

INJECTIVITY AND ACCESSIBLE CATEGORIES

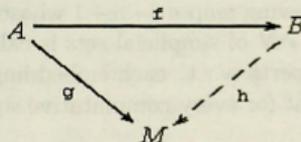
Jiří Rosický

Masaryk University, Department of Mathematics
Janáčkovo nám. 2a, 662 95 Brno,
Czech Republic
rosicky@math.muni.cz

Since its creation by S. Eilenberg and S. MacLane [EM], category theory has brought a number of important concepts. Accessible categories are among them and we are going to show how they can help to treat injectivity in algebra, model theory and homotopy theory.

1 Three situation

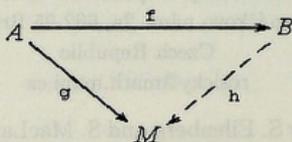
1.1 Injective modules. Injective modules were introduced by R. Baer [B]. A left R -module M is called *injective* if for each injective homomorphism $f : A \rightarrow B$ and each homomorphism $g : A \rightarrow M$ there is a homomorphism $h : B \rightarrow M$ such that $h \cdot f = g$.



The category $R\text{-Mod}$ of left R -modules has enough injectivities, which means that for every R -module A there is an injective homomorphism $A \rightarrow M$ with M injective. This was also proved by Baer [B] using his criterion for injectivity.

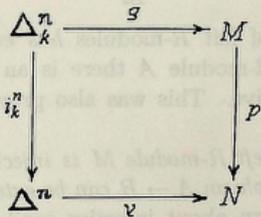
Baer's Criterion. A left R -module M is injective iff for every left ideal A of R , every homomorphism $A \rightarrow R$ can be extended to a homomorphism $R \rightarrow M$. One can learn about injective modules and their use in any monograph about module theory (see, e.g., [F]).

1.2 Saturated models. Let T be a first-order theory of a countable signature Σ . Let $\text{Mod}(T)$ be the category of models of the theory T with elementary embeddings as morphisms. For an uncountable regular cardinal λ , a T -model M is called λ -saturated if for each elementary embedding $f : A \rightarrow B$ with $\text{card}A, \text{card}B < \lambda$ and each elementary embedding $g : A \rightarrow M$ there is an elementary embedding $h : B \rightarrow M$ with $h \cdot f = g$.

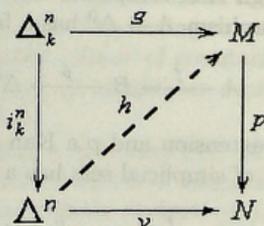


We have not used the original definition of λ -saturated models (due to Morley and Vaught [MV]) but the characterization given in [S] 16.6. The category $\text{Mod}(T)$ has enough λ -saturated models in the sense that each T -model has an elementary embedding into a λ -saturated model.

1.3 Kan fibrations. The category \mathbf{SSet} of simplicial sets is defined as the functor category $\mathbf{Set}^{\Delta^{\text{op}}}$ where Δ is the category of non-zero finite ordinals and order-preserving maps. The simplicial sets Δ^n , $n \geq 0$ are defined as $\Delta^n = Y(n+1)$ where $Y : \Delta \rightarrow \mathbf{SSet}$ is the Yoneda embedding. The simplicial subsets $\Delta_k^n \subseteq \Delta^n$, $n \geq 0$, $0 \leq k \leq n$ are obtained by excluding the identity morphism $\Delta^n \rightarrow \Delta^n$ and the morphism $\Delta^{n-1} \rightarrow \Delta^n$ given by the injective order-preserving map $n \rightarrow n+1$ whose image does not contain k . A morphism $p : M \rightarrow N$ of simplicial sets is called a *Kan fibration* if it has the *right lifting property* w.r.t. each embedding $i_k^n : \Delta_k^n \rightarrow \Delta^n$, $n \geq 0$, $0 \leq k \leq n$. It means that for every commutative square

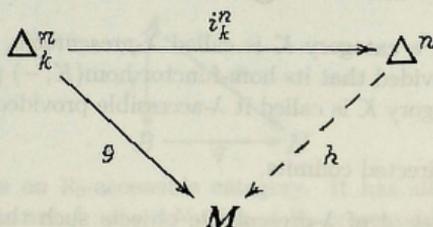


there exists a diagonal

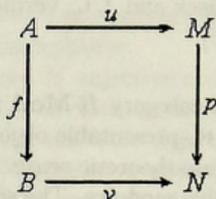


making both triangles commutative.

If $N = \Delta^0$ then the unique morphism $p : M \rightarrow \Delta^0$ (Δ^0 is a terminal object in \mathbf{SSet}) is a Kan fibration iff for each i_k^n , $n \geq 0$, $0 \leq k \leq n$ and for each morphism $g : \Delta_k^n \rightarrow M$ there is a morphism $h : \Delta^n \rightarrow M$ with $h \cdot i_k^n = g$



Such simplicial sets M are called *Kan complexes*. \mathbf{SSet} has enough Kan complexes in the sense that each simplicial set A has an embedding $f : A \rightarrow B$ into a Kan complex. Moreover, this embedding f is an *anodyne extension*, which is defined by having the *left lifting property* w.r.t. each Kan fibration p . It means that for every commutative square



there exists a diagonal h making both triangles commutative. Of course,

every embedding $\Delta_k^n \rightarrow \Delta^n$ is an anodyne extension. The just explained property of having enough Kan complexes can be equivalently formulated in the way that each morphism $A \rightarrow \Delta^0$ has a factorization

$$A \xrightarrow{f} B \xrightarrow{p} \Delta^0$$

where f is an anodyne extension and p a Kan fibrations. More generally, every morphism $A \rightarrow N$ of simplicial sets has a factorization

$$A \xrightarrow{f} B \xrightarrow{p} N$$

where f is an anodyne extension and p a Kan fibration (see, e.g. [GJ]). Kan fibrations were introduced D. M. Kan [K].

2 Accessible categories

An object K of a category \mathcal{K} is called λ -presentable, where λ is a regular cardinal, provided that its hom-functor $\text{hom}(K, -)$ preserves λ -directed colimits. A category \mathcal{K} is called it λ -accessible provided that

- (1) \mathcal{K} has λ -directed colimits,
- (2) \mathcal{K} has a set \mathcal{A} of λ -presentable objects such that every object is a λ -directed colimit of objects of \mathcal{A} .

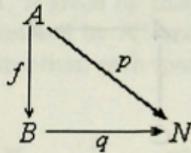
A category is called *accessible* if it is λ -accessible for some regular cardinal λ . Accessible categories were introduced by C. Lair [L] and their theory was created by M. Makkai and R. Paré [MP]. We will use the monograph [AR]. The first steps towards the theory of accessible categories were made by M. Artin, A. Grothendieck and J. L. Verdier [AGV] and especially by P. Gabriel and F. Ulmer [GU].

2.1 Examples. (1) The category $R\text{-Mod}$ is \aleph_0 -accessible for every ring R . It has all colimits and \aleph_0 -presentable objects are finitely presentable R -modules in the usual module-theoretic sense. Every R -module is a directed colimit of finitely presentable modules. The same argument applies to every variety of universal algebras.

(2) The category $\mathbf{Mod}(T)$ is \aleph_1 -accessible for every first-order theory T of a countable signature. It has directed colimits (see [AR] 5.39) and \aleph_1 -presentable objects are T -models having countably many elements. Every T -model is an \aleph_1 -directed colimit of countable T -models. This can be found in [AR] 5.42 but it is an immediate consequence of the downward Löwenheim-Skolem theorem.

(3) The category \mathbf{SSet} is \aleph_0 -accessible. It has all colimits and \aleph_0 -presentable objects are finite colimits of simplicial sets Δ^n , $n \geq 0$. Every simplicial set is a directed colimit of finite colimits of Δ^n , $n \geq 0$. The same argument applies to every functor category $\mathbf{Set}^{\mathcal{X}^{op}}$ where \mathcal{X} is a small category.

(4) Let N be a simplicial set and consider the comma-category $\mathbf{SSet} \downarrow N$. Objects of this category are morphisms $p : A \rightarrow N$ of simplicial sets. Morphisms $(A, p) \rightarrow (B, q)$ are morphisms $f : A \rightarrow B$ of simplicial sets with $q \cdot f = p$



Then $\mathbf{SSet} \downarrow N$ is an \aleph_0 -accessible category. It has all colimits and \aleph_0 -presentable objects are $f : A \rightarrow N$ with A \aleph_0 -presentable in \mathbf{SSet} . Every object in $\mathbf{SSet} \downarrow N$ is a directed colimit of \aleph_0 -presentable objects (see [AR] 1.57).

Let \mathcal{H} be a class of morphisms in a category \mathcal{C} . An object M in \mathcal{C} is called \mathcal{H} -injective if for each morphism $f : A \rightarrow B$ in \mathcal{H} and each morphism $g : A \rightarrow M$ there is a morphism $h : B \rightarrow M$ such that $h \cdot f = g$.

2.2 Examples. (1) Injective R -modules are \mathcal{H} -injective objects in $R\text{-Mod}$ for \mathcal{H} consisting of all monomorphisms.

(2) λ -saturated models are \mathcal{H} -injective objects in $\mathbf{Mod}(T)$ for \mathcal{H} consisting of morphisms $f : A \rightarrow B$ with $\text{card}A, \text{card}B < \lambda$. We recall that these objects are precisely λ -presentable objects.

(3) Kan complexes are \mathcal{H} -injective objects in \mathbf{SSet} for \mathcal{H} consisting of anodyne extensions. In fact, we defined them as being injective w.r.t. embeddings $\Delta_k^n \rightarrow \Delta^n$, $n \geq 0$, $0 \leq k \leq n$ but it immediately follows from

the definition that they are injective w.r.t. every anodyne extension.

(4) Let N be a simplicial set and consider the comma-category $\mathbf{SSet} \downarrow N$. Kan fibrations $p : M \rightarrow N$ are \mathcal{H} -injective objects for \mathcal{H} consisting of morphisms $(A, a) \rightarrow (B, b)$ carried by anodyne extensions $f : A \rightarrow B$. In fact the defining property of a Kan fibration exactly means that

$$\begin{array}{ccc} (A, pu) & \xrightarrow{f} & (B, v) \\ & \searrow u & \swarrow h \\ & (M, p) & \end{array}$$

An accessible category does not need to have all colimits (see, for example, 2.1 (2)). We say that a diagram $D : \mathcal{D} \rightarrow \mathcal{K}$ has a *bound* in a category \mathcal{K} if there is a compatible cocone $(Dd \xrightarrow{c_d} C)_{d \in \mathcal{D} \text{ obj}}$ in \mathcal{K} . We say that \mathcal{K} has *directed bounds* if every directed diagram has a bound in \mathcal{K} and that \mathcal{K} has *pushout bounds* if every diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \nearrow & \\ C & \longrightarrow & \end{array}$$

has a bound in \mathcal{K} .

2.3 Theorem. *Let \mathcal{K} be an accessible category with directed and pushout bounds and \mathcal{H} a set of morphisms in \mathcal{K} . Then every object K in \mathcal{K} has a morphism $K \rightarrow M$ into an \mathcal{H} -injective object L .*

Proof. Following [AR] 2.14 and 2.2 (3), there is a regular cardinal λ such that \mathcal{K} is λ -accessible and every morphism in \mathcal{H} has a λ -presentable domain. Consider an object K in \mathcal{K} . Let \mathcal{X}_K be the set of all spans

$$\begin{array}{ccc} & K & \\ & \uparrow & \\ u & & \\ & C & \xrightarrow{g} D \end{array}$$

with $g \in \mathcal{H}$. We will index these spans by ordinals $i < \mu_K = \text{card } \mathcal{X}_K$.

We define a chain $k_{ij} : K_i \rightarrow K_j$, $i \leq j \leq \mu_K$ by the following transfinite induction:

First step: $K_0 = K$.

Isolated step: K_{i+1} is given by a pushout bound

$$\begin{array}{ccc} K_j & \xrightarrow{k_{i,j+1}} & K_{j+1} \\ \uparrow k_{0,i+1} \cdot u_i & & \uparrow \\ C_i & \xrightarrow{g_i} & D_i \end{array}$$

where $k_{0,i+1} = k_{i,i+1} \cdot k_{0i}$.

Limit step: K_i is a bound of the chain

$$K_0 \xrightarrow{k_{01}} K_1 \xrightarrow{k_{12}} \dots K_j \xrightarrow{k_{j,j+1}} \dots$$

where $j < i$ and $k_{0i} : K_0 \rightarrow K_i$ is given by this bound.

The object K_{μ_K} will be denoted by K^* and the morphism $K_{0\mu_K} : K \rightarrow K^*$ by t_K . Following the construction, each span $(u_i, g_i) \in \mathcal{X}_K$ has a pushout bound

$$\begin{array}{ccc} K & \xrightarrow{t_K} & K^* \\ \uparrow u_i & & \uparrow \\ C_i & \xrightarrow{g_i} & D_i \end{array}$$

We define a chain $m_{ij} : M_i \rightarrow M_j$, $i \leq j \leq \lambda$ by the following transfinite induction:

First step: $M_0 = K$.

Isolated step: $m_{i,i+1} : M_i \rightarrow M_{i+1}$ is $t_{M_i} : M_i \rightarrow M_i^*$.

Limit step: M_i is a directed bound of the chain

$$M_0 \xrightarrow{m_{01}} M_1 \xrightarrow{m_{12}} \dots M_j \xrightarrow{m_{j,j+1}} \dots \tag{1}$$

for $j < i < \lambda$ and M_λ is a colimit of (1) for $i = \lambda$.

We will show that $m_{0\lambda} : K \rightarrow M_\lambda$ is a desired morphism of K into an \mathcal{H} -injective object. Consider a span

$$\begin{array}{ccc}
 & M_\lambda & \\
 u \uparrow & & \\
 C & \xrightarrow{g} & D
 \end{array}$$

Since the object C is λ -presentable and M_λ is a directed colimit of M_i , $i < \lambda$, there is a factorization

$$\begin{array}{ccc}
 & M_\lambda & \\
 u \uparrow & \nearrow m_{i\lambda} & \\
 C & \xrightarrow{u'} & M_i
 \end{array}$$

of u through M_i for some $i < \lambda$. Since the span

$$\begin{array}{ccc}
 & M_i & \\
 u' \uparrow & & \\
 C & \xrightarrow{g} & D
 \end{array}$$

is in the set \mathcal{X}_{M_i} , it has a pushout bound

$$\begin{array}{ccc}
 M_i & \xrightarrow{m_{i+1}} & M_{i+1} \\
 u' \uparrow & & \uparrow v \\
 C & \xrightarrow{g} & D
 \end{array}$$

We have

$$u = m_{i\lambda} \cdot u' = m_{i+1,\lambda} \cdot m_{i,i+1} \cdot u' = m_{i+1,\lambda} \cdot v \cdot g.$$

Hence u factorizes through g , which proves that M_λ is \mathcal{H} -injective. \square

2.4 Examples. (1) The category $R\text{-Mod}$ is \aleph_0 -accessible and has all colimits. Let \mathcal{H} be the set of all embeddings $A \rightarrow R$ where A is a left ideal in

R. Following Baer's Criterion \mathcal{H} -injective modules are precisely injective modules. Following Theorem 2.3 every *R*-module has a homomorphism into an injective *R*-module.

To prove that *R-Mod* has enough injectives, we have to replace the category *R-Mod* by the category *R-Mod*₀ of *R*-modules and injective homomorphisms taken as morphisms. Following [AR] 2.3 (6), *R-Mod*₀ is an accessible category. It has directed colimits (by [AR] 1.62) and pushouts because monomorphisms in *R-Mod* are stable under pushouts. Hence, by applying Theorem 2.3, to the category *R-Mod*₀, we get that *R-Mod* has enough injectives.

(2) Let *T* be a first-order theory of a countable signature and λ an uncountable regular cardinal. The category *Mod*(*T*) has pushout bounds (see [H], p. 288). Hence Theorem 2.3 together with Example 2.1 (2) implies that every *T*-model has an elementary embedding into a λ -saturated *T*-model. Of course, we take for \mathcal{H} the set of all elementary embedding $A \rightarrow B$ with $\text{card}A, \text{card}B < \lambda$.

(3) The category *SSet* is \aleph_0 -accessible and has all colimits. Let \mathcal{H} consist of embeddings $\Delta_k^n \rightarrow \Delta^n, n \geq 0, 0 \leq k \leq n$. Following Theorem 2.3, every simplicial set *A* has a morphism $m : A \rightarrow M$ into a Kan complex *M*.

Since *SSet* is cocomplete, we can use colimits instead of bounds in the proof of Theorem 2.3. Hence *m* belongs to the closure of \mathcal{H} under pushouts, compositions and colimits of chains. Every morphism of this closure belongs to $\square(\mathcal{H}^\square)$ where the box on the right (left) means the use of the right (left) lifting property. Hence *m* is an anodyne extension.

More generally, by applying Theorem 2.3 to the category *SSet* $\downarrow N$ (for \mathcal{H} consisting of morphism carried by embeddings $\Delta_k^n \rightarrow \Delta^n, n \geq 0, 0 \leq k \leq n$), we get that each morphism $A \rightarrow N$ has a factorization

$$A \xrightarrow{f} B \xrightarrow{p} N$$

where *f* is an anodyne extension and *p* a Kan fibration.

The last example gives the essence of essence of the *small object argument* already present in [GZ]. This argument is commonly used in homotopy theory (see [Ho]) but the theory of accessible categories has started to be used in homotopy theory only recently (see T. Beke [B]). Our Theorem 2.3 is a very general formulation of the small object argument. The point is that every object of an accessible category is presentable (= small), which

makes possible to stop the construction of an \mathcal{H} -injective object M for K . The next example shows that it is necessary to assume that \mathcal{H} is a set.

2.5 Example. Let \mathbf{Gr} be the category of groups and \mathcal{H} the class of all injective homomorphisms. Every group K is a subgroup of a simple group $L \neq K$ (see [Sc]). If K is \mathcal{H} -injective, the embedding $f : K \rightarrow L$ splits, i.e., there exists $g : L \rightarrow K$ with $g \cdot f = \text{id}_K$; by applying \mathcal{H} -injectivity to

$$\begin{array}{ccc}
 K & \xrightarrow{\text{id}_K} & K \\
 \searrow f & & \nearrow g \\
 & L &
 \end{array}$$

Since L is simple and $L \not\cong K$, the homomorphism g has to be constant, i.e., $K = \{1\}$. Therefore the trivial group $\{1\}$ is the only injective (= \mathcal{H} -injective) group. Hence the category of groups does not have enough injectives. On the other hand, the category \mathbf{Gr}_0 of groups and injective homomorphisms is accessible (following the same reasons as the category $R\text{-Mod}_0$) and the only obstacle to apply Theorem 2.3 is that \mathcal{H} is not a set.

References

- [AR] J. Adámek and J. Rosický, *Locally presentable and accessible categories*, Cambridge University Press 1994.
- [AGV] M. Artin, A. Grothendieck and J. L. Verdier, *Théorie des topos et cohomologie étale des schémas*, Lect. Notes in Math. 269, Springer-Verlag 1972.
- [B] R. Baer, *Abelian groups that are direct summands of every containing abelian group*, Bull. Amer. Math. Soc. 46 (1940), 800-806.
- [EM] S. Eilenberg and S. MacLane, *General theory of natural equivalences*, Trans. Amer. Math. Soc. 58 (1945), 231-294.
- [F] C. Faith, *Algebra: Rings, Modules and Categories I*, Springer-Verlag 1973.

- [GU] **P. Gabriel and F. Ulmer**, *Lokal Präsentierbare Kategorien*, Lect. Notes in Math. 221, Springer-Verlag 1971.
- [GZ] **P. Gabriel and M. Zisman**, *Calculus of fractions and homotopy theory*, Ergebnisse in Math. and Grenzgebiete 35, Springer-Verlag 1967.
- [GJ] **P. G. Goerss and J. F. Jardine**, *Simplicial homotopy theory*, Birkhäuser 1999.
- [H] **W. Hodges**, *Model theory*, Cambridge University Press 1993.
- [Ho] **M. Hovey**, *Model categories*, Amer. Math. Soc. 1999.
- [K] **D. M. Kan**, *On c.s.s. complexes*, Amer. J. Math. 79 (1957), 449-476.
- [L] **C. Lair**, *Catégories modélables et catégories esquissables*, Diagrammes 6 (1981), 1-20.
- [MP] **M. Makkai and R. Paré**, *Accessible categories: the foundations of categorical model theory*, Contemp. Math., 104, Amer. Math. Soc. 1989.
- [MV] **M. Morley and R. Vaught**, *Homogeneous universal models*, Math. Scand. 11 (1962), 37-57.
- [S] **G. E. Sacks**, *Saturated model theory*, Benjamin 1972.
- [Sc] **W. R. Scott**, *Group theory*, Prentice Hall 1964.