the reciprocal of $\zeta(n)$, the sum of the reciprocals of all the nth powers of the positive integers.

Many years ago, two of my Ph. D. students, David Rearick and Howard Rumsey, considered an interesting generalization of this problem. Let Q be any subset of the unit square lattice. Let $\nu(Q)$ denote the set of those lattice points visible from each point of Q or, what is the same thing, the set of those lattice points that can see each point of Q. *What is the density of the set* $\nu(Q)$? When Q consists of a single point (which we can take to be the origin) the answer is $6/\pi^2$. What about an arbitrary set Q?

David Rearick [11] found the density for several examples of *finite* sets Q. His method of attack was a direct extension of the idea just presented for computing the density of $\nu(0)$, but instead of estimating the partial sums of the Euler phi function, Rearick had to find estimates for the partial sums of other arithmetical functions depending on the set Q. His method requires Q to be finite and does not apply to infinite sets. However, the formulas obtained by Rearick and the probabilistic interpretation of these formulas suggested to Howard Rumsey another way of attacking these problems.

Look again at the case in which Q consists of a single point, the origin, and consider the problem from a probabilistic point of view. The following four statements are logically equivalent:

x is visible from 0;
the components of x are relatively prime;
no prime p divides both components of x;
for all primes p, $x \not\equiv 0 \pmod p$.

In the last statement, vector congruences are to be interpreted componentwise.

Because we are interested in the set of points x visible from the origin, we can think of this set as an "event" in some probability space. The equivalences mentioned above show that this event is the intersection of events depending on primes, thus:

$$\{x : x \text{ is visible from } 0\} = \bigcap_p \{x : x \not\equiv 0 \ (mod\ p)\}.$$

Assuming independence of the events on the right, we have

$$\begin{aligned}\text{Prob}\{x \text{ is visible from } 0\} &= \prod_p \text{Prob}\{x : x \not\equiv 0 \ (mod\ p)\}\\ &= \prod_p (1 - \text{Prob}\{x : x \not\equiv 0 \ (mod\ p)\}).\end{aligned}$$

To compute the probability that $x \equiv 0 \ (mod\ p)$ we need a sample space and an assignment of probabilities. Because we are looking at numbers modulo p, a natural sample space is the set of lattice points (a, b) in the square $0 \le a < p, 0 \le b < p$. This is natural because every lattice point x in the plane is congruent modulo p to one and only one of these points. There are p^2 elements in this sample space and exactly one of them is congruent to the origin modulo p. Therefore if we assign equal probabilities to these points , each one gets probability $1/p^2$, so the event "a lattice point $x \equiv 0 \ (mod\ p)$" has probability $1/p^2$. This gives us the formula

$$\text{Prob}\{x \text{ is visible from } 0\} = \prod_p (1 - p^{-2}).$$

The infinite product on the right is the famous Euler product for the zeta function, or rather, for the reciprocal of the zeta function at 2. (See [1], p. 231.) Thus we arrive at the number $1/\zeta(2) = 6/\pi^2$, as before.

This argument suggests a method for treating the general case. Replace the origin by an arbitrary set of lattice points Q. Then reduce the components of each point of Q modulo p. This gives us a subset Q_p of the sample space. The set Q_p is what Q looks like modulo p. We count the number of points in this subset and denote this number by $r_p(Q)$. Then by exactly the same argument given above we find

$$\text{Prob}\{x \text{ can see } Q\} = \prod_p (1 - r_p(Q)/p^2).$$

This discussion shows that if the set $\nu(Q)$ has a density, then it ought to be given by the formula

$$\text{density of } \nu(Q) = \prod_p (1 - r_p(Q)/p^2).$$

In n-space, the same formula holds with p^2 replaced by p^n.

Howard Rumsey [12] proved that the density actually exists for every finite set Q and is given by this formula. He also showed that the formula is correct for many (but not all) infinite sets Q.

9 Miscellaneous results

. There are dozens of other interesting results concerning lattice points that we have not touched on here, many of which relate to combinatorics. There are also many unsolved problems outlined in Hammer's monograph [6]. We conclude this presentation with some miscellaneous known results about lattice points in the plane that you may attempt to prove as exercises.

For every integer $n \ge 1$, there is a circle with center at the point $P = (\sqrt{2}, \frac{1}{3})$ that contains exactly n lattice points in its interior. Moreover, no point P with both coordinates rational has this property.

Given any integer $n \geq 1$, there is a circular disk that contains exactly n lattice points on its boundary.

Finally, Figure 14 Illustrates a theorem on lattice points that you can discover for yourself

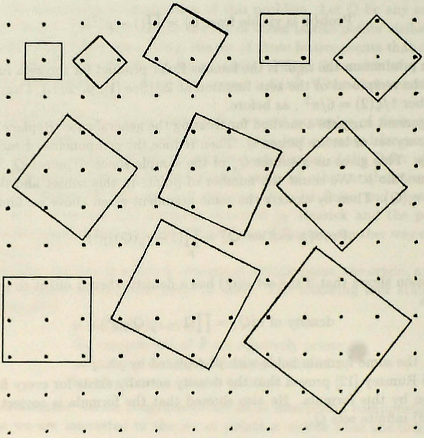

Figure 14.

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