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On algebraic and uniqueness properties of harmonic quaternion fields on 3d manifolds

M.I.Belishev and A.F.Vakulenko
Saint-Petersburg Department of the Steklov Mathematical Institute,
St-Petersburg State University, Supported by the RFBR grant 18-01-00269.
belishev@pdmi.ras.ru, vak@pdmi.ras.ru

ABSTRACT

Let Ω be a smooth compact oriented 3-dimensional Riemannian manifold with boundary. A quaternion field is a pair $q = \{\alpha, u\}$ of a function α and a vector field u on Ω. A field q is *harmonic* if α, u are continuous in Ω and $\nabla\alpha = \operatorname{rot} u$, $\operatorname{div} u = 0$ holds into Ω. The space $\mathscr{Q}(\Omega)$ of harmonic fields is a subspace of the Banach algebra $\mathscr{C}(\Omega)$ of continuous quaternion fields with the point-wise multiplication $qq' = \{\alpha\alpha' - u \cdot u',\ \alpha u' + \alpha' u + u \wedge u'\}$. We prove a Stone-Weierstrass type theorem: the subalgebra $\vee\mathscr{Q}(\Omega)$ generated by harmonic fields is dense in $\mathscr{C}(\Omega)$. Some results on 2-jets of harmonic functions and the uniqueness sets of harmonic fields are provided.

Comprehensive study of harmonic fields is motivated by possible applications to inverse problems of mathematical physics.

RESUMEN

Sea Ω una variedad Riemanniana 3-dimensional suave con borde, orientada y compacta. Un campo cuaterniónico es un par $q = \{\alpha, u\}$ dado por una función α y un campo de vectores u en Ω. Un campo q es *armónico* si α, u son continuos en Ω y $\nabla\alpha = \operatorname{rot} u$, $\operatorname{div} u = 0$ vale en todo Ω. El espacio $\mathscr{Q}(\Omega)$ de campos armónicos es un subespacio del álgebra de Banach $\mathscr{C}(\Omega)$ de campos cuaterniónicos continuos con la multiplicación punto a punto $qq' = \{\alpha\alpha' - u \cdot u',\ \alpha u' + \alpha' u + u \wedge u'\}$. Probamos un teorema de tipo Stone-Weierstrass: la subálgebra $\vee\mathscr{Q}(\Omega)$ generada por campos armónicos es densa en $\mathscr{C}(\Omega)$. Se entregan también algunos resultados acerca de 2-jets de funciones armónicas y los conjuntos de unicidad campos armónicos.

Keywords and Phrases: 3d quaternion harmonic fields, real uniform Banach algebras, Stone-Weierstrass type theorem on density, uniqueness theorems.

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1 Introduction

Motivation

There is an approach to inverse problems of mathematical physics (the so-called Boundary Control method), which was originally based on the relations between inverse problems and the boundary control theory [4, 7, 9]. The BC-method recovers Riemannian manifolds via spectral and/or dynamical boundary data. Later on, its version that makes use of connections with Banach algebras, was proposed in [2, 5, 6].

The problem of recovering the manifold via its DN-map (the so-called Impedance Tomography Problem) in dimensions $\geqslant 3$ isn't yet properly solved. However, beginning from the papers [3, 10] it becomes clear that harmonic quaternion fields may play the key role in the 3d ITP. It is the reason, which has stimulated the study of their properties [8, 11].

Here we consider certain of algebraic and uniqueness properties of the harmonic quaternion fields with hope for their future application to ITP [8]. In the mean time, our results may be of certain independent interest for functional analysis: namely, the real uniform Banach algebras theory [1, 13, 15].

Main result

• Let Ω be a smooth compact oriented 3-dimensional Riemannian manifold with boundary, $T\Omega_x$ the tangent space at $x \in \Omega$, $u \cdot v$ and $u \wedge v$ the inner and vector products in $T\Omega_x$. Elements of the space $H_x := \mathbb{R} \oplus T\Omega_x$ (the pairs $q = \{\alpha, u\}$) endowed with a multiplication $qq' = \{\alpha\alpha' - u \cdot u',\, \alpha u' + \alpha' u + u \wedge u'\}$ are said to be the *geometric quaternions*. As an algebra, H_x is isometrically isomorphic to the quaternion algebra $\mathbb{H}$.

• A *quaternion field* is a pair $q = \{\alpha, u\}$ of a function α and vector field u on Ω; in other words, q is an H_x-valued function on the manifold. The space $C(\Omega; H)$ of continuous quaternion fields endowed with the point-wise linear operations and multiplication, and the relevant sup-norm, is a real uniform Banach algebra [1, 13, 15].

A field $q = \{\alpha, u\} \in C(\Omega; H)$ is *harmonic* if α, u are continuous in Ω and $\nabla\alpha = \operatorname{rot} u$, $\operatorname{div} u = 0$ holds into Ω. The space $\mathscr{Q}(\Omega)$ of harmonic fields is a subspace of $C(\Omega, H)$ (but not a subalgebra!).

• Let $\mathscr{A}$ be an algebra. For a set $A \subset \mathscr{A}$ by $\vee A$ we denote the minimal subalgebra that contains A. The main result of the paper is a Stone-Weierstrass type Theorem 1 which claims that $\vee\mathscr{Q}(\Omega)$ is dense in $C(\Omega; H)$.

More results and comments

• In the course of proving Theorem 1 we show that $\mathscr{Q}(\Omega)$ (and, hence, $\vee\mathscr{Q}(\Omega)$) separates points of Ω. It is quite evident for $\Omega\subset\mathbb{R}^3$ [11] but far from being evident for a 3d-manifold of arbitrary topology. The separation property is derived from the so-called H-controllability of Ω from the boundary, which is much stronger than separability. The H-controllability is proved by the use of the results [18] on existence of the global Green function and the Landis type uniqueness theorems for the second order elliptic equations [16]. The key step in proving Theorem 1 is to show that $\overline{\vee\mathscr{Q}(\Omega)}$ contains the algebra of scalar fields $\big\{\{\alpha,0\}\,|\,\alpha\in C^{\mathbb{R}}(\Omega)\big\}$. The latter resembles the trick applied in [14].

• In sec 4 we prove that the 2-jets of harmonic functions are point-wise controllable from the boundary. The proof also makes use of the elliptic uniqueness theorems. Then this result is applied to show that harmonic functions determine the Riemannian structure of 3d manifold. As we hope, it is a step towards the main prospective goal: application to the 3d impedance tomography problem on Riemannian manifolds.

• One more result, which is of certain independent interest, is the following uniqueness property of harmonic quaternion fields (sec 5). If $q\in\mathscr{Q}(\Omega)$ vanishes on a piece of a smooth surface then it vanishes in Ω identically.

• Everywhere in the paper we deal with *real* functions, fields, spaces, etc. Everywhere *smooth* means C^∞-smooth.

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2 Quaternion fields

Quaternions

• Let E be an oriented 3d euclidean space, $u\cdot v$ and $u\wedge v$ the scalar (inner) and vector products, $|u|=\sqrt{u\cdot u}$. Elements $p=\{\alpha,u\}$ of the space $H:=\mathbb{R}\oplus E$ endowed with the norm $|p|=\sqrt{\alpha^2+|u|^2}$ and a (noncommutative) multiplication

$$pp':=\{\alpha\alpha'-u\cdot u',\ \alpha u'+\alpha' u+u\wedge u'\},\tag{2.1}$$

are said to be *geometric quaternions.*

The norm obeys $|p^2|=|p|^2$,

• Let $\mathbb{H}$ be the algebra of (standard) quaternions. Recall that it is the real algebra generated by $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ with the unit $\mathbf{1}$ and multiplication defined by the table

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}, \quad \mathbf{ij} = \mathbf{k},\ \mathbf{jk} = \mathbf{i},\ \mathbf{ki} = \mathbf{j}\,.$$

• For an orthogonal normalized basis $\varepsilon = \{e_1, e_2, e_3\}$ in E, the correspondence $e_1 \mapsto \mathbf{i}$, $e_2 \mapsto \mathbf{j}$, $e_3 \mapsto \mathbf{k}$ determines an isometric isomorphism $\mu_\varepsilon : H \to \mathbb{H}$,

$$\{\alpha,\ ae_1 + be_2 + ce_3\} \overset{\mu_\varepsilon}{\mapsto} \alpha\mathbf{1} + a\mathbf{i} + b\mathbf{j} + c\mathbf{k}\,, \tag{2.2}$$

(we write $H \cong \mathbb{H}$). Any isometric isomorphism $\mu : H \to \mathbb{H}$ is of the form (2.2) by proper choice of the basis ε.

Vector analysis

In the sequel, the following assumptions are accepted.

Convention 1. *Ω is a smooth compact oriented Riemannian 3d-manifold with the smooth boundary $\partial\Omega$. It is endowed with the metric tensor $g \in C^2$; $d\mu$ is the Riemannian volume 3-form; $\star$ is the Hodge operator.*

On such a manifold, the intrinsic operations of vector analysis are well defined on smooth functions and vector fields (sections of the tangent bundle $T\Omega$). Following [21], Chapter 10, we recall their definitions.

• For a vector field u, one defines the *conjugate* 1*-form* $u_\flat$ by $u_\flat(v) = g(u, v),\ \forall v$. For a 1-form f, the *conjugate field* $f^\flat$ is defined by $g(f^\flat, u) = f(u),\ \forall u$.

• A *scalar product*: {fields} $\times$ {fields} $\overset{\cdot}{\to}$ {functions} is defined point-wise by $u \cdot v = g(u, v)$. A *vector product*: {fields} $\times$ {fields} $\overset{\wedge}{\to}$ {fields} is defined point-wise by $g(u \wedge v, w) = d\mu\,(u, v, w),\ \forall w$.

• A *gradient*: {functions} $\overset{\nabla}{\to}$ {fields} and a *divergence*: {fields} $\overset{\mathrm{div}}{\to}$ {functions} are defined by $\nabla\alpha = (d\alpha)^\flat$ and $\mathrm{div}\, u = \star d \star u_\flat$ respectively, where d is the exterior derivative.

• A *rotor*: {fields} $\overset{\mathrm{rot}}{\to}$ {fields} is defined by $\mathrm{rot}\, u = (\star\, d\, u_\flat)^\flat$. Recall the basic identities: $\mathrm{div}\,\mathrm{rot} = 0$ and $\mathrm{rot}\,\nabla = 0$. The equalities

$$\nabla\alpha = \mathrm{rot}\, u \qquad \text{and} \qquad d\alpha = \star\, d\, u_\flat$$

are equivalent.

• The *Laplacian* {functions} $\overset{\Delta}{\to}$ {functions} is $\Delta = \mathrm{div}\,\nabla$. The *vector Laplacian* {fields} $\overset{\vec{\Delta}}{\to}$ {fields} is $\vec{\Delta} = \nabla\,\mathrm{div} - \mathrm{rot}\,\mathrm{rot}\,$.

Remark 1. *Under the above accepted assumptions on the smoothness of* Ω *and* g*, the (harmonic) functions and fields, which obey* $\Delta\alpha = 0$ *and* $\vec{\Delta}u = 0$ *in the relevant weak sense, do belong to the class* $C^2_{\rm loc}$*: see, e.g, [12], Part II, Chapter 1.*

Fields

Let $\dot{\Omega} := \Omega \setminus \partial\Omega$ be the set of the inner points, $C(\Omega)$ and $\vec{C}(\Omega)$ the spaces of continuous functions and vector fields. Let $H_x := \mathbb{R} \oplus T\Omega_x, \ \ x \in \Omega$ be the point-wise geometric quaternion algebras.

- A quaternion *field* is a pair $p = \{\alpha, u\}$ with the components $\alpha \in C(\Omega)$ and $u \in \vec{C}(\Omega)$, the values $p(x) = \{\alpha(x), u(x)\} \in H_x$ being regarded as geometric quaternions.

By $C(\Omega; H)$ we denote the space of continuous quaternion fields. One can regard them as sections of the bundle $C(\Omega; H) = \cup_{x\in\Omega} H_x$.

- Elements of the subspace

$$\mathscr{Q}(\Omega) := \left\{p \in C(\Omega; H) \,\middle|\, \nabla\alpha = \operatorname{rot} u, \ \operatorname{div} u = 0 \ \text{ in } \ \dot{\Omega}\right\}$$

are called *harmonic fields*. To be rigorous, here the conditions on the components of p are understood in the relevant sense of distributions but imply $\Delta\alpha = 0$ and $\vec{\Delta}u = 0$, so that α and u are automatically smooth enough by Remark 1.

3 Density theorem

Algebra $C(\Omega; H)$

The space $C(\Omega; H)$ with the point-wise multiplication (2.1) and the norm

$$\|p\| = \sup_{x\in\Omega} |p(x)| = \sup_{x\in\Omega} \sqrt{|\alpha(x)|^2 + |u(x)|^2_{T\Omega_x}}$$

satisfying $\|qp\| \leqslant \|q\|\|p\|$, $\|p^2\| = \|p\|^2$ is a real uniform noncommutative Banach algebra.

- The fields $\{\alpha, 0\}$ constitute a subalgebra $C(\Omega; \mathbb{R})$ of $C(\Omega; H)$, which is isometrically isomorphic to the real continuous function algebra on Ω:

$$C(\Omega; \mathbb{R}) \cong C^{\mathbb{R}}(\Omega). \tag{3.1}$$

We say $\{\alpha, 0\}$ to be the scalar fields and often identify them with functions α via the map $\alpha \mapsto \{\alpha, 0\}$, which embeds $C^{\mathbb{R}}(\Omega)$ in $C(\Omega; H)$.

- The harmonic subspace $\mathscr{Q}(\Omega) \subset C(\Omega; H)$ is not an algebra since, in general, $p, q \in \mathscr{Q}(\Omega)$ does not imply $pq \in \mathscr{Q}(\Omega)$. It is easy to see that

$$\mathscr{Q}(\Omega) \cap C(\Omega; \mathbb{R}) = \{\{c, 0\} \mid c \text{ is a constant function}\},$$

whereas $\{1, 0\}$ is the unit of $C(\Omega; H)$.

Main result

For an algebra $\mathscr{A}$ and a set $S \subset \mathscr{A}$ by $\vee S$ we denote a minimal (sub)algebra in $\mathscr{A}$, which contains S. Our main results is the following.

Theorem 1. *The algebra* $\vee\mathscr{Q}(\Omega)$ *is dense in* $C(\Omega; H)$.

The proof occupies the rest of sec 3.

Green function

• A well-known in geometry fact is that the assumptions of Convention 1, in particular, provide the existence of a compact 3-dimensional C^∞- manifold $\Omega' \Supset \Omega$ endowed with the tensor $g' \in C^2$ such that $g'|_\Omega = g$. This enables one to apply the results by M.Mitrea and M.Taylor [18] (existence of the fundamental solution, Green function, Poisson formula, etc) which are valid for much weaker smoothness restrictions on g and $\partial\Omega$. Also, one can apply the results on the uniqueness of continuation of solutions to the elliptic PDE [12, 16].

• The following results are mostly taken from [18]. Also we use some well-known facts of the elliptic 2-nd order equations theory [17, 12, 16]. By $W^l_p(\Omega)$ we denote the Sobolev space of functions which possess the (generalized) derivatives of the order $l = 1, 2, \ldots$ belonging to $L_p(\Omega)$ $(p \geqslant 1)$. Recall that $\dot\Omega = \Omega \setminus \partial\Omega$. Also we put $D := \{(x, y) \in \Omega \times \Omega \,|\, x = y\}$. The distance in Ω is denoted by r_{xy}. Let $\mathscr{D}(\dot\Omega)$ be a space of the smooth compactly supported into Ω functions (test functions) endowed with the standard topology, $\mathscr{D}'(\dot\Omega)$ the corresponding distributions.

For an $h \in L_2(\Omega)$, the Dirichlet problem

$$\begin{aligned} &\Delta v = h && \text{in } \dot\Omega \\ &v = 0 && \text{on } \partial\Omega \end{aligned}$$

has a unique solution $v^h \in W^2_2(\Omega)$ vanishing at the boundary. The solution is represented in the form

$$v^h(x) = \int_\Omega G(x, y)\, h(y)\, d\mu(y), \qquad x \in \Omega \tag{3.2}$$

via the *Green function* G, which possesses the following properties.

1. $G \in C^2_{\mathrm{loc}}([\Omega \times \Omega] \setminus D)$; $\quad G(x, y) = G(y, x)$, $\ (x, y) \notin D$;

$$G(x, \cdot)|_{\partial\Omega} = 0, \qquad x \in \dot\Omega\,. \tag{3.3}$$

For the closed sets $K, K' \subset \Omega$ provided $K \cap K' = \emptyset$ the map $y \mapsto G(\cdot, y)$ is continuous from K to $C^2(K')$.

2. The estimates

$$G(x,y) \leqslant \frac{c}{r_{xy}}\,, \qquad |\nabla_y G(x,y)| \leqslant \frac{c}{r^2_{xy}}$$

hold and imply $G(x,\cdot) \in W^1_p(\Omega)$ for $x \in \Omega,\ 1 \leqslant p < \frac{3}{2}$.

3. As a distribution of the class $\mathscr{D}'(\dot\Omega)$ on the test functions (of the variable y) of the class $\mathscr{D}(\dot\Omega)$, the Green function satisfies

$$\Delta_y G(x,\cdot) \;=\; \delta_x, \tag{3.4}$$

where δ_x is the Dirac measure supported at x. Note that in (3.4), and below in (3.8), (3.9), the variable $x \in \dot\Omega$ plays the role of parameter.

4. For $f \in C^\infty(\partial\Omega)$, the inhomogeneous boundary value problem

$$\Delta w = 0 \qquad \text{in } \dot\Omega \tag{3.5}$$

$$w = f \qquad \text{on } \partial\Omega \tag{3.6}$$

has a unique classical solution $w = w^f(x)$, which is represented in the form

$$w^f(x) \;=\; \int_{\partial\Omega} \partial_{\nu_y} G(x,y)\, f(y)\, d\sigma(y), \qquad x \in \dot\Omega\,, \tag{3.7}$$

where ν_y is the outward unit normal at the boundary, $d\sigma$ is the boundary surface element. This is a Poisson formula derived from (3.2) by integration by parts. Function f in (3.6) is said to be a *boundary control*.

- Fix a point $x \in \dot\Omega$ and a vector $e \in T\Omega_x,\ |e| = 1$. Let γ_e be the geodesic that emanates from x in direction e. Define a functional $\partial^x_e \delta_x \in \mathscr{D}'(\dot\Omega)$ by

$$\langle \partial^x_e \delta_x, \varphi \rangle := \lim_{\gamma_e \ni\, x' \to x} \frac{\varphi(x') - \varphi(x)}{r_{xx'}} = \left\langle \lim_{\gamma_e \ni\, x' \to x} \frac{\delta_{x'} - \delta_x}{r_{xx'}}\,,\ \varphi \right\rangle = e \cdot \nabla \varphi(x)\,.$$

The relevant limit passage in (3.4) determines a derivative $\partial^x_e G(x,\cdot) \in \mathscr{D}'(\dot\Omega)$ which satisfies

$$\Delta_y [\partial^x_e G(x,\cdot)] \;=\; \partial^x_e \delta_x\,. \tag{3.8}$$

In the mean time, by the properties 1 and 2, $\partial^x_e G(\cdot, y)$ is a (classical) function belonging to $L_p(\Omega)$ for $1 \leqslant p < \frac{3}{2}$. Moreover it is *harmonic* (and hence C^2-smooth) in $\Omega \setminus \{x\}$ and satisfies

$$\partial^x_e G(x,\cdot)|_{\partial\Omega} = 0\,, \qquad x \in \dot\Omega. \tag{3.9}$$

- The relevant limit passage in the Poisson formula (3.7) implies

$$e \cdot \nabla w^f(x) \;=\; \int_{\partial\Omega} \partial_{\nu_y} [\partial^x_e G(x,y)]\; f(y)\, d\sigma(y), \qquad x \in \dot\Omega\,. \tag{3.10}$$

H-controllability

• The following result plays the key role in the proof of Theorem 1. Recall that $H_x = \mathbb{R} \oplus T\Omega_x \cong \mathbb{H}$, and Ω obeys Convention 1.

For a set of points $A = \{a_1, \ldots, a_N\} \subset \Omega$ define a $4N$-dimensional space $H_A := \oplus \sum_{i=1}^N H_{a_i}$ and a map $M_A : C^\infty(\partial\Omega) \to H_A$:

$$f \mapsto \oplus \sum_{i=1}^{N} \{w^f(a_i), \nabla w^f(a_i)\}$$

(each summand $\{w^f(a_i), \nabla w^f(a_i)\}$ belongs to the corresponding H_{a_i}). We say Ω to be H-*controllable from boundary* if this map is surjective for any finite set A.

Lemma 1. *The manifold Ω is H-controllable from boundary.*

Proof. The opposite means that $H_A \ominus \mathrm{Ran}\, M_A \neq \{0\}$, i.e. there is a nonzero element $\oplus \sum_{i=1}^N \{\alpha_i, \beta_i e_i\} \in H_A$ $(\alpha_i, \beta_i \in \mathbb{R},\ |e_i| = 1)$ such that

$$\sum_{i=1}^{N} \alpha_i w^f(a_i) + \beta_i\, e_i \cdot \nabla w^f(a_i) = 0 \tag{3.11}$$

holds for all $f \in C^\infty(\partial\Omega)$. Show that such an assumption leads to contradiction.

1. Let $A \subset \dot{\Omega}$, i.e., all a_i are the interior points. A function

$$\Phi(y) := \sum_{i=1}^{N} \alpha_i G(a_i, y) + \beta_i \partial^x_{e_i} G(a_i, y) \tag{3.12}$$

satisfies

$$\Delta\Phi = 0 \qquad \text{in } \Omega \setminus A \tag{3.13}$$

$$\Phi|_{\partial\Omega} = 0 \tag{3.14}$$

by (3.3), (3.4), (3.8), and (3.9).

The relations (3.7), (3.10) and (3.11) easily follow to

$$\int_{\partial\Omega} \partial_\nu \Phi(y)\, f(y)\, d\sigma(y) = 0$$

that implies

$$\partial_\nu \Phi|_{\partial\Omega} = 0 \tag{3.15}$$

by arbitrariness of f.

2. So, Φ is harmonic in $\Omega \setminus A$ and has the *zero* Cauchy data at the boundary: see (3.14) and (3.15). By the well-known uniqueness property of solutions to elliptic PDE (see, e.g., [16], sec. 4.3, Remark 4.17), we get $\Phi = 0$ in $\Omega \setminus A$, i.e., almost everywhere in Ω.

Since $G(a_i, \cdot) \in W^1_p(\Omega)$ and $\partial_{e_i} G(a_i, \cdot) \in L_p(\Omega)$, we have $\Phi \in L_p(\Omega)$ for some $p \geqslant 1$. Therefore, Φ is a summable function equal zero a.e. in Ω. Thus, $\Phi = 0$ as a distribution of the class $\mathscr{D}'(\dot\Omega)$.

In the mean time, by (3.4) and (3.8) one has

$$\Delta\Phi = \sum_{i=1}^{N} \alpha_i \delta_{a_i} + \beta_i \partial^x_{e_i} \delta_{a_i} \neq 0,$$

i.e., Φ is a *nonzero* element of $\mathscr{D}'(\dot\Omega)$. We arrive at the contradiction that proves the Lemma for $A \in \dot\Omega$.

3. Let A contain the points of $\partial\Omega$. The smoothness assumptions on Ω enable one to provide Ω', g' obeying Convention 1 and such that $\Omega \Subset \Omega'$ and $g'|_\Omega = g$ holds. Then one has $A \subset \dot\Omega'$ that reduces this case to the previous one. $\square$

Note that relations between controllability and uniqueness theorems (like the one used in the proof) are widely exploited in control theory for PDE (see, e.g., [9]).

• Recall that w^f is a harmonic function that solves (3.5), (3.6). As immediate consequence of Lemma 1 we have

Corollary 1. *The algebra* $\vee\left\{|\nabla w^f|^2 \mid f \in C^\infty(\Omega)\right\}$ *is dense in* $C^{\mathbb{R}}(\Omega)$.

Indeed, by Lemma 1, for any $a, b \in \Omega$ there is a smooth f such that $|\nabla w^f(a)|^2 \neq |\nabla w^f(b)|^2$, i.e., the functions $|\nabla w^f(\cdot)|^2$ separate points of Ω. In the mean time, by the same Lemma, there is no $x_0 \in \Omega$, at which all these functions vanish simultaneously. Hence, by the classical Stone-Weierstrass Theorem (see, e.g., [19]), the above mentioned density does hold.

Note that $\{0, \nabla w^f\} \in \mathscr{Q}(\Omega)$ and $\{0, \nabla w^f\}^2 = -\{|\nabla w^f(\cdot)|^2, 0\} \in \vee\mathscr{Q}(\Omega)$. Hence, the algebra $\vee\left\{\{|\nabla w^f|^2, 0\} \mid f \in C^\infty(\Omega)\right\}$ is a subalgebra in $\vee\mathscr{Q}(\Omega)$. By (3.1), Corollary 1 implies that this algebra is dense in $C(\Omega; \mathbb{R})$. As a result, denoting

$$\mathscr{C} := \overline{\vee\mathscr{Q}(\Omega)}$$

we arrive at the important relation

$$\mathscr{C} \supset C(\Omega; \mathbb{R}). \tag{3.16}$$

Strong separation

We say that a family $\mathscr{F} \subset C(\Omega; H)$ *strongly separates* points (of Ω) if for any $a, b \in \Omega$ and $h_a \in H_a$, $h_b \in H_b$ there is a $p \in \mathscr{F}$ such that $p(a) = h_a$ and $p(b) = h_b$ holds [13].

Lemma 2. *The space $\mathscr{Q}(\Omega)$ strongly separates points.*

Proof. • Let $\vec{L}_2(\Omega)$ be the space of square-integrable vector fields and $\mathscr{H} := \{v \in \vec{L}_2(\Omega) \mid \operatorname{div} v = 0,\ \operatorname{rot} v = 0\}$ its harmonic subspace. The well-known Hodge-Morrey-Friedrichs decomposition claims that

$$\mathscr{H} = \mathscr{G} \oplus \mathscr{N} = \mathscr{R} \oplus \mathscr{D}, \tag{3.17}$$

where

$$\mathscr{G} := \{v \in \mathscr{H} \mid v = \nabla\alpha\}, \quad \mathscr{N} := \{v \in \mathscr{H} \mid v \cdot \nu = 0\},$$
$$\mathscr{R} := \{v \in \mathscr{H} \mid v = \operatorname{rot} u\}, \quad \mathscr{D} := \{v \in \mathscr{H} \mid v \wedge \nu = 0\}.$$

(see, e.g., [21], Corollary 3.5.2). The subspaces $\mathscr{N}$ and $\mathscr{D}$ determined by the boundary conditions are called the Neumann and Dirichlet spaces respectively. Their *finite* dimensions are equal to the Betti numbers: $\dim \mathscr{N} = \beta_1$, $\dim \mathscr{D} = \beta_2$ [21]. Note that $\mathscr{N} \cap \mathscr{D} = \{0\}$ [3, 21]. Also note that $\dim \mathscr{G} = \dim \mathscr{R} = \infty$.

• As a consequence of (3.17), a field $v \in \mathscr{H}$ is represented in the form $v = \nabla\alpha = \operatorname{rot} u$ if and only if $v \in \mathscr{G} \cap \mathscr{R}$ or, equivalently, $v \perp [\mathscr{N} \dot{+} \mathscr{D}]$.

If $w = w^f(x)$ solves (3.5), (3.6) then for any $d \in \mathscr{D}$ one has

$$(\nabla w^f, d) = \int_\Omega \nabla w^f \cdot d \, d\mu = \int_{\partial\Omega} f \, d \cdot \nu \, d\sigma .$$

In the mean time, since $\nabla w^f \in \mathscr{G}$, the representation $\nabla w^f = \operatorname{rot} u$ holds if and only if $\nabla w^f \perp \mathscr{D}$, which is equivalent to

$$\int_{\partial\Omega} f \, d \cdot \nu \, d\sigma = 0, \qquad d \in \mathscr{D}. \tag{3.18}$$

In particular, taking $f = 1$ one has $w^f = 1$ in Ω and gets

$$\int_{\partial\Omega} d \cdot \nu \, d\sigma = 0, \qquad d \in \mathscr{D}. \tag{3.19}$$

• Now, fix two distinct points $a, b \in \Omega$ and elements $h_a = \{c_a, k_a\} \in H_a$, $h_b = \{c_b, k_b\} \in H_b$. To prove the Lemma we need to show that there is a smooth f, which provides

$$w^f(a) = c_a,\ w^f(b) = c_b; \quad \nabla w^f = \operatorname{rot} u; \quad u(a) = h_a,\ u(b) = h_b . \tag{3.20}$$

Step 1. At first assume $a, b \in \dot\Omega$. Let $P_x(y) := \partial_{\nu_y} G(x,y)$ be the Poisson kernel. By (3.7) for $f = 1$ we have

$$\int_{\partial\Omega} P_x(y)\, d\sigma(y) \;=\; 1\,, \qquad x \in \Omega\,. \tag{3.21}$$

In accordance with (3.7) and (3.18), to satisfy the relations $w^f(a) = c_a$, $w^f(b) = c_b$; $\nabla w^f = \operatorname{rot} u$ in (3.20) we need to find f provided

$$\int_{\partial\Omega} P_a(y)\, f(y)\, d\sigma(y) = c_a\,, \quad \int_{\partial\Omega} P_b(y)\, f(y)\, d\sigma(y) = c_b\,;$$
$$\int_{\partial\Omega} f(y)\, d(y)\cdot\nu\, d\sigma(y) \;=\; 0\,, \qquad d \in \mathscr{D},$$

or, equivalently,

$$(P_a, f) = c_a,\ (P_b, f) = c_b\,,\ \ f \perp \nu\cdot\mathscr{D} \tag{3.22}$$

(the inner products in $L_2(\partial\Omega)$), where $\nu\cdot\mathscr{D} := \{\nu\cdot d \,|\, d \in \mathscr{D}\}$.

Comparing (3.19) with (3.21), we conclude that neither P_a nor P_b belong to $\nu\cdot\mathscr{D}$. In the mean time, $P_a \neq P_b$ as elements of $L_2(\partial\Omega)$. Indeed, otherwise we'd have $w^f(a) = w^f(b)$ for any f that is impossible by Lemma 2. Hence, $\operatorname{span}\{P_a, P_b\} \cap \nu\cdot\mathscr{D}$ may consist of $\{c(P_a - P_b) \,|\, c \in \mathbb{R}\}$ *only*. As a result, to proof the solvability of the linear system (3.22) (with respect to f) in the case of $c_a \neq c_b$ we must show that $P_a - P_b \notin \nu\cdot\mathscr{D}$.

Step 2. Assume the opposite: there is a $d \in \mathscr{D}$ such that $P_a - P_b = d\cdot\nu$, and show that this assumption leads to a contradiction.

Compare the fields $\nabla[G(a,\cdot) - G(b,\cdot)]$ and d. Since $G(a,\cdot) = G(b,\cdot) = 0$ on $\partial\Omega$ both of them are *normal* on the boundary. Hence, by the assumption, they are *equal* on $\partial\Omega$. In the mean time, the field $\nabla[G(a,\cdot) - G(b,\cdot)]$ is harmonic in $\dot\Omega \setminus [\{a\}\cup\{b\}]$, whereas d is harmonic in the whole $\dot\Omega$. The coincidence at the boundary implies the coincidence in the domain of harmonicity. Hence, $\nabla[G(a,\cdot) - G(b,\cdot)]$ can be extended by continuity to the whole Ω and $\nabla[G(a,\cdot) - G(b,\cdot)] = d$ everywhere. However, the latter is impossible since

$$\operatorname{div}\nabla[G(a,\cdot) - G(b,\cdot)] = \Delta[G(a,\cdot) - G(b,\cdot)] = \delta_a - \delta_b\,,$$

whereas $\operatorname{div} d = 0$ everywhere in $\dot\Omega$. This contradiction shows that $P_a - P_b \notin \nu\cdot\mathscr{D}$.

Step 3. The case of a and/or b belonging to the boundary is reduced to the previous one by the collar theorem arguments, which were applied at the end of the proof of Lemma 1. □

Corollary 2. *The algebra* $\vee\mathscr{Q}(\Omega) \subset C(\Omega; H)$ *strongly separates points of* Ω.

This property plays important role in proving density theorems [13].

Completing the proof of Theorem 1

Recall that $\mathscr{C} = \overline{\vee\mathscr{Q}(\Omega)}$ and prove that $\mathscr{C} = C(\Omega; H)$. The fact, which will play the key role, is the embedding $\mathscr{C} \supset C(\Omega; \mathbb{R}) \cong C^{\mathbb{R}}(\Omega)$: see (3.16).

- Fix an $x \in \Omega$ and choose the smooth boundary controls f^x_1, f^x_2, f^x_3 such that $\nabla w^{f^x_1}(x), \nabla w^{f^x_2}(x), \nabla w^{f^x_3}(x)$ constitute a basis of $T\Omega_x$. It is possible owing to Lemma 1. By continuity, there is a ball $B_{r(x)}[x] \subset \Omega$ centered at x, of (small enough) radius $r(x)$, such that $\nabla w^{f^x_1}(y), \nabla w^{f^x_2}(y), \nabla w^{f^x_3}(y)$ is a basis of $T\Omega_y$ for each $y \in B_{r(x)}[x]$.

Let such a choice be done for each $x \in \Omega$.

- The balls provide an open cover $\Omega = \cup_{x\in\Omega} B_{r(x)}[x]$. By compactness there is a finite subcover $\Omega = \cup_{n=1}^N B_{r_n}[x_n]$, where $r_n := r(x_n)$. Let $\eta_1, \dots, \eta_N$ be a partition of unit subordinated to the subcover, so that

$$\eta_1, \dots, \eta_N \in C^\infty(\Omega), \quad \operatorname{supp} \eta_n \subset B_{r_n}[x_n], \quad \sum_{n=1}^N \eta_n \equiv 1 \text{ in } \Omega$$

holds.

- Take $p = \{\alpha, u\} \in C(\Omega; H)$ and represent

$$p = \sum_{n=1}^N \eta_n p = \{\sum_{n=1}^N \eta_n \alpha, \sum_{n=1}^N \eta_n u\} = \sum_{n=1}^N \{\eta_n \alpha, 0\} + \sum_{n=1}^N \{0, \eta_n u\}$$

with $\{\eta_n \alpha, 0\} \in C(\Omega; \mathbb{R}) \subset \mathscr{C}$. In the mean time, one has

$$\eta_n u = \sum_{k=1}^3 \varkappa^n_k \, \nabla w^{f^{x_n}_k}$$

with the certain $\varkappa^n_k \in C^{\mathbb{R}}(\Omega)$ supported in $B_{r_n}[x_n]$. Note that $\{\varkappa^n_k, 0\} \in C(\Omega; \mathbb{R}) \subset \mathscr{C}$.

Summarizing, we arrive at the representation

$$p = \sum_{n=1}^N \{\eta_n \alpha, 0\} + \sum_{n=1}^N \sum_{k=1}^3 \{\varkappa^n_k, 0\}\{0, \nabla w^{f^{x_n}_k}\},$$

where all cofactors and summands do belong to $\mathscr{C}$. Thus $p \in \mathscr{C}$ and, hence, $C(\Omega; H) = \mathscr{C}$.

Theorem 1 is proved.

Remark 2. *Analyzing the proof, it is easy to recognize that the family* $\mathscr{W} := \big\{\{0, \nabla w^f\} \mid f \textit{ is smooth}\big\}$, *which is smaller than* $\mathscr{Q}(\Omega)$, *also generates the whole of the continuous field algebra:* $\overline{\vee\mathscr{W}} = C(\Omega; H)$.

4 Controllability of 2-jets

Fix an $a \in \dot\Omega$; let x^1, x^2, x^3 be the local coordinates in a neighborhood $\omega \ni a$. With a smooth function ϕ one associates the row of its 0,1,2-order derivatives

$$\begin{aligned} j_a[\phi] := \{&\phi(a);\; \phi_{x^1}(a), \phi_{x^2}(a), \phi_{x^3}(a);\\ &\phi_{x^1x^1}(a), \phi_{x^1x^2}(a), \phi_{x^1x^3}(a), \phi_{x^2x^2}(a), \phi_{x^2x^3}(a), \phi_{x^3x^3}(a)\} \in \mathbb{R}^{10},\end{aligned}$$

which provides a coordinate representation of its *second jet* at the point a [20]. For short, we say $j_a[\phi]$ to be a 2-jet of ϕ at a and consider $\mathbb{R}^{10}$ with the (standard) inner product $\langle j, j'\rangle$ as a space of 2-jets.

Recall that in coordinates the Laplacian acts by

$$\Delta\phi \;=\; g^{-\frac{1}{2}}[g^{\frac{1}{2}} g^{ik}\phi_{x^k}]_{x^i}\,,$$

where $\{g^{ik}\}$ is the inverse to the metric tensor matrix $\{g_{ik}\}$ and $g = \det\{g_{ik}\}$ (summation over repeating indexes is in the use). We say the row

$$\begin{aligned}&\lambda_a :=\\ &= \{0; g^{-\frac{1}{2}}[g^{\frac{1}{2}}g^{i1}]_{x^i}, g^{-\frac{1}{2}}[g^{\frac{1}{2}}g^{i2}]_{x^i}, g^{-\frac{1}{2}}[g^{\frac{1}{2}}g^{i3}]_{x^i}; g^{11}, 2g^{12}, 2g^{13}, g^{22}, 2g^{23}, g^{33}\}\big|_{x=a}\end{aligned}$$

to be the *Laplace jet* and represent $(\Delta\phi)(a) = \langle \lambda_a, j_a[\phi]\rangle$.

The harmonicity $\Delta w = 0$ is equivalent to the orthogonality $\langle j_a[w], \lambda_a\rangle = 0,\;\; a \in \omega$. Therefore one has $j_a[w] \in \mathbb{R}^{10} \ominus \operatorname{span}\lambda_a$. Let us show that the 2-jets of harmonic functions exhaust the subspace $\mathbb{R}^{10} \ominus \operatorname{span}\lambda_a$. This result may be interpreted as a point-wise boundary controllability of 2-jets by harmonic functions. Recall that w^f is a solution to (3.5), (3.6).

Lemma 3. *For any* $a \in \Omega$ *and* $s \in \mathbb{R}^{10} \ominus \operatorname{span}\lambda_a$ *there is a smooth* f *such that* $j_a[w^f] = s$.

Proof. Taking into account the structure of the Laplace jet, we may deal with $s = \{0; s_1, s_2, s_3; s_{11}, \dots, s_{33}\}$, and let it be such that $0 \neq s \in \mathbb{R}^{10} \ominus \operatorname{span}\lambda_a$ but $\langle s, j_a[w^f]\rangle = 0$ for any smooth f. Show that such an assumption leads to contradiction.

• For a differential operator L with smooth coefficients in Ω, by L^* we denote its adjoint by Lagrange that is defined by

$$(L\eta, \zeta)_{L_2(\Omega)} \;=\; (\eta, L^*\zeta)_{L_2(\Omega)}, \qquad \eta, \zeta \in \mathscr{D}(\dot\Omega)\,.$$

For a distribution $h \in \mathscr{D}'(\dot\Omega)$ one defines Lh by $(Lh, \eta) := (h, L^*\eta)_{L_2(\Omega)},\; \eta \in \mathscr{D}(\dot\Omega)$.

Let S be a differential operator, which acts by

$$\begin{aligned}
&(Sv)(x) = \\
&= [s_1 v_{x^1} + s_2 v_{x^2} + s_3 v_{x^3} + s_{11} v_{x^1x^1} + s_{12} v_{x^1x^2} + \cdots + s_{33} v_{x^3x^3}](x) = \\
&= \langle s, j_x[v]\rangle, \qquad x \in \omega
\end{aligned}$$

in a coordinate neighborhood ω of $a \in \dot{\Omega}$, where the (constant) coefficients are the components of the above chosen jet s.

- Let $\delta_a \in \mathscr{D}'(\dot{\Omega})$ be the Dirac measure supported at the point $a \in \dot{\Omega}$. Consider the problem

$$\Delta H = S^*\delta_a \tag{4.1}$$

$$H\big|_{\partial\Omega} = 0\,. \tag{4.2}$$

The equation is understood as a relation in $\mathscr{D}'(\dot{\Omega})$; its r.h.s. is a distribution acting by $(S^*\delta_a, \eta)_{L_2(\Omega)} = (S\eta)(a)$. The boundary condition does make sense since H is harmonic outside $\operatorname{supp} S^*\delta_a = \{a\}$. Also, the normal derivative $\partial_\nu H$ is a smooth function on $\partial\Omega$.

Formally by Green, for a function $v \in C^2(\Omega)$ one has

$$\begin{aligned}
&\langle s, j_a[v]\rangle = (Sv)(a) = \int_\Omega \delta_a\, Sv\, d\mu = \int_\Omega S^*\delta_a\, v\, d\mu \overset{(4.1)}{=} \int_\Omega \Delta H\, v\, d\mu = \\
&\overset{(4.2)}{=} \int_\Omega H\, \Delta v\, d\mu + \int_{\partial\Omega} \partial_\nu H\, v\, d\sigma\,.
\end{aligned}$$

To justify the final equality

$$\langle s, j_a[v]\rangle = \int_\Omega H\, \Delta v\, d\mu + \int_{\partial\Omega} \partial_\nu H\, v\, d\sigma \tag{4.3}$$

one can use the standard regularization technique, approximating δ_a by $\delta_a^\varepsilon \in \mathscr{D}(\dot{\Omega})$ supported near a.

- By the choice of s, for $v = w^f$ the equality (4.3) provides

$$\int_{\partial\Omega} \partial_\nu H\, w^f\, d\sigma = \int_{\partial\Omega} \partial_\nu H\, f\, d\sigma = 0\,.$$

By arbitrariness of f we get $\partial_\nu H = 0$ on $\partial\Omega$. So, H is harmonic in $\Omega \setminus \{a\}$ and has the zero Cauchy data on the boundary. By the uniqueness theorem, H vanishes everywhere outside a. Hence, the distribution H is supported at a. The well-known fact of the distribution theory is that such an H is a linear combination of δ_a and its derivatives. In the mean time, comparing the orders of singularities in the left and right hand sides of (4.1), one easily concludes that

$$H = c\delta_a$$

with $c = \text{const} \neq 0$. Indeed, otherwise ΔH contains the derivatives of δ_a of the order $\geqslant 3$ that makes the equality (4.1) impossible.

For an $\eta \in \mathscr{D}(\dot\Omega)$ one has

$$\langle s, j_a[\eta]\rangle = (\delta_a, S\eta) = (S^*\delta_a, \eta) \overset{(4.1)}{=} (\Delta c\delta_a, \eta) = (c\delta_a, \Delta\eta) = \langle c\lambda_a, j_a[\eta]\rangle\,.$$

Comparing the beginning with the end and referring to the evident $\{j_a[\eta]\,|\,\eta \in \mathscr{D}(\dot\Omega)\} = \mathbb{R}^{10}$, we arrive at $s = c\lambda_a$ that contradicts to the starting assumption $s \perp \lambda_a$.

• The case $a \in \partial\Omega$ is reduced to the previous one by means of the trick already used at the end of the proof of Lemma 1: embedding $\Omega \Subset \Omega'$. $\square$

As is easy to recognize, Lemma 3 implies the assertion of Lemma 1 for the case of the single point a. However, Lemma 3 may be generalized on the finite set $a_1, \dots, a_N$ so that the relevant boundary controllability of 2-jets of harmonic functions holds up to the natural defect in $\oplus \sum_i \mathbb{R}^{10}_{a_i}$.

Determination of metric from harmonic functions

The metric on Ω determines the family of harmonic functions. The converse is also true in the following sense.

• Let $c > 0$ be a smooth function on Ω and cg a conformal deformation of the metric g. By Δ_{cg} and Δ_g we denote the corresponding Laplacians. A simple calculation leads to the relation

$$\Delta_{cg} y \;=\; c^{-1}\Delta_g y - 2^{-1}\,\nabla c^{-1}\cdot\nabla y\,, \tag{4.4}$$

which is specific for the 3d case. Taking $y = w^f$, we see that the metrics cg and g have the same reserve of harmonic functions w^f if and only if $\nabla c^{-1}\cdot\nabla w^f = 0$ holds for any smooth f. In the mean time, by Lemma 1 the gradients $\nabla w^f = 0$ constitute the local bases in Ω. Hence, the latter equality implies $\nabla c^{-1} = 0$, i.e., $c = \text{const}$.

• Fix a point a in a coordinate neighborhood $\omega \ni a$. By λ^g_a we denote the Laplace jet of the given metric g. By Lemma 3, the space of jets is

$$\mathbb{R}^{10}_a = \{j_a[\phi]\,|\,\phi \text{ is smooth}\} = \{j_a[w^f]\,|\,f \text{ is smooth}\} \oplus \operatorname{span}\lambda^g_a\,. \tag{4.5}$$

Therefore, writing $(\Delta w^f)(a) = 0$ in the form

$$\langle\lambda^g_a, j_a[w^f]\rangle = 0, \qquad f \text{ is smooth}$$

and varying $f = f_1, f_2, \dots$, we get a linear homogeneous algebraic system with respect to the components of the jet λ^g_a, which determines them up to a factor, which may depend on a. Along with the components, we determine the tensor g up to a factor, possibly depending on a. However, by the above mentioned geometric reasons, this factor is a constant.

Thus, the family $\{w^f \mid f \text{ is smooth}\}$ determines the metric g up to a constant positive factor. If g is known at least at a single point $x_0 \in \Omega$, then it is uniquely determined everywhere.

Notice in addition that in two-dimensional case relation (4.4) is of the form $\Delta_{cg} y = c^{-1} \Delta_g y$, so that the metrics cg and g determine the same reserve of harmonic functions. It is the reason, because of which in 2d impedance tomography problem the metric is recovered up to conformal equivalence [2].

• Here we describe a trick, which is used in dynamical/spectral inverse problems and 2d impedance tomography problem, for recovering the metric via boundary data[9]. The hope is that it may be useful in future investigation of 3d ITP.

Assume that a topological space $\tilde{\Omega}$ is *homeomorphic* to Ω via a homeomorphism $\beta : \Omega \to \tilde{\Omega}$. Also assume that the family of functions

$$\{\tilde{w}^f = w^f \circ \beta^{-1} \mid f \text{ is smooth}\}$$

is given. The following procedure enables one to determine the metric $\tilde{g} = \beta_* g$ in $\tilde{\Omega}$.

1. Fix a point $a \in \tilde{\Omega}$ and choose its neighborhood $\tilde{\omega}$ with the coordinates x^1, x^2, x^3. By the way, Lemma 1 enables one to use the images $\tilde{w}^f$ as local coordinates.

2. Find $\operatorname{span} \lambda_a^{\tilde{g}}$ by (4.5) (replacing functions w^f on ω with $\tilde{w}^f$ on $\tilde{\omega}$). As was shown above, the family of these subspaces given for $a \in \tilde{\omega}$ determines the metric up to a constant factor. So, $c\tilde{g}$ is recovered. Assuming $\tilde{g}$ to be known at least at a single point $a_0 \in \tilde{\omega}$, one recovers $\tilde{g}$ uniquely.

3. Covering $\tilde{\Omega}$ by the coordinate neighborhoods and repeating the previous steps, we determine $\tilde{g}$ in $\tilde{\Omega}$.

5 Uniqueness properties of harmonic fields

Roughly speaking, the following result means that the set of zeros of a harmonic quaternion field may be at most of dimension 1.

Lemma 4. *Let $\Sigma \in \Omega$ be a C^2-smooth surface (2-dim submanifold). If $p \in \mathscr{Q}(\Omega)$ obeys $p|_\Sigma = 0$ then $p = 0$ in the whole Ω.*

Proof. Since the claimed result is of local character, we assume Σ to be a both-side surface endowed with a smooth field of the unit normals ν. Also, Σ possesses the (induced) Riemannian metric and is provided with the corresponding operations on vector fields. In particular, a divergence, which is denoted by $\operatorname{div}_\Sigma$, is well defined.

- For a point $x \in \Sigma$ and vector $v \in T\Omega_x$ we represent

$$v = v_\theta + v_\nu : \qquad v_\nu = v \cdot \nu\, \nu, \quad v_\theta = v - v_\nu$$

and, by default, identify v_θ with the proper vector of $T\Sigma_x$. By the latter, for a smooth vector field v given in a neighborhood of Σ, the value $[\operatorname{div}_\Sigma v_\theta](x)$ is of clear meaning. Also, recall the well-known vectot analysis relation

$$\nu \cdot \operatorname{rot} v \,=\, \operatorname{div}_\Sigma \nu \wedge v_\theta \qquad \text{on} \;\; \Sigma \tag{5.1}$$

(see, e.g. [21]).

- Begin with the case $\Sigma \subset \dot{\Omega}$. Let $p = \{\alpha, u\} \in \mathscr{Q}(\Omega)$, so that

$$\nabla\alpha \,=\, \operatorname{rot} u, \quad \operatorname{div} u \,=\, 0 \qquad \text{in} \;\; \dot{\Omega} \tag{5.2}$$

holds. Let $p|_\Sigma = 0$. Since $\alpha|_\Sigma = 0$, we have $(\nabla\alpha)_\theta|_\Sigma = 0$ that implies $(\operatorname{rot} u)_\theta\big|_\Sigma = 0$ by (5.2). In the mean time, $u|_\Sigma = 0$ is equivalent to $u_\theta = u_\nu = 0$ on Σ; hence $(\operatorname{rot} u)_\nu|_\Sigma = \operatorname{div}_\Sigma \nu \wedge u_\theta = 0$ by virtue of (5.1). Thus we get $(\operatorname{rot} u)_\theta|_\Sigma = (\operatorname{rot} u)_\nu|_\Sigma = 0$, i.e. $\operatorname{rot} u|_\Sigma = 0$.

The latter equality and (5.2) lead to $(\nabla\alpha)|_\Sigma = 0$ (along with $\alpha|_\Sigma = 0$). So, α is a harmonic function with the zero Cauchy data on Σ. Therefore $\alpha = 0$ in Ω by the elliptic uniqueness theorems [16].

As a result, $\operatorname{rot} u = \nabla\alpha = 0$ everywhere in Ω. Since $\operatorname{div} u = 0$, the vector field u is harmonic in Ω and vanishes on Σ. Therefore, locally near the points $x \in \Sigma$ one represents $u = \nabla\varphi$ with a *harmonic* function φ provided $\nabla\varphi|_\Sigma = 0$. Such a function is a constant; hence $u = 0$ near Σ. By its harmonicity, u vanishes globally in Ω.

So, we have $p = 0$ in Ω.

- The case $\Sigma \subset \partial\Omega$ is reduced to the previous one by means of the trick already used at the end of the proof of Lemma 1: embedding $\Omega \Subset \Omega'$. $\square$

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