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Postulation of general unions of lines and +lines in positive characteristic

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ABSTRACT

A +line is a scheme $R \subset \mathbb{P}^r$ with a line as its reduction $L = R_{red}$ which is the union of L and a tangent vector $\nu \nsubseteq L$ with $\nu_{red} \in L$. Here we prove in arbitrary characteristic that for $r \geq 4$ a general union of lines and +lines has maximal rank. We use the case $r = 3$ proved by myself in a previous paper and adapt the characteristic zero proof of the case $r > 3$ given in the same paper.

RESUMEN

Una +línea es un esquema $R \subset \mathbb{P}^r$ con una línea como su reducción $L = R_{red}$ que es la unión de L y un vector tangente $\nu \nsubseteq L$, con $\nu_{red} \in L$. Acá demostramos que para $r \geq 4$ una unión general de líneas y +líneas tiene rango máximo en característica arbitraria. Usamos el caso $r = 3$ demostrado por el autor en un artículo anterior y adaptamos la demostración en característica cero del caso $r > 3$ dado en el mismo artículo anterior.

Keywords and Phrases: Hilbert function; decorated line; disjoint unions of lines.

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1 Introduction

The aim of this note is to extend to the positive characteristic case a results in [1]. This extension is sufficient to extend [2, 3] to the positive characteristic case.

A scheme $X \subset \mathbb{P}^r$ is said to have *maximal rank* if $h^0(\mathcal{I}_X(t)) \cdot h^1(\mathcal{I}_X(t)) = 0$ for all $t \in \mathbb{N}$. Fix a line $L \subset \mathbb{P}^r$, $r \geq 2$, and $P \in L$. A tangent vector of $\mathbb{P}^r$ with P as its support is a zero-dimensional scheme $Z \subset \mathbb{P}^r$ such that $\deg(Z) = 2$ and $Z_{red} = \{P\}$. The tangent vector Z is uniquely determined by P and the line $\langle Z \rangle$ spanned by Z. Conversely, for each line $D \subset \mathbb{P}^r$ with $P \in D$ there is a unique tangent vector ν with $\nu_{red} = P$ and $\langle \nu \rangle = D$. A +line $M \subset \mathbb{P}^r$ supported by L and with nilradical supported by P is the union $\nu \cup L$ of L and a tangent vector ν with P as its support and spanning a line $\langle \nu \rangle \neq L$. The set of all +lines of $\mathbb{P}^r$ supported by L and with a nilradical at P is an irreducible variety of dimension $r-1$ (the complement of L in the $(r-1)$-dimensional projective space of all lines of $\mathbb{P}^r$ containing P). Hence the set of all +lines of $\mathbb{P}^r$ supported by L is parametrized by an irreducible variety of dimension r. For any +line R and every integer $k > 0$ we have $h^0(\mathcal{O}_R(k)) = k+2$ and $h^1(\mathcal{O}_R(k)) = 0$.

For any integers $r \geq 3$, $t \geq 0$, $c \geq 0$ with $(t, c) \neq (0, 0)$ let $L(r, t, c)$ be the set of all schemes $X \subset \mathbb{P}^r$ which are the disjoint union of t lines and c +lines. Every element of $L(r, c, t)$ has the map $k \mapsto (k+1)t + (k+2)c$ as its Hilbert function.

Consider the following statement.

Theorem 1.1. *For all integers $r \geq 3$, $a \geq 0$ and $b \geq 0$, $(a, b) \neq (0, 0)$, a general union $X \subset \mathbb{P}^r$ of a lines and b +lines has maximal rank,*

This statement was proved in [1] when either $r = 3$ or $r \geq 4$ and the algebraically closed base field has characteristic zero. The aim of this note is to prove Theorem 1.1 in positive characteristic (using the case $r = 3$ proved in [1]). Hence we may assume $r \geq 4$. We also use numerical lemmas and elementary remarks contained in [1]. We only need to change all parts which quote [4, Lemma 1.4] or [6], the only characteristic zero tool used in [1]. We recall that the case $c = 0$ is due to R. Hartshorne and A. Hirschowitz ([7]).

2 Proof of Theorem 1.1

For all integers $r \geq 3$ and $k \geq 0$ let $H_{r,k}$ denote the following statement:

Assertion $H_{r,k}$, $r \geq 3$, $k \geq 0$: Fix $(t, c) \in \mathbb{N}^2 \setminus \{(0, 0)\}$ and take a general $X \in L(r, t, c)$. If $(k+1)t + (k+2)c \geq \binom{r+k}{k}$, then $h^0(\mathcal{I}_X(k)) = 0$. If $(k+1)t + (k+2)c \leq \binom{r+k}{k}$, then $h^1(\mathcal{I}_X(k)) = 0$.

For all integers $r \geq 3$ and $k \geq 0$ define the integers $m_{r,k}$ and $n_{r,k}$ by the relations

$$(k+1)m_{r,k} + n_{r,k} = \binom{r+k}{r}, \; 0 \leq n_{r,k} \leq k \tag{2.1}$$

From (2.1) for the pairs (r,k) and $(r,k-1)$ we get

$$m_{r,k-1} + (k+1)(m_{r,k} - m_{r,k-1}) + n_{r,k} - n_{r,k-1} = \binom{r+k-1}{r-1} \tag{2.2}$$

for all $k > 0$.

For all integers $r \geq 3$ and $k \geq 0$ set $u_{r,k} := \lceil \binom{r+k}{r}/(k+2) \rceil$ and $v_{r,k} := (k+2)u_{r,k} - \binom{r+k}{r}$. We have

$$(k+2)(u_{r,k} - v_{r,k}) + (k+1)v_{r,k} = \binom{r+k}{r}, \; 0 \leq v_{r,k} \leq k+1 \tag{2.3}$$

As in [1] we need the following assumption $B_{r,k}$:

Assumption $B_{r,k}$, $r \geq 4$, $k > 0$. Fix a hyperplane $H \subset \mathbb{P}^r$. There is $X \in L(r, m_{r,k} - n_{r,k}, n_{r,k})$ such that the support of the nilradical sheaf of X is contained in H and $h^0(\mathcal{I}_X(k)) = 0$.

For all $X \in L(r, m_{r,k} - n_{r,k}, n_{r,k})$ we have $h^0(\mathcal{O}_X(k)) = \binom{r+k}{r}$ and so $h^1(\mathcal{I}_X(k)) = h^0(\mathcal{I}_X(k))$.

Lemma 2.1. *We have* $m_{r,k} - m_{r,k-1} \geq n_{r,k-1} + n_{r,k}$ *if* $r \geq 4$ *and* $k \geq 2$.

Proof. Assume $m_{r,k} - m_{r,k-1} \leq n_{r,k-1} + n_{r,k} - 1$. From (2.1) we get

$$m_{r,k-1} + kn_{r,k-1} + (k+2)n_{r,k} - k - 1 \geq \binom{r+k-1}{r-1}$$

Since $n_{r,k-1} \leq k-1$ and $n_{r,k} \leq k$, we get $m_{r,k-1} \geq \binom{r+k-1}{r-1} - 2k^2 + 1$. Since $km_{r,k-1} \leq \binom{r+k-1}{r}$ and $k\binom{r+k-1}{r-1} - \binom{r+k-1}{r} = (r-1)\binom{r+k-1}{r}$, we get

$$2k^3 - k \geq (r-1)\binom{r+k-1}{r} \tag{2.4}$$

This inequality is false if $r = 4$ and $k \geq 2$, because it is equivalent to the inequality $k(2k^2-1) \geq (k+3)(k+2)(k+1)k/8$. Since the right hand side of (2.4) is an increasing function of r, we conclude for all $r \geq 5$ and $k \geq 2$. □

Lemma 2.2. *Fix an integer* $r \geq 4$ *and assume that Theorem 1.1 is true in* $\mathbb{P}^{r-1}$*. Then* $B_{r,k}$ *is true for all* $k > 0$.

Proof. Since the case $k = 1$ is true ([1, Remark 3]), we may assume $k \geq 2$ and use induction on k. By Lemma 2.1 we have $m_{r,k} - m_{r,k-1} \geq n_{r,k-1} + n_{r,k}$. Fix a solution $X \in L(r, m_{r,k-1} - n_{r,k-1}, n_{r,k-1})$ of $B_{r,k-1}$, say $X = A \sqcup B$ with $A \in L(r, m_{r,k-1} - n_{r,k-1}, 0)$, $B \in L(r, 0, n_{r,k-1})$ and the tangent vectors of B have support $S \subset H$. By semicontinuity we may assume that no irreducible component of X_{red} is contained in H, that no tangent vector associated to the nilradical of B is contained in H and that S is a general subset of H with cardinality $n_{r,k-1}$. Let $C_1 \subset H$ be a general union of $m_{r,k} - m_{r,k-1} - n_{r,k} - n_{r,k-1}$ lines. Let $C_2 \subset H$ be a general union of $n_{r,k-1}$ lines, each of them containing a different point of S. Let $E \subset H$ be a general union of $n_{r,k}$ +lines. Since S is general, $C_1 \cup C_2 \cup E$ is a general element of $L(r-1, m_{r,k} - m_{r,k-1} - n_{r,k}, r_{n,k})$. Since Theorem 1.1 is true in $\mathbb{P}^{r-1}$, by (2.2) we get $h^1(H, \mathcal{I}_{C_1\cup C_2\cup F}(k)) = 0$ and $h^0(H, \mathcal{I}_{C_1\cup C_2\cup E}(k)) = m_{r,k-1} - n_{r,k-1}$. Deforming A with $B \cup C_1 \cup C_2 \cup E$ fixed, we may assume $A \cap (B \cup C_1 \cup C_2 \cup E) = \emptyset$ and that $h^i(H, \mathcal{I}_{C_1\cup C_2\cup E\cup(A\cap H)}(k)) = 0$, $i = 0, 1$. Since $A \cap (B \cup C_1 \cup C_2 \cup E) = \emptyset$, $Y := A \cup B \cup C_1 \cup C_2 \cup E$ is a disjoint union of $n_{r,k}$ +lines with support in H (even contained in H), $m_{r,k} - 2n_{r,k-1} - n_{r,k}$ lines and $n_{r,k-1}$ sundials in the sense of [5]. Hence Y is a flat limit of a family of elements $L(r, m_{r,k-1} - n_{r,k-1}, n_{r,k-1})$ whose nilpotent sheaf is contained in H ([7], [5]). By the semicontinuity theorem to prove $B_{r,k}$ it is sufficient to prove that $h^0(\mathcal{I}_Y(k)) = 0$. Since no tangent vector of B is contained in H, then $\mathrm{Res}_H(Y) = X$ and $Y \cap H = C_1 \cup C_2 \cup E \cup (A \cap H)$. Since $h^0(\mathcal{I}_X(k-1)) = 0$ and $h^0(H, \mathcal{I}_{C_1\cup C_2\cup E\cup(A\cap H)}(k)) = 0$, a residual exact sequence gives $h^0(\mathcal{I}_Y(k)) = 0$. □

Lemma 2.3. *Assume* $r \geq 4$ *and that Theorem 1.1 is true in* $H = \mathbb{P}^{r-1}$. *Fix an integer* $k \geq 2$ *and assume that* $H_{r,k-1}$ *is true. Fix integers* $a \geq 0$, $b \geq 0$, $e \geq 0$ *such that* $e \leq 2\lfloor (k+2)/2 \rfloor$, $(k+2)a + (k+1)b + 4\lfloor (k+2)/2 \rfloor \leq \binom{r+k-1}{r-1}$. *Let* $X \subset H$ *be a general union of* a *+lines,* b *lines and* e *tangent vectors. Then* $h^1(H, \mathcal{I}_X(k)) = 0$.

Proof. It is sufficient to do the case $e = \lfloor (k+2)/2 \rfloor$. Let $A \subset H$ be a general union of a lines and b 2-lines.

First assume that k is even. Let $L_1, L_2 \subset H$ be general lines. Fix a general $S_i \subset L_i$ with $\sharp(S_i) = k/2$ and a general $P_i \in L_i$, $i = 1, 2$. Let $v_i \subset H$ be a general tangent vector of H with P_i as its support; in particular we assume $v_i \nsubseteq L_i$. Let $E_i \subset L_i$ be the union of the $k/2$ tangent vectors of L_i with $(E_i)_{red} = S_i$. Set $Y := A \cup E_1 \cup v_1 \cup E_2 \cup v_2$. Let R_i the +lines with L_i as their supports and with v_i as the tangent vectors associated to their nilpotent sheaf. We have $h^0(\mathcal{O}_{A\cup E_1\cup E_2\cup v_1\cup v_2}(k)) = h^0(\mathcal{O}_{A\cup R_1\cup R_2}(k))$, $h^1(\mathcal{O}_{A\cup E_1\cup E_2\cup v_1\cup v_2}(k)) = h^1(\mathcal{O}_{A\cup R_1\cup R_2}(k))$ and $h^0(\mathcal{I}_{A\cup E_1\cup E_2\cup v_1\cup v_2}(k)) = h^0(\mathcal{I}_{A\cup R_1\cup R_2}(k))$. Therefore we have $h^1(\mathcal{I}_{A\cup E_1\cup E_2\cup v_1\cup v_2}(k)) = h^1(\mathcal{I}_{A\cup R_1\cup R_2}(k))$. Since $(k+2)a + (k+1)b + 2(k+2) \leq \binom{r+k-1}{r-1}$ and Theorem 1.1 is true in $\mathbb{P}^{r-1}$, we have $h^1(\mathcal{I}_{A\cup R_1\cup R_2}(k)) = 0$. Hence $h^1(\mathcal{I}_{A\cup E_1\cup E_2\cup v_1\cup v_2}(k)) = 0$. The semicontinuity theorem gives $h^1(H, \mathcal{I}_X(k)) = 0$.

Now assume that k is even. Let $F_i \subset L_i$ be any disjoint union of $(k+1)/2$ tangent vectors. We have $h^0(\mathcal{O}_{A\cup F_1\cup F_2}(k)) = h^0(\mathcal{O}_{A\cup L_1\cup L_2}(k))$, $h^1(\mathcal{O}_{A\cup F_1\cup F_2}(k)) = h^1(\mathcal{O}_{A\cup L_1\cup L_2}(k))$ and $h^0(\mathcal{I}_{A\cup F_1\cup F_2}(k)) = h^0(\mathcal{I}_{A\cup L_1\cup L_2}(k))$. Therefore we obtain $h^1(\mathcal{I}_{A\cup F_1\cup F_2}(k)) = h^1(\mathcal{I}_{A\cup L_1\cup L_2}(k))$. Since $(k+2)a + (k+1)b + 2(k+1) \leq \binom{r+k-1}{r-1}$ and Theorem 1.1 is true in $\mathbb{P}^{r-1}$, we have $h^1(\mathcal{I}_{A\cup L_1\cup L_2}(k)) = 0$. Therefore $h^1(\mathcal{I}_{A\cup F_1\cup F_2}(k)) = 0$. The semicontinuity theorem gives $h^1(H, \mathcal{I}_X(k)) = 0$. □

Proof of Theorem 1.1: By [1] we may assume $r \geq 4$. By induction on r we may also assume that Theorem 1.1 is true in $\mathbb{P}^{r-1}$. By [1, Remark 3] it is sufficient to prove $H_{r,k}$ for all integers $k \geq 1$. $H_{r,1}$ is true ([1, Lemma 3]). Hence we may assume $k \geq 2$ and that $H_{r,k-1}$ is true. By [1, Remark 4] it is sufficient to prove $H_{r,k}$ for the pairs (t, c) such that either $t = 0$ and $\binom{r+k}{r} - k - 1 \leq c(k+2) \leq \binom{r+k}{r}$ or $t(k+1) + (k+2)c = \binom{r+k}{r}$ and $c > 0$; in the former case either

$\nu_{r,k} = 0$ and $c = u_{r,k}$ or $\nu_{r,k} > 0$ and $c = u_{r,k} - 1$; in the latter case we have $t + c \geq u_{r,k}$. If $c < n_{r,k-1}$, then we use step (b) of the proof of Theorem 1 in [1], because we gave a characteristic free proof of $B_{r,k}$ (Lemma 2.2). The case $c \geq n_{r,k-1}$ and $t \geq m_{r,k-1} - n_{r,k-1}$ was proved as step (a1) without using the characteristic zero assumption. Hence we may assume $c \geq n_{r,k-1}$ and $t < m_{r,k-1} - n_{r,k-1}$, i.e. the case of step (a2) of the proof in [1].

(i) Assume $t = 0$ and hence either $\nu_{r,k} = 0$ and $c = u_{r,k}$ or $\nu_{r,k} > 0$ and $c = u_{r,k} - 1$. Fix a general $U \in L(r, 0, \nu_{r,k-1}, u_{r,k-1} - \nu_{r,k-1})$, say $U = A \sqcup B$ with A the union of the $\nu_{r,k-1}$ lines. By $H_{r,k-1}$ we have $h^i(\mathcal{I}_U(k-1)) = 0$, $i = 0, 1$. It is easy to check using (2.3) that $u_{r,k} > u_{r,k-1}$. Hence $c \geq u_{r,k-1}$. Let $E \subset H$ be a general union of $c - u_{r,k-1}$ +lines. We may assume $E \cap (H \cap U)) = \emptyset$. Let $G \subset H$ be a general union of $\nu_{r,k-1}$ tangent vectors of H with the only restriction that $G_{red} = A \cap H$. For general A (and hence a general $A \cap H$)) the scheme $E \cup G$ is a general union inside H of $u_{r,k} - u_{k-1}$ +lines and $\nu_{r,k-1}$ tangent vectors. We have $\nu_{r,k-1} \leq k$. Using (2.3) for the integer $k - 1$ is easy to check that if $\nu_{r,k-1} > 0$, then $u_{r,k-1} - \nu_{r,k-1} \geq 2(k + 2) - 2\nu_{r,k-1}$. Hence Lemma 2.3 gives $h^1(H, \mathcal{I}_{E\cup G}(k)) = 0$. Since $B \cap H$ is a general union of

(ii) Assume $t > 0$, $c > 0$, $t(k+1) + (k+2)c = \binom{r+k}{r}$ and $t < m_{r,k-1} - n_{r,k-1}$. First assume $t \leq 2\lfloor (k + 2)/2 \rfloor$. In this case we may use the proof given in [1] (step (a2)) quoting Lemma 2.3 instead of [4, Lemma 1.4] for the postulation of the t tangent vectors, because $m_{r,k-1} - t \geq 2k + 2$ in this case. Therefore we may assume $t \geq k + 1$. Since $t < m_{r,k-1} - n_{r,k-1}$, we have $k \geq 3$ and $kt < \binom{r+k-1}{r}$. Set $d := \lfloor (\binom{r+k-1}{r} - kt)/(k + 1) \rfloor$ and $z := (k + 1)d + kt - \binom{r+k}{r}$. We have $0 \leq z \leq k + 1$. Fix a general $W \in L(r, t, d)$. Since $H_{r,x}$ holds for $x = k - 1, k - 2$, we have $h^0(\mathcal{I}_W(k - 2)) = 0$ and $h^1(\mathcal{I}_W(k)) = 0$ and $h^0(\mathcal{I}_W(k)) = z$. Since S is general in H and $\sharp(S) = z$, we get $h^i(\mathcal{I}_{W\cup S}(k-1)) = 0$, $i = 0, 1$. Since $kt+(k+1)t+z = \binom{r+k-1}{r}$ and $t(k+1)+(k+2)c = \binom{r+k}{r}$, we get

$$t + d + (k + 2)(c - d - z) + (k + 1)z = \binom{r + k - 1}{r - 1} \tag{2.5}$$

Claim 1: We have $c \geq d + z$.

Proof of Claim 1: Assume $c \leq d+z-1$. From (2.5) we get $t+d+(k+1)z-(k+1) \geq \binom{r+k-1}{r-1}$ and hence $k(t + d) + (k + 1)kz - (k + 1)k \geq k\binom{r+k-1}{r-1}$. Since $kt + (k + 1)d + z = \binom{r+k-1}{r}$ and $z \leq k$, we get $(k + 1)k^2 - k(k + 1) - k \geq k\binom{r+k-1}{r-1} - \binom{r+k-1}{r}$, i.e. $k^3 - 2k \geq (r - 1)\binom{r+k-1}{r}$. Call $\phi(r, k)$ the difference between the right hand side and the left hand side of this inequality. We have $\phi(r, k) = (r - 1)\binom{r+k-1}{r} - k^3 + 2k$, which is positive if $r \geq 4$ and $k \geq 2$.

Let $M \subset H$ be a general union of $c - d - z$ +lines of H. Let $N \subset H$ be z general lines of H, each of them containing a different point of Z. Since S is general, $M \cup N$ has the Hilbert function of a general element of $L(r - 1, z, c - d - z)$ and hence it has maximal rank. By (2.5) we have $h^1(H, \mathcal{I}_{M\cup N}(k)) = 0$ and $h^0(\mathcal{I}_{M\cup N}(k)) = t + d$. Let $Z \subset \mathbb{P}^r$ be a general union of z +lines of $\mathbb{P}^r$ with N as their support. We have $G \cap H = N$ and $\mathrm{Res}_H(Z) = S$. Since $W \cup M \cup Z \in L(r, t, c)$, it is sufficient to prove that $h^i(\mathcal{I}_{W\cup M\cup Z}(k)) = 0$, $i = 0, 1$. Since $\mathrm{Res}_H(W \cup M \cup Z) = W \cup S$, we have $h^i(\mathcal{I}_{\mathrm{Res}_H(W\cup M\cup Z)}(k - 1)) = 0$. Since $W \cap H$ is a general union of $d + c$ points of H and $(W \cup M \cup Z) = (W \cap H) \cup M \cup N$ as schemes, (2.5) gives $h^i(H, \mathcal{I}_{H\cap(W\cup M\cup Z)}(k)) = 0$. Apply the

Castelnuovo's lemma. □

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