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Quantitative Approximation by a Kantorovich-Shilkret quasi-interpolation neural network operator

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ABSTRACT

In this article we present multivariate basic approximation by a Kantorovich-Shilkret type quasi-interpolation neural network operator with respect to supremum norm. This is done with rates using the multivariate modulus of continuity. We approximate continuous and bounded functions on $\mathbb{R}^N$, $N \in \mathbb{N}$. When they are additionally uniformly continuous we derive pointwise and uniform convergences.

RESUMEN

En este artículo presentamos un resultado de aproximación básico multivariado a través de un operador de cuasi-interpolación en red neuronal de tipo Kantorovich-Shilkret con respecto a la norma del supremo. Esto se realiza con tasas usando el módulo de continuidad multivariado. Aproximamos funciones continuas y acotadas en $\mathbb{R}^N$, $N \in \mathbb{N}$. Cuando ellas son adicionalmente uniformemente continuas, derivamos convergencias puntuales y uniformes.

Keywords and Phrases: error function based activation function, multivariate quasi-interpolation neural network approximation, Kantorovich-Shilkret type operator.

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1 Introduction

The author here performs multivariate error function based neural network approximation to continuous functions over $\mathbb{R}^N$, $N \in \mathbb{N}$, and then he extends his results to complex valued functions. The convergences here are with rates expressed via the multivariate modulus of continuity of the involved function and give by very tight Jackson type inequalities.

The author comes up with the "right" precisely defined flexible quasi-interpolation Baskakov-Shilkret type integral coefficient neural network operator associated to the error function.

Feed-forward neural network (FNNs) with one hidden layer with deal with, are expressed mathematicaly as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma(\langle a_j \cdot x \rangle + b_j), \quad x \in \mathbb{R}^s, \ s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of a_j and x, and σ is the activation function of the network. In many fundamental neural network models the activation function is error function generated.

About neural networks in general you may read [4], [5], [6]. In recent years non-additive integrals, like the N. Shilkret one [7], have become fashionable and more useful in Economic theory, etc.

2 Background

Here we follow [7].

Let $\mathcal{F}$ be a σ-field of subsets of an arbitrary set Ω. An extended non-negative real valued function μ on $\mathcal{F}$ is called maxitive if $\mu(\varnothing) = 0$ and

$$\mu(\cup_{i \in I} E_i) = \sup_{i \in I} \mu(E_i), \tag{1}$$

where the set I is of cardinality at most countable. We also call μ a maxitive measure. Here f stands for a non-negative measurable function on Ω. In [7], Niel Shilkret developed his non-additive integral defined as follows:

$$(N^*) \int_D f d\mu := \sup_{y \in Y} \{y \cdot \mu(D \cap \{f \geq y\})\}, \tag{2}$$

where $Y = [0, m]$ or $Y = [0, m)$ with $0 < m \leq \infty$, and $D \in \mathcal{F}$. Here we take $Y = [0, \infty)$.

It is easily proved that

$$(N^*) \int_D f d\mu = \sup_{y > 0} \{y \cdot \mu(D \cap \{f > y\})\}. \tag{3}$$

The Shilkret integral takes values in $[0, \infty]$.

The Shilkret integral ([7]) has the following properties:

$$(N^*) \int_{\Omega} \chi_E \, d\mu = \mu(E), \tag{4}$$

where χ_E is the indicator function on $E \in \mathcal{F}$,

$$(N*) \int_D cf d\mu = c (N^*) \int_D f d\mu, \quad c \geq 0, \tag{5}$$

$$(N^*) \int_D \sup_{n \in \mathbb{N}} f_n \, d\mu = \sup_{n \in \mathbb{N}} (N^*) \int_D f_n \, d\mu, \tag{6}$$

where f_n, $n \in \mathbb{N}$, is an increasing sequence of elementary (countably valued) functions converging uniformly to f. Furthermore we have

$$(N^*) \int_D f d\mu \geq 0, \tag{7}$$

$$f \geq g \text{ implies } (N^*) \int_D f d\mu \geq (N^*) \int_D g d\mu, \tag{8}$$

where $f, g : \Omega \to [0, \infty]$ are measurable.

Let $a \leq f(\omega) \leq b$ for almost every $\omega \in E$, then

$$a\mu(E) \leq (N^*) \int_E f d\mu \leq b\mu(E);$$

$$(N^*) \int_E 1 d\mu = \mu(E);$$

$f > 0$ almost everywhere and $(N^*) \int_E f d\mu = 0$ imply $\mu(E) = 0$;

$(N^*) \int_{\Omega} f d\mu = 0$ if and only $f = 0$ almost everywhere;

$(N^*) \int_{\Omega} f d\mu < \infty$ implies that

$$\overline{N}(f) := \{\omega \in \Omega | f(\omega) \neq 0\} \text{ has } \sigma\text{-finite measure}; \tag{9}$$

$$(N^*) \int_D (f + g) \, d\mu \leq (N^*) \int_D f d\mu + (N^*) \int_D g d\mu;$$

and

$$\left| (N^*) \int_D f d\mu - (N^*) \int_D g d\mu \right| \leq (N^*) \int_D |f - g| \, d\mu. \tag{10}$$

From now on in this article we assume that $\mu : \mathcal{F} \to [0, +\infty)$.

3 Main Results

We consider here the (Gauss) error special function ([1], [3])

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \quad x \in \mathbb{R}, \tag{11}$$

which is a sigmoidal type function and a strictly increasing function.

It has the properties

$$\operatorname{erf}(0) = 0, \quad \operatorname{erf}(-x) = \operatorname{erf}(x), \ \operatorname{erf}(+\infty) = 1, \quad \operatorname{erf}(-\infty) = -1,$$

and

$$(\operatorname{erf}(x))' = \frac{2}{\sqrt{\pi}} e^{-x^2}, \quad x \in \mathbb{R},$$

$$\int \operatorname{erf}(x)\, dx = x \operatorname{erf}(x) + \frac{e^{-x^2}}{\sqrt{\pi}} + C,$$

where C is a constant.

The error function is related to the cumulative probability distribution function of the standard normal distribution

$$\Phi(x) = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right).$$

We consider the activation function

$$\chi(x) = \frac{1}{4}\left(\operatorname{erf}(x+1) - \operatorname{erf}(x-1)\right), \quad x \in \mathbb{R}, \tag{12}$$

and we notice that

$$\chi(-x) = \chi(x), \tag{13}$$

and even function.

Clearly $\chi(x) > 0$, all $x \in \mathbb{R}$.

We see that

$$\chi(0) = \frac{\operatorname{erf}(1)}{2} \simeq 0.4215. \tag{14}$$

Let $x > 0$, we have that

$$\chi'(x) < 0, \quad \text{for } x > 0. \tag{15}$$

That is χ is strictly decreasing on $[0, \infty)$ and is strictly increasing on $(-\infty, 0]$, and $\chi'(0) = 0$.

Clearly the x-axis is the horizontal asymptote on χ.

Conclusion, χ is a bell symmetric function with maximum $\chi(0) \simeq 0.4215$.

We further need

Theorem 3.1. *([2]) We have that*

$$\sum_{i=-\infty}^{\infty} \chi(x-i) = 1, \text{ all } x \in \mathbb{R}. \tag{16}$$

Theorem 3.2. *([2]) It holds*

$$\int_{-\infty}^{\infty} \chi(x)\, dx = 1. \tag{17}$$

So $\chi(x)$ is a density function on $\mathbb{R}$.

Theorem 3.3. *([2]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} \geq 3$.*

It holds

$$\sum_{\left\{\begin{array}{c} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{array}\right.}^{\infty} \chi(nx-k) < \frac{1}{2\sqrt{\pi}\left(n^{1-\alpha}-2\right) e^{\left(n^{1-\alpha}-2\right)^2}}. \tag{18}$$

Remark 3.4. *We introduce*

$$Z(x_1, ..., x_N) := Z(x) := \prod_{i=1}^{N} \chi(x_i), \tag{19}$$

$x = (x_1, ..., x_N) \in \mathbb{R}^N$, $N \in \mathbb{N}$.

It has the properties:

(i)

$$Z(x) > 0, \ \forall\, x \in \mathbb{R}^N, \tag{20}$$

(ii)

$$\sum_{k=-\infty}^{\infty} Z(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} Z(x_1-k_1, ..., x_N-k_N) = 1, \tag{21}$$

where $k := (k_1, ..., k_N) \in \mathbb{Z}^N$, $\forall\, x \in \mathbb{R}^N$,

hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z(nx-k) = 1, \forall\, x \in \mathbb{R}^N,\ n \in \mathbb{N}, \tag{22}$$

and

(iv)

$$\int_{\mathbb{R}^N} Z(x)\, dx = 1, \tag{23}$$

that is Z *is a multivariate density function.*

Here $\|x\|_\infty := \max\{|x_1|, ..., |x_N|\}$, $x \in \mathbb{R}^N$, *also set* $\infty := (\infty, ..., \infty)$, $-\infty = (-\infty, ..., -\infty)$ *upon the multivariate context.*

It is also clear that (see (18))

(v)

$$\sum_{\substack{k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta}}}^{\infty} Z(nx-k) \leq \frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right) e^{\left(n^{1-\beta}-2\right)^2}}, \tag{24}$$

$0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} \geq 3$, $x \in \mathbb{R}^N$.

For $f \in C_B^+\left(\mathbb{R}^N\right)$ (continuous and bounded functions from $\mathbb{R}^N$ into $\mathbb{R}_+$), we define the first modulus of continuity

$$\omega_1(f,h) := \sup_{\substack{x,y \in \mathbb{R}^N \\ \|x-y\|_\infty \leq h}} |f(x) - f(y)|, \quad h > 0. \tag{25}$$

Given that $f \in C_U^+\left(\mathbb{R}^N\right)$ (uniformly continuous from $\mathbb{R}^N$ into $\mathbb{R}_+$), we have that

$$\lim_{h \to 0} \omega_1(f,h) = 0. \tag{26}$$

We make

Definition 3.5. *Let* $\mathcal{L}$ *be the Lebesgue* σ*-algebra on* $\mathbb{R}^N$, $N \in \mathbb{N}$, *and the maxitive measure* $\mu : \mathcal{L} \to [0, +\infty)$, *such that for any* $A \in \mathcal{L}$ *with* $A \neq \varnothing$, *we get* $\mu(A) > 0$.

For $f \in C_B^+\left(\mathbb{R}^N\right)$, *we define the multivariate Kantorovich-Shilkret type neural network operator for any* $x \in \mathbb{R}^N$:

$$T_n^\mu(f,x) = T_n^\mu(f, x_1, ..., x_N) :=$$

$$\sum_{k=-\infty}^{\infty} \left(\frac{(N^*)\int_{\left[0,\frac{1}{N}\right]^N} f\left(t+\frac{k}{n}\right) d\mu(t)}{\mu\left(\left[0,\frac{1}{n}\right]^N\right)} \right) Z(nx-k) =$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \left(\frac{(N^*)\int_0^{\frac{1}{n}} ... \int_0^{\frac{1}{n}} f\left(t_1+\frac{k_1}{n}, t_2+\frac{k_2}{n}, ..., t_N+\frac{k_N}{n}\right) d\mu(t_1, ..., t_N)}{\mu\left(\left[0,\frac{1}{n}\right]^N\right)} \right) \tag{27}$$

$$\cdot \left(\prod_{i=1}^N Z(nx_i - k_i) \right),$$

where $x = (x_1, ..., x_N) \in \mathbb{R}^N$, $k = (k_1, ..., k_N)$, $t = (t_1, ..., t_N)$, $n \in \mathbb{N}$.

Clearly here $\mu\left(\left[0,\frac{1}{n}\right]^N\right) > 0$, $\forall\, n \in \mathbb{N}$.

Above we notice that

$$\left\|T_n^\mu(f)\right\|_\infty \leq \|f\|_\infty, \tag{28}$$

so that $T_n^\mu(f,x)$ *is well-defined.*

Remark 3.6. *Let* $t \in \left[0, \frac{1}{n}\right]^N$ *and* $x \in \mathbb{R}^N$, *then*

$$f\left(t+\frac{k}{n}\right) = f\left(t+\frac{k}{n}\right) - f(x) + f(x) \leq \left|f\left(t+\frac{k}{n}\right) - f(x)\right| + f(x), \tag{29}$$

hence

$$(N^*)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\frac{k}{n}\right) d\mu(t) \leq$$
$$(N^*)\int_{\left[0,\frac{1}{n}\right]^N} \left|f\left(t+\frac{k}{n}\right) - f(x)\right| d\mu(t) + f(x)\,\mu\left(\left[0,\frac{1}{n}\right]^N\right). \tag{30}$$

That is

$$(N^*)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\frac{k}{n}\right) d\mu(t) - f(x)\,\mu\left(\left[0,\frac{1}{n}\right]^N\right) \leq \tag{31}$$
$$(N^*)\int_{\left[0,\frac{1}{n}\right]^N} \left|f\left(t+\frac{k}{n}\right) - f(x)\right| d\mu(t).$$

Similarly we have

$$f(x) = f(x) - f\left(t+\frac{k}{n}\right) + f\left(t+\frac{k}{n}\right) \leq \left|f\left(t+\frac{k}{n}\right) - f(x)\right| + f\left(t+\frac{k}{n}\right),$$

hence

$$(N^*)\int_{\left[0,\frac{1}{n}\right]^N} f(x)\,d\mu(t) \leq (N^*)\int_{\left[0,\frac{1}{n}\right]^N} \left|f\left(t+\frac{k}{n}\right) - f(x)\right| d\mu(t)$$
$$+ (N^*)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\frac{k}{n}\right) d\mu(t).$$

That is

$$f(x)\,\mu\left(\left[0,\frac{1}{n}\right]^N\right) - (N^*)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\frac{k}{n}\right) d\mu(t) \leq \tag{32}$$
$$(N^*)\int_{\left[0,\frac{1}{n}\right]^N} \left|f\left(t+\frac{k}{n}\right) - f(x)\right| d\mu(t).$$

By (31) and (32) we derive

$$\left|(N^*)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\frac{k}{n}\right) d\mu(t) - f(x)\,\mu\left(\left[0,\frac{1}{n}\right]^N\right)\right| \leq$$
$$(N^*)\int_{\left[0,\frac{1}{n}\right]^N} \left|f\left(t+\frac{k}{n}\right) - f(x)\right| d\mu(t). \tag{33}$$

In particular it holds

$$\left|\frac{(N^*)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\frac{k}{n}\right) d\mu(t)}{\mu\left(\left[0,\frac{1}{n}\right]^N\right)} - f(x)\right| \leq$$
$$\frac{(N^*)\int_{\left[0,\frac{1}{n}\right]^N} \left|f\left(t+\frac{k}{n}\right) - f(x)\right| d\mu(t)}{\mu\left(\left[0,\frac{1}{n}\right]^N\right)}. \tag{34}$$

We present

Theorem 3.7. *Let* $f \in C_B^+(\mathbb{R}^N)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$; $N, n \in \mathbb{N}$ *with* $n^{1-\beta} \geq 3$. *Then*

i)

$$\sup_{\mu} |T_n^{\mu}(f, x) - f(x)| \leq \omega_1\left(f, \frac{1}{n} + \frac{1}{n^{\beta}}\right) + \frac{\|f\|_{\infty}}{\sqrt{\pi}\left(n^{1-\beta} - 2\right) e^{\left(n^{1-\beta}-2\right)^2}} =: \lambda_n, \tag{35}$$

ii)

$$\sup_{\mu} \|T_n^{\mu}(f) - f\|_{\infty} \leq \lambda_n. \tag{36}$$

Given that $f \in \left(C_U^+(\mathbb{R}^N) \cap C_B^+(\mathbb{R}^N)\right)$, *we obtain* $\lim_{n \to \infty} T_n^{\mu}(f) = f$, *uniformly.*

Proof. We observe that

$$|T_n^{\mu}(f, x) - f(x)| =$$

$$\left| \sum_{k=-\infty}^{\infty} \left(\frac{(N^*) \int_{[0,\frac{1}{n}]^N} f\left(t + \frac{k}{n}\right) d\mu(t)}{\mu\left(\left[0, \frac{1}{n}\right]^N\right)} \right) Z(nx - k) - \sum_{k=-\infty}^{\infty} f(x) Z(nx - k) \right| =$$

$$\left| \sum_{k=-\infty}^{\infty} \left(\left(\frac{(N^*) \int_{[0,\frac{1}{n}]^N} f\left(t + \frac{k}{n}\right) d\mu(t)}{\mu\left(\left[0, \frac{1}{n}\right]^N\right)} \right) - f(x) \right) Z(nx - k) \right| \leq \tag{37}$$

$$\sum_{k=-\infty}^{\infty} \left| \left(\frac{(N^*) \int_{[0,\frac{1}{n}]^N} f\left(t + \frac{k}{n}\right) d\mu(t)}{\mu\left(\left[0, \frac{1}{n}\right]^N\right)} \right) - f(x) \right| Z(nx - k) \overset{(34)}{\leq}$$

$$\sum_{k=-\infty}^{\infty} \left(\frac{(N^*) \int_{[0,\frac{1}{n}]^N} \left| f\left(t + \frac{k}{n}\right) - f(x) \right| d\mu(t)}{\mu\left(\left[0, \frac{1}{n}\right]^N\right)} \right) Z(nx - k) = \tag{38}$$

$$\sum_{\substack{k=-\infty \\ : \left\|\frac{k}{n} - x\right\|_{\infty} \leq \frac{1}{n^{\beta}}}}^{\infty} \left(\frac{(N^*) \int_{[0,\frac{1}{n}]^N} \left| f\left(t + \frac{k}{n}\right) - f(x) \right| d\mu(t)}{\mu\left(\left[0, \frac{1}{n}\right]^N\right)} \right) Z(nx - k) +$$

$$\sum_{\substack{k=-\infty \\ : \left\|\frac{k}{n} - x\right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\infty} \left(\frac{(N^*) \int_{[0,\frac{1}{n}]^N} \left| f\left(t + \frac{k}{n}\right) - f(x) \right| d\mu(t)}{\mu\left(\left[0, \frac{1}{n}\right]^N\right)} \right) Z(nx - k) \leq$$

$$\sum_{\substack{k=-\infty \\ : \left\|\frac{k}{n} - x\right\|_{\infty} \leq \frac{1}{n^{\beta}}}}^{\infty} \left(\frac{(N^*) \int_{[0,\frac{1}{n}]^N} \omega_1\left(f, \|t\|_{\infty} + \left\|\frac{k}{n} - x\right\|_{\infty}\right) d\mu(t)}{\mu\left(\left[0, \frac{1}{n}\right]^N\right)} \right) Z(nx - k) \tag{39}$$

$$+2\|f\|_{\infty}\left(\sum_{\substack{k=-\infty \\ \left\{ :\left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}\right.}}^{\infty} Z(nx-k)\right) \overset{(24)}{\leq}$$

$$\omega_1\left(f, \frac{1}{n}+\frac{1}{n^{\beta}}\right)+\frac{\|f\|_{\infty}}{\sqrt{\pi}\left(n^{1-\beta}-2\right) e^{\left(n^{1-\beta}-2\right)^2}}, \tag{40}$$

proving the claim. ∎

Additionally we give

Definition 3.8. *Denote by* $C_B^{+}\left(\mathbb{R}^N, \mathbb{C}\right)=\{f: \mathbb{R}^N \rightarrow \mathbb{C} \mid f=f_1+if_2,$ *where* $f_1, f_2 \in C_B^{+}\left(\mathbb{R}^N\right)$, $N \in \mathbb{N}\}$. *We set for* $f \in C_B^{+}\left(\mathbb{R}^N, \mathbb{C}\right)$ *that*

$$T_n^{\mu}(f, x):=T_n^{\mu}\left(f_1, x\right)+i T_n^{\mu}\left(f_2, x\right), \tag{41}$$

$\forall\, n \in \mathbb{N}$, $x \in \mathbb{R}^N$, $i=\sqrt{-1}$.

Theorem 3.9. *Let* $f \in C_B^{+}\left(\mathbb{R}^N, \mathbb{C}\right)$, $f=f_1+if_2$, $N \in \mathbb{N}$, $0<\beta<1$, $x \in \mathbb{R}^N$; $n \in \mathbb{N}$ *with* $n^{1-\beta} \geq 3$. *Then*

i)

$$\sup_{\mu}\left|T_n^{\mu}(f, x)-f(x)\right| \leq\left[\omega_1\left(f_1, \frac{1}{n}+\frac{1}{n^{\beta}}\right)+\omega_1\left(f_2, \frac{1}{n}+\frac{1}{n^{\beta}}\right)\right]$$
$$+\frac{\left(\left\|f_1\right\|_{\infty}+\left\|f_2\right\|_{\infty}\right)}{\sqrt{\pi}\left(n^{1-\beta}-2\right) e^{\left(n^{1-\beta}-2\right)^2}}=: \psi_n, \tag{42}$$

and

ii)

$$\sup_{\mu}\left\|T_n^{\mu}(f)-f\right\| \leq \psi_n. \tag{43}$$

Proof.

$$\left|T_n^{\mu}(f, x)-f(x)\right|=\left|T_n^{\mu}\left(f_1, x\right)+i T_n^{\mu}\left(f_2, x\right)-f_1(x)-i f_2(x)\right|=$$
$$\left|\left(T_n^{\mu}\left(f_1, x\right)-f_1(x)\right)+i\left(T_n^{\mu}\left(f_2, x\right)-f_2(x)\right)\right| \leq$$
$$\left|T_n^{\mu}\left(f_1, x\right)-f_1(x)\right|+\left|T_n^{\mu}\left(f_2, x\right)-f_2(x)\right| \overset{(35)}{\leq} \tag{44}$$
$$\left(\omega_1\left(f_1, \frac{1}{n}+\frac{1}{n^{\beta}}\right)+\frac{\left\|f_1\right\|_{\infty}}{\sqrt{\pi}\left(n^{1-\beta}-2\right) e^{\left(n^{1-\beta}-2\right)^2}}\right)+$$
$$\left(\omega_1\left(f_2, \frac{1}{n}+\frac{1}{n^{\beta}}\right)+\frac{\left\|f_2\right\|_{\infty}}{\sqrt{\pi}\left(n^{1-\beta}-2\right) e^{\left(n^{1-\beta}-2\right)^2}}\right)=$$

$$\left[\omega_1\left(f_1, \frac{1}{n}+\frac{1}{n^{\beta}}\right)+\omega_1\left(f_2, \frac{1}{n}+\frac{1}{n^{\beta}}\right)\right]+$$

$$\frac{\left(\|f_1\|_{\infty}+\|f_2\|_{\infty}\right)}{\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}, \tag{45}$$

proving the claim. ∎

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