

On Fractional Integro-differential Equations with State-Dependent Delay and Non-Instantaneous Impulses

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ABSTRACT

In this paper, we prove the existence of mild solution of the fractional integro-differential equations with state-dependent delay with not instantaneous impulses. The existence results are obtained under the conditions in respect of Kuratowski's measure of non-compactness. An example is also given to illustrate the results.

RESUMEN

En este artículo, demostramos la existencia de soluciones mild de ecuaciones integro-diferenciales fraccionarias con retardo dependiente del estado e impulsos no instantáneos. Los resultados de existencia se obtienen bajo condiciones respecto de la medida de Kuratowski de no compacidad. También se entrega un ejemplo para ilustrar los resultados.

Keywords and Phrases: Non-instantaneous impulsive conditions, fractional integro-differential equations, Caputo fractional derivative, mild solution, fixed point, state-dependent delay.

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1 Introduction

Fractional differential equations play the crucial and significant role in the field of science and engineering. Most importantly non-integer order differential equations have ability to describe the real behavior and memory effects of the system and processes. For more details about fractional differential equations and its applications refer the monographs of Abbas *et al.* [1, 2, 3], Baleanu *et al.* [12], Diethelm [18], Hilfer [24], Kilbas *et al.* [26], Miller and Ross [32], Samko *et al.* [37], Tarasov [38], and Zhou [39] and the references therein.

Most of the research papers deal with the existence of solutions for differential equations with instantaneous impulsive conditions see [6, 7, 10, 11, 14, 28, 31]. But many times it has been seen that certain dynamics of evolution processes cannot be described by instantaneous impulses. For instance: Pharmacotherapy, high or low levels of glucose, this situation can be interpreted as an impulsive action which starts abruptly at certain point of time and continues with a finite time interval. Such type of systems are known as non-instantaneous impulsive systems which are more suitable to study the dynamics of evolution processes [4].

This theory of a new class of impulsive differential equation was initiated by Hernández *et al.* [23]. Afterwards, Pierri *et al.* [35] continued the work in this field and extended the theory of [23] in a PC_α normed Banach space. The existence of solutions for non-instantaneous impulsive fractional differential equations has also been discussed in [8, 19, 27, 29, 34].

Recently, Benchohra *et al.* [15] investigated the existence and uniqueness of solutions on a compact interval for non-linear fractional integro-differential equations with state-dependent delay and noninstantaneous impulses. Anguraj and Kanjanadevi [9] studied the existence and uniqueness of fractional neutral differential equations with state-dependent delay subject to non-instantaneous impulsive conditions.

Motivated by the papers cited above, in this paper, we consider the existence of mild solutions for fractional integro-differential equations with state-dependent delay and non-instantaneous impulses described by the form

$$\begin{aligned}
 {}^C D_t^q x(t) + Ax(t) &= \int_0^t a(t,s)f(s, x_{\rho(s,x_s)}, x(s)) ds, \quad \text{a.e. } t \in (s_i, t_{i+1}] \subset J, i = 0, \dots, N, \\
 x(t) &= h_i(t, x_{\rho(t,x_t)}, x(t)), \quad t \in (t_i, s_i], i = 1, \dots, N, \\
 x_0 &= \phi \in \mathcal{B},
 \end{aligned} \tag{1.1}$$

where ${}^C D_t^q$ is the Caputo fractional derivative of order $0 < q < 1$, $A : D(A) \subset X \rightarrow X$ is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t \geq 0}$ of uniformly bounded linear operators on X , $f : J \times \mathcal{B} \times X \rightarrow X$, $J = [0, T]$, $T > 0$, and $\rho : J \times \mathcal{B} \rightarrow (-\infty, T]$ are appropriate functions, $a : D \rightarrow \mathbb{R}$ ($D = \{(t, s) \in J \times J : t \geq s\}$). Here $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \dots < t_{N-1} \leq s_{N-1} \leq t_N \leq t_{N+1} = T$ are pre-fixed numbers, and $h_i \in C((t_i, s_i] \times \mathcal{B} \times X, X)$, for all $i = 1, 2, \dots, N$. For

any continuous function x defined on $(-\infty, T]$ and any $t \in J$, we denote by x_t the element of \mathcal{B} defined by

$$x_t(\theta) = x(t + \theta), \quad \theta \in (-\infty, 0].$$

Here x_t represents the history of the state up to the present time t and $\phi \in \mathcal{B}$ to be specified later.

2 Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space.

$C = C(J, X)$ be the space of all X -valued continuous functions on J .

$L(X)$ be the Banach space of all linear and bounded operators on X .

$L^1(J, X)$ the space of X -valued Bochner integrable functions on J with the norm

$$\|y\|_{L^1} = \int_0^T \|y(t)\| dt.$$

$L^\infty(J, \mathbb{R})$ is the Banach space of measurable functions which are essentially bounded, normed by

$$\|y\|_{L^\infty} = \inf\{d > 0 : |y(t)| \leq d, \text{ a.e. } t \in J\}.$$

We need some basic definitions of the fractional calculus theory which are used in this paper.

Definition 2.1. Let $\alpha > 0$ and $f : \mathbb{R}_+ \rightarrow X$ be in $L^1(\mathbb{R}_+, X)$. Then the Riemann–Liouville integral is given by:

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

For more details on the Riemann–Liouville fractional derivative, we refer the reader to [17].

Definition 2.2. [36] The Caputo derivative of order α for a function $f : [0, +\infty) \rightarrow X$ can be written as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds = I^{n-\alpha} f^{(n)}(t), \quad t > 0, \quad n-1 \leq \alpha < n.$$

If $0 \leq \alpha < 1$, then

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f^{(1)}(s)}{(t-s)^\alpha} ds.$$

Obviously, The Caputo derivative of a constant is equal to zero.

Definition 2.3. A function $f : J \times \mathcal{B} \times X \rightarrow X$ is said to be an Carathéodory function if it satisfies :

- (i) for each $t \in J$ the function $f(t, \cdot, \cdot) : \mathcal{B} \times X \rightarrow X$ is continuous;

(ii) for each $(v, w) \in \mathcal{B} \times X$ the function $f(\cdot, v, w) : J \rightarrow X$ is measurable .

Next we give the concept of a measure of noncompactness [13].

Definition 2.4. Let B be a bounded subset of a Banach space Y . The Kuratowski measure of noncompactness of B is defined as

$$\alpha(B) = \inf\{d > 0 : B \text{ has a finite cover by sets of diameter } \leq d\}.$$

We note that this measure of noncompactness satisfies the properties ([13]).

Lemma 2.5.

1. If $A \subseteq B$ then $\alpha(A) \leq \alpha(B)$,
2. $\alpha(A) = \alpha(\bar{A})$, where \bar{A} denotes the closure of A ,
3. $\alpha(A) = 0 \Leftrightarrow \bar{A}$ is compact (A is relatively compact),
4. $\alpha(\lambda A) = |\lambda| \alpha(A)$, with $\lambda \in \mathbb{R}$,
5. $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$,
6. $\alpha(A + B) \leq \alpha(A) + \alpha(B)$, where

$$A + B = \{x + y : x \in A, y \in B\},$$

7. $\alpha(A + a) = \alpha(A)$ for any $a \in X$,
8. $\alpha(\overline{\text{conv}}A) = \alpha(A)$, where $\overline{\text{conv}}A$ is the closed convex hull of A .

For $H \subset C(J, X)$, we define

$$\int_0^t H(s) ds = \left\{ \int_0^t u(s) ds : u \in H \right\} \quad \text{for } t \in J, \quad (2.1)$$

where $H(s) = \{u(s) \in X : u \in H\}$.

Lemma 2.6. [13] If $H \subset C(J, X)$ is a bounded, equicontinuous set, then

$$\alpha_C(H) = \sup_{t \in J} \alpha(H(t)). \quad (2.2)$$

Lemma 2.7. [21] If $\{u_n\}_{n=1}^\infty \subset L^1(J, X)$ and there exists $m \in L^1(J, \mathbb{R}^+)$ such that $\|u_n(t)\| \leq m(t)$, a.e. $t \in J$, then $\alpha(\{u_n(t)\}_{n=1}^\infty)$ is integrable and

$$\alpha \left(\left\{ \int_0^t u_n(s) ds \right\}_{n=1}^\infty \right) \leq 2 \int_0^t \alpha(\{u_n(s)\}_{n=1}^\infty) ds. \quad (2.3)$$

In this paper, we will employ an axiomatic definition for the phase space \mathcal{B} which is similar to those introduced by Hale and Kato [20]. Specifically, \mathcal{B} will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$, and satisfies the following axioms:

(A1) If $x : (-\infty, T] \rightarrow X$ is continuous on J and $x_0 \in \mathcal{B}$, then $x_t \in \mathcal{B}$ and x_t is continuous in $t \in J$ and

$$\|x(t)\| \leq C\|x_t\|_{\mathcal{B}}, \tag{2.4}$$

where $C \geq 0$ is a constant.

(A2) There exist a continuous function $C_1(t) > 0$ and a locally bounded function $C_2(t) \geq 0$ in $t \geq 0$ such that

$$\|x_t\|_{\mathcal{B}} \leq C_1(t) \sup_{s \in [0, t]} \|x(s)\| + C_2(t)\|x_0\|_{\mathcal{B}}, \tag{2.5}$$

for $t \in [0, T]$ and x as in (A1).

(A3) The space \mathcal{B} is complete.

Remark 2.8. Condition (2.4) in (A1) is equivalent to $\|\phi(0)\| \leq C\|\phi\|_{\mathcal{B}}$, for all $\phi \in \mathcal{B}$.

Example 2.9. The phase space $C_r \times L^p(g, X)$.

Let $r \geq 0, 1 \leq p < \infty$, and let $g : (-\infty, -r) \rightarrow \mathbb{R}$ be a nonnegative measurable function which satisfies the conditions (g-5), (g-6) in the terminology of [25]. Briefly, this means that g is locally integrable and there exists a nonnegative, locally bounded function Λ on $(-\infty, 0]$, such that $g(\xi + \theta) \leq \Lambda(\xi)g(\theta)$, for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero.

The space $C_r \times L^p(g, X)$ consists of all classes of functions $\varphi : (-\infty, 0] \rightarrow X$, such that φ is continuous on $[-r, 0]$, Lebesgue-measurable, and $g\|\varphi\|^p$ on $(-\infty, -r)$. The seminorm in $\|\cdot\|_{\mathcal{B}}$ is defined by

$$\|\varphi\|_{\mathcal{B}} = \sup_{\theta \in [-r, 0]} \|\varphi(\theta)\| + \left(\int_{-\infty}^{-r} g(\theta) \|\varphi(\theta)\|^p d\theta \right)^{\frac{1}{p}}.$$

The space $\mathcal{B} = C_r \times L^p(g, X)$ satisfies axioms (A1), (A2), (A3). Moreover, for $r = 0$ and $p = 2$, this space coincides with $C_0 \times L^2(g, X)$, $H = 1, M(t) = \Lambda(-t)^{\frac{1}{2}}, K(t) = 1 + \left(\int_{-r}^0 g(\tau) d\tau \right)^{\frac{1}{2}}$, for $t \geq 0$ (see [25], Theorem 1.3.8 for details).

For our purpose we will only need the following fixed point theorems.

Theorem 2.10. [5, 33] Let \mathcal{U} be a bounded, closed and convex subset of a Banach space, and let N be a continuous mapping of \mathcal{U} into itself. If the implication

$$V = \overline{\text{conv}}N(V) \text{ or } V = N(V) \cup \{0\} \implies \alpha(V) = 0$$

holds for every subset V of \mathcal{U} , then N has a fixed point.

A continuous map $N : D \subseteq E \rightarrow E$ is said to be a α -contraction if there exists a constant $\nu \in [0, 1)$ such that $\alpha(N(C)) \leq \nu\alpha(C)$ for any bounded closed subset $C \subseteq D$.

Theorem 2.11. (Darbo–Sadovskii)[13] *Let E be a Banach space. If $D \subseteq E$ is bounded closed and convex, the continuous map $N : D \rightarrow D$ is a α -contraction, then the map N has at least one fixed point in D .*

Consider the space

$$\text{PC}(J, X) = \left\{ x : J \rightarrow X, x \in C \left(J \cap \left(\bigcup_{k=0}^N (t_k, s_k] \right), X \right), \right. \\ \left. \text{and } x(t_k^+), x(s_k^-) \text{ exist with, } x(s_k^-) = x(s_k), k = 1, \dots, N \right\}.$$

Obviously, $\text{PC}(J, X)$ is a Banach space with the norm

$$\|x\|_{\text{PC}} = \sup_{t \in J} \|x(t)\|.$$

3 Existence Results

In this section, we prove the existence of mild solution of (1.1).

Definition 3.1. *A function $x : (-\infty, T] \rightarrow X$ is said to be a mild solution of the equation (1.1) if $x_0 = \phi$ on $(-\infty, T]$, $x|_{[0, T]} \in \text{PC}([0, T], X)$ and x satisfies*

$$x(t) = \begin{cases} Q(t)\phi(0) + \int_0^t \int_0^s R(t-s)a(s, \tau)f(\tau, x_{\rho(\tau, x_\tau)}, x(\tau))d\tau ds, & t \in [0, t_1], \\ h_i(t, x_{\rho(t, x_t)}, x(t)), & t \in (t_i, s_i], i = 1, 2, \dots, N, \\ Q(t-s_i)h_i(s_i, x_{\rho(s_i, x_{s_i})}, x(s_i)) \\ + \int_0^t \int_0^s R(t-s)a(s, \tau)f(\tau, x_{\rho(\tau, x_\tau)}, x(\tau))d\tau ds, & t \in (s_i, t_{i+1}], \end{cases} \quad (3.1)$$

where

$$Q(t) = \int_0^\infty \xi_q(\sigma)S(t^q \sigma)d\sigma, \quad R(t) = q \int_0^\infty \sigma t^{q-1} \xi_q(\sigma)S(t^q \sigma)d\sigma$$

and ξ_q is a probability density function defined on $(0, \infty)$ such that

$$\xi_q(\sigma) = \frac{1}{q} \sigma^{-1 - (\frac{1}{q})} \omega_q(\sigma^{-\frac{1}{q}}) \geq 0,$$

where

$$\omega_q(\sigma) = \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^{k-1} \sigma^{-qk-1} \frac{\Gamma(kq+1)}{k!} \sin(k\pi q), \quad \sigma \in (0, \infty).$$

Remark 3.2. Note that $\{S(t)\}_{t \geq 0}$ is a uniformly bounded i.e

there exists a constant $M > 0$ such that $\|S(t)\|_{L(X)} \leq M$ for all $t \geq 0$.

Remark 3.3. According to [30], direct calculation gives that

$$\|R(t)\| \leq C_{q,M} t^{q-1}, \quad t > 0, \tag{3.2}$$

where $C_{q,M} = \frac{qM}{\Gamma(1+q)}$.

Set

$$\mathcal{R}(\rho^-) = \{\rho(s, \varphi) : (s, \varphi) \in J \times \mathcal{B}, \rho(s, \varphi) \leq 0\}.$$

We always assume that $\rho : J \times \mathcal{B} \rightarrow (-\infty, T]$ is continuous. Additionally, we introduce following hypothesis:

(H $_{\varphi}$) The function $t \rightarrow \varphi_t$ is continuous from $\mathcal{R}(\rho^-)$ into \mathcal{B} and there exists a continuous and bounded function $L^{\varphi} : \mathcal{R}(\rho^-) \rightarrow (0, \infty)$ such that

$$\|\phi_t\|_{\mathcal{B}} \leq L^{\varphi}(t) \|\phi\|_{\mathcal{B}} \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

Remark 3.4. Condition (H $_{\varphi}$), is frequently verified by the continuous and bounded functions. For more details see [25].

Remark 3.5. In the rest of this section, C_1^* and C_2^* are the constants

$$C_1^* = \sup_{s \in J} C_1(s) \quad \text{and} \quad C_2^* = \sup_{s \in J} C_2(s).$$

Lemma 3.6. [22] If $\chi : \mathbb{R} \rightarrow X$ is a function such that $\chi_0 = \phi$, then

$$\|\chi_s\|_{\mathcal{B}} \leq (C_2^* + L^{\varphi}) \|\phi\|_{\mathcal{B}} + C_1^* \sup\{|\chi(\theta)|; \theta \in [0, \max\{0, s\}]\}, \quad s \in \mathcal{R}(\rho^-) \cup J,$$

where $L^{\varphi} = \sup_{t \in \mathcal{R}(\rho^-)} L^{\varphi}(t)$.

Let us introduce the following hypotheses:

(H1) $f : J \times \mathcal{B} \times X \rightarrow X$ satisfies the Carathéodory conditions.

(H2) There exist functions $\mu, \mu^* \in L^1(J, \mathbb{R}^+)$ and continuous nondecreasing functions $\psi, \psi^* : \mathbb{R}^+ \rightarrow (0, +\infty)$ such that

$$\|f(t, x, y)\| \leq \mu(t) \psi(\|x\|_{\mathcal{B}} + \|y\|), \quad (t, x, y) \in J \times \mathcal{B} \times X,$$

$$\|h_i(t, x, y)\| \leq \mu^*(t) \psi^*(\|x\|_{\mathcal{B}} + \|y\|), \quad (t, x, y) \in J \times \mathcal{B} \times X,$$

(H3) For any bounded sets $D_1 \subset \mathcal{B}, D_2 \subset X$, and $0 \leq s \leq t \leq T$, there exists an integrable positive function η such that

$$\alpha(\mathcal{R}(t-s)f(\tau, D_1, D_2)) \leq \eta_t(s, \tau) \left(\alpha(D_2) + \sup_{-\infty < \theta \leq 0} \alpha(D_1(\theta)) \right),$$

where $\eta_t(s, \tau) = \eta(t, s, \tau)$ and $\sup_{t \in J} \int_0^t \int_0^s \eta_t(s, \tau) d\tau ds = \eta^* < \infty$.

(H4) There exists a constant $L > 0$ such that, for each bounded sets $D_1 \subset \mathcal{B}, D_2 \subset X$,

$$\alpha(h_i(\tau, D_1, D_2)) \leq L \left(\alpha(D_2) + \sup_{-\infty < \theta \leq 0} \alpha(D_1(\theta)) \right).$$

(H5) For each $t \in J$, $\mathbf{a}(t, s)$ is measurable on $[0, t]$ and $\mathbf{a}(t) = \text{ess sup}\{\mathbf{a}(t, s), 0 \leq s \leq t\}$ is bounded on J . The map $t \rightarrow \mathbf{a}_t$ is continuous from J to $L^\infty(J, \mathbb{R})$, here, $\mathbf{a}_t(s) = \mathbf{a}(t, s)$.

Set $\mathbf{a} = \sup_{t \in J} \mathbf{a}(t)$.

Our first result is based on the Mönch fixed point theorem.

Theorem 3.7. *Suppose that the assumptions (H $_\varphi$), (H1) – (H5) hold, and if*

$$2ML + 16 \mathbf{a} \eta^* < 1, \tag{3.3}$$

then the problem (1.1) has at least one mild solution.

Proof. Let $Y = \{u \in PC(X) : u(0) = \phi(0) = 0\}$ endowed with the uniform convergence topology and define the operator $P : Y \rightarrow Y$ by

$$P(x)(t) = \begin{cases} Q(t)\phi(0) + \int_0^t \int_0^s \mathcal{R}(t-s)\mathbf{a}(s, \tau)f(\tau, \bar{x}_{\rho(\tau, \bar{x}_\tau)}, \bar{x}(\tau))d\tau ds, & t \in [0, t_1], \\ h_i(t, \bar{x}_{\rho(t, \bar{x}_t)}, \bar{x}(t)), & t \in (t_i, s_i], i = 1, 2, \dots, N, \\ Q(t-s_i)h_i(s_i, \bar{x}_{\rho(s_i, \bar{x}_{s_i})}, \bar{x}(s_i)) \\ + \int_{s_i}^t \int_0^s \mathcal{R}(t-s)\mathbf{a}(s, \tau)f(\tau, \bar{x}_{\rho(\tau, \bar{x}_\tau)}, \bar{x}(\tau))d\tau ds, & t \in (s_i, t_{i+1}], \end{cases}$$

where $x : (-\infty, T] \rightarrow X$ is such that $x_0 = \phi$ and $\bar{x} = x$ on J . Let $\bar{\phi} : (-\infty, T] \rightarrow X$ be the extension of ϕ to $(-\infty, T]$ such that $\bar{\phi}(\theta) = \phi(0) = 0$ on J .

Choose

$$r \geq M \|\mu^*\|_{L^1} \psi^* ((C_2^* + L^\Phi) \|\phi\|_{\mathcal{B}} + (C_1^* + 1)r) + \mathbf{a} C_{q, M} \|\mu\|_{L^1} \frac{T^q}{q} \psi ((C_2^* + L^\Phi) \|\phi\|_{\mathcal{B}} + (C_1^* + 1)r),$$

and define the set

$$B_r = \{x \in Y : \|x\|_{PC} \leq r\},$$

then B_r is a bounded, closed-convex subset in Y .

Step 1: P is continuous.

Let $\{x^k\}_{k \in \mathbb{N}}$ be a sequence such that $x^k \rightarrow x$ in B_r as $k \rightarrow \infty$.

Case 1. For each $t \in [0, t_1]$, we have

$$\begin{aligned} \|P(x^k)(t) - P(x)(t)\| &\leq \int_0^t \int_0^s \|R(t-s)\| \|a(s, \tau)\| \|f(\tau, \bar{x}_{\rho(\tau, \bar{x}_\tau^k)}, \bar{x}^k(\tau)) \\ &\quad - f(\tau, \bar{x}_{\rho(\tau, \bar{x}_\tau)}, \bar{x}(\tau))\| d\tau ds \\ &\leq \alpha C_{q, M} \int_0^t \int_0^s (t-s)^{q-1} \|f(\tau, \bar{x}_{\rho(\tau, \bar{x}_\tau^k)}, \bar{x}^k(\tau)) \\ &\quad - f(\tau, \bar{x}_{\rho(\tau, \bar{x}_\tau)}, \bar{x}(\tau))\| d\tau ds. \end{aligned}$$

Case 2. For each $t \in [t_i, s_i], i = 1, 2, \dots, N$, we have

$$\begin{aligned} \|P(x^k)(t) - P(x)(t)\| &= \|h_i(t, \bar{x}_{\rho(t, \bar{x}_t^k)}, \bar{x}^k(t)) - h_i(t, \bar{x}_{\rho(t, \bar{x}_t)}, \bar{x}(t))\| \\ &\rightarrow 0 \quad k \rightarrow \infty. \end{aligned}$$

Case 3. For each $t \in (s_i, t_{i+1}], i = 1, 2, \dots, N$, we obtain

$$\begin{aligned} \|P(x^k)(t) - P(x)(t)\| &\leq \|Q(t-s_i)\| \|h_i(s_i, \bar{x}_{\rho(s_i, \bar{x}_{s_i}^k)}, \bar{x}^k(s_i)) - h_i(s_i, \bar{x}_{\rho(s_i, \bar{x}_{s_i})}, \bar{x}(s_i))\| \\ &\quad + \int_{s_i}^t \int_0^s \|R(t-s)\| \|a(s, \tau)\| \|f(\tau, \bar{x}_{\rho(\tau, \bar{x}_\tau^k)}, \bar{x}^k(\tau)) \\ &\quad - f(\tau, \bar{x}_{\rho(\tau, \bar{x}_\tau)}, \bar{x}(\tau))\| d\tau ds \\ &\leq M \|h_i(s_i, \bar{x}_{\rho(s_i, \bar{x}_{s_i}^k)}, \bar{x}^k(s_i)) - h_i(s_i, \bar{x}_{\rho(s_i, \bar{x}_{s_i})}, \bar{x}(s_i))\| \\ &\quad + \alpha C_{q, M} \int_{s_i}^t \int_0^s (t-s)^{q-1} \|f(\tau, \bar{x}_{\rho(\tau, \bar{x}_\tau^k)}, \bar{x}^k(\tau)) \\ &\quad - f(\tau, \bar{x}_{\rho(\tau, \bar{x}_\tau)}, \bar{x}(\tau))\| d\tau ds. \end{aligned}$$

Since the function h_i is continuous and f is of Carathéodory type, we have by the Lebesgue dominated convergence theorem that

$$\|P(x^k)(t) - P(x)(t)\| \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

which shows the operator P is continuous.

Step 2: P maps B_r into itself.

Case 1. For all $t \in [0, t_1]$, we get

$$\begin{aligned}
 \|P(x)(t)\| &\leq \|Q(t)\phi(0)\| + \int_0^t \int_0^s \|R(t-s)a(s,\tau)f(\tau, \bar{x}_{\rho(\tau, \bar{x}_\tau)}, \bar{x}(\tau))\| d\tau ds \\
 &\leq MC\|\phi\|_{\mathcal{B}} + aC_{q,M} \int_0^t \int_0^s (t-s)^{q-1} \mu(\tau) \psi(\|\bar{x}_{\rho(\tau, \bar{x}_\tau)}\|_{\mathcal{B}} + \|\bar{x}\|) d\tau ds \\
 &\leq MC\|\phi\|_{\mathcal{B}} + aC_{q,M} \int_0^t \int_0^s (t-s)^{q-1} \mu(\tau) \\
 &\quad \times \psi((C_2^* + L^\Phi)\|\phi\|_{\mathcal{B}} + C_1^*r + r) d\tau ds \\
 &\leq MC\|\phi\|_{\mathcal{B}} + aC_{q,M} \|\mu\|_{L^1} \frac{T^q}{q} \psi((C_2^* + L^\Phi)\|\phi\|_{\mathcal{B}} + (C_1^* + 1)r) \\
 &\leq r.
 \end{aligned}$$

Case 2. For all $t \in [t_i, s_i], i = 1, 2, \dots, N$, we have

$$\begin{aligned}
 \|P(x)(t)\| &\leq \|h_i(t, \bar{x}_{\rho(t, \bar{x}_t)}, \bar{x}(t))\| \\
 &\leq \mu^*(t) \psi^*(\|\bar{x}_{\rho(t, \bar{x}_t)}\|_{\mathcal{B}} + \|\bar{x}\|) \\
 &\leq \|\mu^*\|_{L^1} \psi^*((C_2^* + L^\Phi)\|\phi\|_{\mathcal{B}} + (C_1^* + 1)r) \\
 &\leq r.
 \end{aligned}$$

Case 3. For all $t \in (s_i, t_{i+1}], i = 1, 2, \dots, N$, we obtain

$$\begin{aligned}
 \|P(x)(t)\| &\leq \|Q(t-s_i)h_i(s_i, \bar{x}_{\rho(s_i, \bar{x}_{s_i})}, \bar{x}(s_i))\| \\
 &\quad + \int_{s_i}^t \int_0^s \|R(t-s)a(s,\tau)f(\tau, \bar{x}_{\rho(\tau, \bar{x}_\tau)}, \bar{x}(\tau))\| d\tau ds, \\
 &\leq M\|\mu^*\|_{L^1} \psi^*((C_2^* + L^\Phi)\|\phi\|_{\mathcal{B}} + (C_1^* + 1)r) \\
 &\quad + aC_{q,M} \|\mu\|_{L^1} \frac{T^q}{q} \psi((C_2^* + L^\Phi)\|\phi\|_{\mathcal{B}} + (C_1^* + 1)r) \\
 &\leq r.
 \end{aligned}$$

Step 3: $P(B_r)$ is bounded and equicontinuous.

Case 1. For each $t \in [0, t_1], 0 \leq \tau_2 \leq \tau_1 \leq t_1$, and $x \in B_r$. Then we have

$$\|P(x)(\tau_1) - P(x)(\tau_2)\| \leq I_1 + I_2 + I_3,$$

where

$$\begin{aligned}
 I_1 &= \|Q(\tau_1) - Q(\tau_2)\| \|\phi(0)\| \\
 I_2 &= \left\| \int_0^{\tau_2} \int_0^s [R(\tau_1-s) - R(\tau_2-s)] a(s,\tau) f(\tau, \bar{x}_{\rho(\tau, \bar{x}_\tau)}, \bar{x}(\tau)) d\tau ds \right\| \\
 I_3 &= \left\| \int_{\tau_2}^{\tau_1} \int_0^s R(\tau_1-s) a(s,\tau) f(\tau, \bar{x}_{\rho(\tau, \bar{x}_\tau)}, \bar{x}(\tau)) d\tau ds \right\|.
 \end{aligned}$$

I_1 tends to zero as $\tau_2 \rightarrow \tau_1$, since $S(t)$ is uniformly continuous operator.

For I_2 , using (3.2) and (H2), we have

$$\begin{aligned} I_2 &\leq \alpha\psi((C_2^* + L^\Phi)\|\phi\|_{\mathcal{B}} + (C_1^* + 1)r)\|\mu\|_{L^1} \int_0^{\tau_2} [R(\tau_1 - s) - R(\tau_2 - s)]ds \\ &\leq \alpha\psi((C_2^* + L^\Phi)\|\phi\|_{\mathcal{B}} + (C_1^* + 1)r)\|\mu\|_{L^1} \\ &\quad \times \int_0^{\tau_2} \left[q \int_0^\infty \sigma(\tau_1 - s)^{q-1} \xi_q(\sigma) S((\tau_1 - s)^q \sigma) d\sigma \right. \\ &\quad \left. - q \int_0^\infty \sigma(\tau_2 - s)^{q-1} \xi_q(\sigma) S((\tau_2 - s)^q \sigma) d\sigma \right] ds \\ &\leq \alpha\psi((C_2^* + L^\Phi)\|\phi\|_{\mathcal{B}} + (C_1^* + 1)r)\|\mu\|_{L^1} \\ &\quad \times \left[q \int_0^{\tau_2} \int_0^\infty \sigma \|[(\tau_1 - s)^{q-1} - (\tau_2 - s)^{q-1}] \xi_q(\sigma) S((\tau_1 - s)^q \sigma) \right. \\ &\quad \left. + q \int_0^{\tau_2} \int_0^\infty \sigma(\tau_2 - s)^{q-1} \xi_q(\sigma) \|S((\tau_1 - s)^q \sigma) - S((\tau_2 - s)^q \sigma)\| \right] \\ &\leq \alpha\psi((C_2^* + L^\Phi)\|\phi\|_{\mathcal{B}} + (C_1^* + 1)r)\|\mu\|_{L^1} \\ &\quad \times [C_{q,M} \int_0^{\tau_2} |(\tau_1 - s)^{q-1} - (\tau_2 - s)^{q-1}| ds \\ &\quad + q \int_0^{\tau_2} \int_0^\infty \sigma(\tau_2 - s)^{q-1} \xi_q(\sigma) \|S((\tau_1 - s)^q \sigma) - S((\tau_2 - s)^q \sigma)\| d\sigma ds]. \end{aligned}$$

Clearly, the first term on the right-hand side of the above inequality tends to zero as $\tau_2 \rightarrow \tau_1$. From the continuity of $S(t)$ in the uniform operator topology for $t > 0$, The second term on the right-hand side of the above inequality tends to zero as $\tau_2 \rightarrow \tau_1$.

In view of (H2), we have

$$\begin{aligned} I_3 &\leq \alpha C_{q,M} \int_{\tau_2}^{\tau_1} \int_0^s (\tau_1 - s)^{q-1} \|f(\tau, \bar{x}_\rho(\tau, \bar{x}_\tau), \bar{x}(\tau))\| d\tau ds \\ &\leq \alpha C_{q,M} \psi((C_2^* + L^\Phi)\|\phi\|_{\mathcal{B}} + (C_1^* + 1)r)\|\mu\|_{L^1} \int_{\tau_2}^{\tau_1} (\tau_1 - s)^{q-1} ds. \end{aligned}$$

As $\tau_2 \rightarrow \tau_1$, I_3 tends to zero.

Case 2. For each $t \in [t_i, s_i], i = 1, 2, \dots, N, t_i \leq \tau_2 \leq \tau_1 \leq s_i$, and $x \in B_r$. Then we have

$$\begin{aligned} \|P(x)(\tau_1) - P(x)(\tau_2)\| &= \|h_i(\tau_1, \bar{x}_\rho(\tau_1, \bar{x}_{\tau_1}), \bar{x}(\tau_1)) - h_i(\tau_2, \bar{x}_\rho(\tau_2, \bar{x}_{\tau_2}), \bar{x}(\tau_2))\| \\ &\rightarrow 0 \text{ as } \tau_2 \rightarrow \tau_1. \end{aligned}$$

Case 3. For each $t \in (s_i, t_{i+1}], i = 1, 2, \dots, N, s_i \leq \tau_2 \leq \tau_1 \leq t_{i+1}$, and $x \in B_r$. Then we have

$$\|P(x)(\tau_1) - P(x)(\tau_2)\| \leq \|Q(\tau_1 - s_i) - Q(\tau_2 - s_i)\| \|h_i(s_i, \bar{x}_\rho(s_i, \bar{x}_{s_i}), \bar{x}(s_i))\| + I_1 + I_2 + I_3.$$

Since $S(t)$ is uniformly continuous operator, so

$$\lim_{\tau_2 \rightarrow \tau_1} \|Q(\tau_1 - s_i) - Q(\tau_2 - s_i)\| = 0, \quad i = 1, \dots, N.$$

Consequently

$$\lim_{\tau_2 \rightarrow \tau_1} \|P(x)(\tau_1) - P(x)(\tau_2)\| = 0.$$

Thus, $P(B_r)$ is equicontinuous.

Now let V be a subset of B_r such that $V \subset \overline{\text{conv}}(P(V) \cup \{0\})$. Moreover, for any $\varepsilon > 0$ and bounded set D , we can take a sequence $\{v_n\}_{n=1}^\infty \subset D$ such that $\alpha(D) \leq 2\alpha(\{v_n\}) + \varepsilon$ ([16], P. 125). Thus, for $\{v_n\}_{n=1}^\infty \subset V$, and using lemmas 2.5-2.7 and (H3), we have, for $t \in [0, t_1]$,

$$\begin{aligned} \alpha(PV) &\leq 2\alpha(\{Pv_n\}) + \varepsilon \\ &= 2 \sup_{t \in J} \alpha(\{Pv_n(t)\}) + \varepsilon \\ &= 2 \sup_{t \in J} \alpha \left(\left\{ \int_0^t \mathcal{R}(t-s) \int_0^s a(s, \tau) f(\tau, y_\tau + v_{n\tau}, y(\tau) + v_n(\tau)) d\tau ds \right\} \right) + \varepsilon \\ &\leq 4 \sup_{t \in J} \int_0^t \alpha \left(\left\{ \mathcal{R}(t-s) \int_0^s a(s, \tau) f(\tau, y_\tau + v_{n\tau}, y(\tau) + v_n(\tau)) d\tau ds \right\} \right) + \varepsilon \\ &\leq 8 \sup_{t \in J} \int_0^t \int_0^s \alpha(\{\mathcal{R}(t-s)a(s, \tau)f(\tau, y_\tau + v_{n\tau}, y(\tau) + v_n(\tau))d\tau ds\}) + \varepsilon \\ &\leq 8 a \sup_{t \in J} \int_0^t \int_0^s \alpha(\{\mathcal{R}(t-s)f(\tau, y_\tau + v_{n\tau}, y(\tau) + v_n(\tau))d\tau ds\}) + \varepsilon \\ &\leq 8 a \sup_{t \in J} \int_0^t \int_0^s \eta_t(s, \tau) \left[\alpha(v_n(\tau)) + \sup_{-\infty < \theta \leq 0} \alpha(v_n(\theta + \tau)) \right] d\tau ds + \varepsilon \\ &\leq 8 a \sup_{t \in J} \int_0^t \int_0^s \eta_t(s, \tau) \left[\alpha(v_n) + \sup_{0 < \mu \leq \tau} \alpha(v_n(\mu)) \right] d\tau ds + \varepsilon \\ &\leq 16 a \alpha(v_n) \sup_{t \in J} \int_0^t \int_0^s \eta_t(s, \tau) d\tau ds + \varepsilon \\ &\leq 16 a \eta^* \alpha(V) + \varepsilon. \end{aligned}$$

For any $t \in [t_i, s_i], i = 1, 2, \dots, N$, we get

$$\begin{aligned} \alpha(PV) &= \alpha(h_i(t, \bar{x}_\rho(t, \bar{x}_t), \bar{x}(t))) \\ &\leq L \left(\alpha(v_n(t)) + \sup_{-\infty < \theta \leq 0} \alpha(v_n(\theta + t)) \right) \\ &\leq L \left(\alpha(v_n) + \sup_{0 < \mu \leq \tau} \alpha(v_n(\mu)) \right) \\ &\leq 2L\alpha(v_n) \\ &\leq 2L\alpha(V). \end{aligned}$$

In the same way, for any $t \in (s_i, t_{i+1}]$, $i = 1, 2, \dots, N$, we obtain

$$\begin{aligned}
 \alpha(PV) &\leq 2\alpha(\{Pv_n\}) + \varepsilon \\
 &= 2 \sup_{t \in J} \alpha(\{Pv_n(t)\}) + \varepsilon \\
 &= 2 \sup_{t \in J} \alpha \left(Q(t - s_i) h_i(s_i, \bar{x}_{\rho(s_i, \bar{x}_{s_i})}, \bar{x}(s_i)) \right) \\
 &\quad + 2 \sup_{t \in J} \alpha \left(\left\{ \int_{s_i}^t R(t-s) \int_0^s a(s, \tau) f(\tau, y_\tau + v_{n\tau}, y(\tau) + v_n(\tau)) d\tau ds \right\} \right) + \varepsilon \\
 &\leq 2ML\alpha(v_n) \\
 &\quad + 4 \sup_{t \in J} \int_{s_i}^t \alpha \left(\left\{ R(t-s) \int_0^s a(s, \tau) f(\tau, y_\tau + v_{n\tau}, y(\tau) + v_n(\tau)) d\tau ds \right\} \right) + \varepsilon \\
 &\leq 2ML\alpha(v_n) \\
 &\quad + 8 \sup_{t \in J} \int_{s_i}^t \int_0^s \alpha(\{R(t-s)a(s, \tau)f(\tau, y_\tau + v_{n\tau}, y(\tau) + v_n(\tau))d\tau ds\}) + \varepsilon \\
 &\leq 2ML\alpha(v_n) + 8 a \sup_{t \in J} \int_{s_i}^t \int_0^s \alpha(\{R(t-s)f(\tau, y_\tau + v_{n\tau}, y(\tau) + v_n(\tau))d\tau ds\}) + \varepsilon \\
 &\leq 2ML\alpha(v_n) + 8 a \sup_{t \in J} \int_{s_i}^t \int_0^s \eta_t(s, \tau) \left[\alpha(v_n(\tau)) + \sup_{-\infty < \theta \leq 0} \alpha(v_n(\theta + \tau)) \right] d\tau ds + \varepsilon \\
 &\leq 2ML\alpha(v_n) + 8 a \sup_{t \in J} \int_{s_i}^t \int_0^s \eta_t(s, \tau) \left[\alpha(v_n) + \sup_{0 < \mu \leq \tau} \alpha(v_n(\mu)) \right] d\tau ds + \varepsilon \\
 &\leq 2ML\alpha(v_n) + 16 a \alpha(v_n) \sup_{t \in J} \int_0^t \int_0^s \eta_t(s, \tau) d\tau ds + \varepsilon \\
 &\leq 2ML\alpha(V) + 16 a \eta^* \alpha(V) + \varepsilon \\
 &\leq (2ML + 16 a \eta^*) \alpha(V) + \varepsilon.
 \end{aligned}$$

Therefore, in view of Lemma 2.5, we have

$$\alpha(V) \leq \alpha(PV) \leq (2ML + 16 a \eta^*) \alpha(V) + \varepsilon,$$

since ε is arbitrary we obtain that

$$\alpha(V) \leq (2ML + 16 a \eta^*) \alpha(V).$$

This means that

$$\alpha(V) (1 - (2ML + 16 a \eta^*)) \leq 0.$$

By (3.3) it follows that $\alpha(V) = 0$. In view of the Ascoli-Arzelà theorem, V is relatively compact in B_r . Applying now Theorem 2.10, we conclude that P has a fixed point which is a solution of the problem (1.1).

The second result is established using the Darbo's fixed point theorem.

Theorem 3.8. *Assume that (H1)–(H5) are satisfied, then the problem (1.1) has at least one mild solution.*

Proof. In what follows we show that the operator $P : Y \rightarrow Y$ is a strict set contraction. We know that $P : Y \rightarrow Y$ is bounded and continuous, we need to prove that there exists a constant $0 \leq \nu < 1$ such that $\alpha(PV) \leq \nu\alpha(V)$ for $V \subset B_r$.

Using the same method as the proof of Theorem 3.7, for $t \in [0, T]$, we have

$$\alpha(PV) \leq (2ML + 16 a \eta^*) \alpha(V) + \varepsilon,$$

since ε is arbitrary we obtain that

$$\alpha(PV) \leq \nu\alpha(V).$$

Hence P is a set contraction. According to Theorem 2.11 the operator P has at least one fixed point which is obviously a mild solution of the problem (2.4). This completes the proof.

4 An Example

We consider the fractional integro-differential equations with state-dependent delay and non-instantaneous impulses of the form

$$\begin{aligned} \frac{\partial_t^q}{\partial t^q} v(t, \zeta) + \frac{\partial^2}{\partial \zeta^2} v(t, \zeta) &= \int_0^t (t-s)^2 \int_{-\infty}^s \gamma(\tau-s) v(\tau - \rho_1(s) \rho_2(|v(s, \zeta)|), \zeta) d\tau ds \\ &\quad + \int_0^t (t-s)^2 \cos |v(s, \zeta)| ds, \quad (t, x) \in \mathbb{N} \in \cup_{i=1}^n [s_i, t_{i+1}] \times [0, \pi], \\ v(t, 0) = v(t, \pi) &= 0, \quad t \in [0, T], \\ v(\tau, \zeta) &= v_0(\theta, \zeta), \quad \theta \in (-\infty, 0], x \in [0, \pi] \\ v(t, \zeta) &= H_i(t, v(t - \rho_1(t) \rho_2(|v(t, \zeta)|), \zeta), \zeta), \quad (t, x) \in (t_i, s_i] \times [0, \pi], \quad i = 1, 2, \dots, N, \end{aligned} \tag{4.1}$$

where $0 < q < 1, 0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \dots < t_{N-1} \leq s_N \leq t_N \leq t_{N+1} = T$ are prefixed real numbers and the functions $\gamma : \mathbb{R} \rightarrow \mathbb{R}, \rho_i : [0, +\infty) \rightarrow [0, +\infty), i = 1, 2$ are continuous functions.

Let $X = L^2([0, \pi])$ and define the operator $A : D(A) \subset X \rightarrow X$ by $A\omega = \omega''$ with domain

$$D(A) = \{\omega \in E : \omega, \omega' \text{ are absolutely continuous, } \omega'' \in E, \omega(0) = \omega(\pi) = 0\}.$$

Then

$$A\omega = \sum_{n=1}^{\infty} n^2(\omega, \omega_n)\omega_n, \quad \omega \in D(A),$$

where $\omega_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$, $n \in \mathbb{N}$ is the orthogonal set of eigenvectors of A . It is well known that A is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t \geq 0}$ in X and is given by

$$S(t)\omega = \sum_{n=1}^{\infty} e^{-n^2 t} (\omega, \omega_n) \omega_n, \quad \forall \omega \in X, \text{ and every } t > 0.$$

From these expressions, it follows that $\{S(t)\}_{t \geq 0}$ is a uniformly bounded compact semigroup on X . For the phase space, we choose $\mathcal{B} = C_0 \times L^2(g, X)$, see Example 2.9 for details.

Set

$$\begin{aligned} x(t)(\zeta) &= v(t, \zeta), \\ \phi(\theta)(\zeta) &= v_0(\theta, \zeta), \\ a(t, s) &= (t - s)^2 \\ f(t, \varphi, x(t))(\zeta) &= \int_{-\infty}^0 \gamma(s) \varphi(t, \zeta) ds + \cos |x(t)(\zeta)|, \\ h_i(t, \varphi, x(t))(\zeta) &= H_i(t, v(t - \rho_1(t) \rho_2(|x(t)|), \zeta), \zeta) \\ \rho(t, \varphi) &= t - \rho_1(t) \rho_2(|\varphi(0)|). \end{aligned}$$

Under the above conditions, we can represent the problem (4.1) by the abstract problem (1.1).

Proposition 4.1. *Let $\varphi \in \mathcal{B}$ be such that (H_φ) holds, and let $t \rightarrow \varphi_t$ be continuous on $\mathcal{R}(\rho^-)$. Then there exists a mild solution of (4.1).*

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