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Totally umbilical proper slant submanifolds of para-Kenmotsu manifold

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ABSTRACT

In this paper, we study slant submanifolds of a para-Kenmotsu manifold. We prove that totally umbilical slant submanifold of a para-Kenmotsu manifold is either invariant or anti-invariant or dimension of submanifold is 1 or the mean curvature vector H of the submanifold lies in the invariant normal subbundle.

RESUMEN

En este paper estudiamos subvariedades inclinadas en variedades para-Kenmotsu. Demostramos que una subvariedad inclinada en una variedad para-Kenmotsu totalmente umbilical es invariante, o anti-invariante, o una subvariedad de dimensión 1, o el vector de curvatura media H de la subvariedad vive en el fibrado normal invariante.

Keywords and Phrases: para-Kenmotsu manifold; totally umbilical; slant submanifold.
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1 Introduction

The study of submanifolds of an almost contact manifold is one of the utmost interesting topics in differential geometry. According to the behaviour of the tangent bundle of a submanifold with respect to action of the almost contact structure ϕ of the ambient manifold, there are two well known classes of submanifolds, namely, invariant and anti-invariant submanifolds. Chen [4], introduced the notion of slant submanifolds of the almost Hermitian manifolds. The contact version of slant submanifolds were given by Lotta [12]. Since then many research articles have been appeared on the existence of different contact and lorentzian manifolds (See. [1, 3, 7, 14, 15]).

Motivated by the above studies, in the present paper we study slant submanifolds of almost para-Kenmotsu manifold and give a classification of results. Also we prove that totally umbilical slant submanifolds of para-Kenmotsu manifolds are totally geodesic.
The paper is organized as follows: In section 2, we review some basic concepts of para-Kenmotsu manifold and submanifold theory. Section 3 is the main section of this paper. In this section we give the classification result of totally umbilical slant submanifolds of para-Kenmotsu manifold. Further, we prove that totally umbilical slant submanifolds of a para-Kenmotsu manifold are totally geodesic.

2 Preliminaries

Let $\tilde{M}$ be a $(2m+1)$-dimensional smooth manifold, ϕ a tensor field of type $(1,1)$, ξ a vector field and η a 1-form. We say that (ϕ, ξ, η) is an almost para contact structure on $\tilde{M}$ if [18]

$$\phi\xi = 0, \quad \eta\cdot\phi = 0, \quad rank(\phi) = 2m, \tag{2.1}$$

$$\phi^2 = I - \eta\otimes\xi, \quad \eta(\xi) = 1. \tag{2.2}$$

If an almost paracontact manifold admits a pseudo Riemannian metric g of signature $(m+1, m)$ satisfying

$$g(\phi\cdot, \phi\cdot) = -g + \eta\otimes\eta \tag{2.3}$$

called almost para contact metric manifold. Examples of almost para contact metric structure are given in [6] and [9].

Analogous to the definition of Kenmotsu manifold [10], Welyczko [17] introduced para-Kenmotsu structure for three dimensional normal almost para contact metric structures. The similar notion called p-Kenmotsu structure appears in the Sinha and Sai Prasad [16].

Definition 2.1. *An almost para contact metric manifold $M(\phi, \xi, \eta, g)$ is para-Kenmotsu manifold if the Levi-Civita connection $\tilde{\nabla}$ of g satisfies*

$$(\tilde{\nabla}_X\phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.4}$$

for any $X, Y \in \chi(M)$, *(where* $\chi(M)$ *is the set of all differential vector fields on* M*).*

From (2.4), we have

$$\tilde{\nabla}_X \xi = X - \eta(X)\xi. \tag{2.5}$$

Assume M is a submanifold of a para-Kenmotsu manifold $\tilde{M}$. Let g and ∇ be the induced Riemannian metric and connections of M, respectively. Then the Gauss and Weingarten formulae are given respectively, by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{2.6}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^\perp_X N, \tag{2.7}$$

for all X, Y on TM and $N \in T^\perp M$, where $\nabla^\perp$ is the normal connection and A is the shape operator of M with respect to the unit normal vector N. The second fundamental form σ and the shape operator A are related by:

$$g(\sigma(X, Y), N) = g(A_N X, Y). \tag{2.8}$$

Now for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, we write

$$\phi X = PX + FX, \tag{2.9}$$

$$\phi V = pV + fV. \tag{2.10}$$

For $X, Y \in \Gamma(TM)$, it is easy to observe from (2.1) and (2.9) that

$$g(PX, Y) = -g(X, PY). \tag{2.11}$$

The covariant derivatives of the endomorphisms ϕ, P and F are defined respectively as

$$(\tilde{\nabla}_X \phi)Y = \tilde{\nabla}_X \phi Y - \phi \tilde{\nabla}_X Y, \ \ \forall X, Y \in \Gamma(T\tilde{M}), \tag{2.12}$$

$$(\tilde{\nabla}_X P)Y = \nabla_X PY - P\nabla_X Y, \ \ \forall X, Y \in \Gamma(TM), \tag{2.13}$$

$$(\tilde{\nabla}_X F)Y = \nabla_X FY - F\nabla_X Y, \ \ \forall X, Y \in \Gamma(TM). \tag{2.14}$$

The structure vector field ξ has been considered to be tangential to M throughout this paper, else M is simply anti-invariant [12]. Since $\xi \in TM$, for any $X \in \Gamma(TM)$ by virtue of (2.5) and (2.6), we have

$$\nabla_X \xi = X - \eta(X)\xi \ \ \text{and} \ \ \sigma(X, \xi) = 0. \tag{2.15}$$

Making use of (2.4), (2.6), (2.7), (2.9), (2.10) and (2.12)-(2.14), we obtain

$$(\tilde{\nabla}_X P)Y = p\sigma(X, Y) + A_{FY}X + g(PX, Y)\xi - \eta(Y)PX, \tag{2.16}$$

$$(\tilde{\nabla}_X F)Y = f\sigma(X, Y) - \sigma(X, PY) - \eta(Y)FX. \tag{2.17}$$

A submanifold M of an almost para contact metric manifold $\tilde{M}$ is said to be totally umbilical if

$$\sigma(X, Y) = g(X, Y)H, \tag{2.18}$$

where H is the mean curvature vector of M. Further M is totally geodesic if $\sigma(X, Y) = 0$ and minimal if $H = 0$.

3 Slant submanifolds of an almost contact metric manifold

For any $x \in M$ and $X \in T_xM$ such that X, ξ are linearly independent, the angle $\theta(x) \in [0, \frac{\pi}{2}]$ between ϕX and T_xM is a constant θ, that is θ does not depend on the choice of X and $x \in M$. θ is called the slant angle of M in $\tilde{M}$. Invariant and anti-invariant submanifolds are slant submanifolds with slant angle θ equal to 0 and $\frac{\pi}{2}$, respectively [5]. A slant submanifold which is neither invariant nor anti-invariant is called a proper slant submanifold.
We mention the following results for later use.

Theorem 3.1. *[1] Let M be a submanifold of an almost contact metric manifold $\tilde{M}$ such that $\xi \in TM$. Then, M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$P^2 = -\lambda(I - \eta \otimes \xi). \tag{3.1}$$

Further more, if θ is the slant angle of M, then $\lambda = \cos^2\theta$.

Corolary 1. *[1] Let M be a slant submanifold of an almost contact metric manifold $\tilde{M}$ with slant angle θ. Then, for any $X, Y \in TM$, we have*

$$g(PX, PY) = -\cos^2\theta(g(X, Y) - \eta(X)\eta(Y)), \tag{3.2}$$

$$g(FX, FY) = -\sin^2\theta(g(X, Y) - \eta(X)\eta(Y)). \tag{3.3}$$

Theorem 3.2. *Let M be a totally umbilical slant submanifold of a para-Kenmotsu manifold $\tilde{M}$. Then either one of the following statements is true:*
(i) M is invariant;
(ii) M is anti-invariant;
(iii) M is totally geodesic;
(iv) dim$M$= 1;
(v) If M is proper slant, then $H \in \Gamma(\mu)$;
where H is the mean curvature vector of M.

Proof. Suppose M is totally umbilical slant submanifold, then we have

$$\sigma(PX, PX) = g(PX, PX)H = \cos^2\theta\{-\|X\|^2 + \eta^2(X)\}H.$$

By virtue of (2.6), one can get

$$\cos^2\theta\{-\|X\|^2 + \eta^2(X)\}H = \tilde{\nabla}_{PX}PX - \nabla_{PX}PX.$$

From (2.9), we have

$$\cos^2\theta\{-\|X\|^2 + \eta^2(X)\}H = \tilde{\nabla}_{PX}\phi X - \tilde{\nabla}_{PX}FX - \nabla_{PX}PX.$$

Applying (2.7) and (2.12), we get

$$\cos^2\theta\{-\|X\|^2+\eta^2(X)\}H = (\tilde{\nabla}_{PX}\phi)X + \phi\tilde{\nabla}_{PX}X + A_{FX}PX - \nabla^{\perp}_{PX}FX - \nabla_{PX}PX.$$

Using (2.4) and (2.6), we obtain

$$\begin{aligned}\cos^2\theta\{-\|X\|^2+\eta^2(X)\}H &= g(\phi PX, X)\xi - \eta(X)\phi PX + \phi(\nabla_{PX}X + \sigma(X, PX)) \\ &\quad + A_{FX}PX - \nabla^{\perp}_{PX}FX - \nabla_{PX}PX.\end{aligned}$$

From (2.9), (2.11), (2.18) and the fact that X and PX are orthogonal vector fields on M, we arrive at

$$\begin{aligned}\cos^2\theta\{-\|X\|^2+\eta^2(X)\}H &= -g(PX, PX)\xi - \eta(X)P^2X - \eta(X)FPX + P\nabla_{PX}X + F\nabla_{PX}X \\ &\quad + A_{FX}PX - \nabla^{\perp}_{PX}FX - \nabla_{PX}PX.\end{aligned}$$

Then applying (3.1) and (3.2), we obtain

$$\begin{aligned}\cos^2\theta\{-\|X\|^2+\eta^2(X)\}H &= \cos^2\theta\{\|X\|^2-\eta^2(X)\}\xi + \cos^2\theta\eta(X)\{X-\eta(X)\}\xi - \eta(X)FPX \\ &\quad + P\nabla_{PX}X + F\nabla_{PX}X + A_{FX}PX - \nabla^{\perp}_{PX}FX - \nabla_{PX}PX. \qquad (3.4)\end{aligned}$$

Taking inner product with PX in (3.4), we get

$$0 = g(P\nabla_{PX}X, PX) + g(A_{FX}PX, PX) - g(\nabla_{PX}PX, PX). \qquad (3.5)$$

By virtue of (3.2), the first term of (3.5) can be written as

$$g(P\nabla_{PX}X, PX) = -\cos^2\theta\{g(\nabla_{PX}X, X) - \eta(X)g(\nabla_{PX}X, \xi)\}. \qquad (3.6)$$

We simplify the third term of (3.5) as follows

$$\begin{aligned}g(\nabla_{PX}PX, PX) &= g(\tilde{\nabla}_{PX}PX, PX) = \frac{1}{2}PXg(PX, PX). \\ &= \frac{1}{2}PX[-\cos^2\theta\{(g(X,X)-\eta^2(X))\}] \\ &= -\frac{1}{2}\cos^2\theta[PXg(X,X) - P(X)(g(X,\xi)g(X,\xi))] \\ &= -\frac{1}{2}\cos^2\theta[PXg(X,X) - 2\eta(X)P(X)g(X,\xi)] \\ &= -\frac{1}{2}\cos^2\theta[2g(\tilde{\nabla}_{PX}X, X) - 2\eta(X)\{g(\tilde{\nabla}_{PX}X, \xi) + g(X, \tilde{\nabla}_{PX}\xi)\}].\end{aligned}$$

Using (2.5), (2.6), (3.6) and the fact that X and PX are orthogonal vector fields on M, we derive

$$\begin{aligned}g(\nabla_{PX}PX, PX) &= -\cos^2\theta[g(\nabla_{PX}X, X) - \eta(X)g(\nabla_{PX}X, \xi) \\ &\quad -\eta(X)g(X, PX - \eta(PX)\xi)] \\ &= -\cos^2\theta[g(\nabla_{PX}X, X) - \eta(X)g(\nabla_{PX}X, \xi)] \\ \rightarrow g(\nabla_{PX}PX, PX) &= g(P\nabla_{PX}X, PX).\end{aligned}$$

Using this fact in (3.5), we obtain

$$0 = g(A_{FX}PX, PX) = g(\sigma(PX, PX), FX).$$

As M is totally umbilical slant, then from (2.18) and (3.2), we obtain

$$0 = -\cos^2\theta\{\|X\|^2 - \eta^2(X)\}g(H, FX). \tag{3.7}$$

Thus from (3.7), we conclude that either $\theta = \frac{\pi}{2}$ that is M is anti-invariant which part (ii) or the vector field X is parallel to the structure vector field ξ, i.e., M is 1-dimensional submanifold which is fourth part of the theorem or $H \perp FX$, for all $X \in \Gamma(TM)$, i.e., $H \in \Gamma(\mu)$ which is the last part of the theorem or $H = 0$, i.e., M is totally geodesic which is (iii) or $FX = 0$, i.e., M is invariant which is part (i). This completes the proof of the theorem. □

Theorem 3.3. *Every totally umbilical proper slant submanifold of a para-Kenmotsu manifold is totally geodesic.*

Proof. Let M be a totally umbilical proper slant submanifold of a para-Kenmotsu manifold $\tilde{M}$, then for any $X, Y \in \Gamma(TM)$, we have

$$\tilde{\nabla}_X\phi Y - \phi\tilde{\nabla}_X Y = g(\phi X, Y)\xi - \eta(Y)\phi X.$$

Using equations (2.6) and (2.9), we get

$$\tilde{\nabla}_X PY + \tilde{\nabla}_X FY - \phi(\nabla_X Y + \sigma(X, Y)) = g(PX, Y)\xi - \eta(Y)PX - \eta(Y)FX.$$

Again from (2.6), (2.7) and (2.9), we obtain

$$\begin{aligned} g(PX, Y)\xi - \eta(Y)PX - \eta(Y)FX &= \nabla_X PY + \sigma(X, PY) - A_{FY}X \\ &\quad + \nabla^{\perp}_X FY - P\nabla_X Y - F\nabla_X Y - \phi\sigma(X, Y). \end{aligned}$$

As M is totally umbilical, then

$$\begin{aligned} g(PX, Y)\xi - \eta(Y)PX - \eta(Y)FX &= \nabla_X PY + g(X, PY)H - A_{FY}X + \nabla^{\perp}_X FY \\ &\quad - P\nabla_X Y - F\nabla_X Y - g(X, Y)\phi H. \end{aligned} \tag{3.8}$$

Taking the inner product with ϕH in (3.8) and from the fact that $H \in \Gamma(\mu)$, we obtain

$$g(\nabla^{\perp}_X FY, \phi H) = -g(X, Y)\|H\|^2.$$

Applying (2.7) and the property of Riemannian connection the above equation takes the form

$$g(FY, \nabla^{\perp}_X \phi H) = g(X, Y)\|H\|^2. \tag{3.9}$$

Now for any $X \in \Gamma(TM)$, we have

$$\tilde{\nabla}_X \phi H = (\tilde{\nabla}_X \phi)H + \phi \tilde{\nabla}_X H.$$

Using the fact $H \in \Gamma(\mu)$ and by virtue of (2.4), (2.7) and (2.9), we obtain

$$-A_{\phi H}X + \nabla^{\perp}_X \phi H = -PA_H X - FA_H X + \phi \nabla^{\perp}_X H. \tag{3.10}$$

Also for any $X \in \Gamma(TM)$, we have

$$g(\nabla^{\perp}_X H, FX) = g(\tilde{\nabla}_X H, FX) = -g(H, \tilde{\nabla}_X FX).$$

Using (2.9), we obtain

$$g(\nabla^{\perp}_X H, FX) = -g(H, \tilde{\nabla}_X \phi X) + g(H, \tilde{\nabla}_X PX).$$

Further from (2.6) and (2.12), we derive

$$g(\nabla^{\perp}_X H, FX) = -g(H, (\tilde{\nabla}_X \phi)X) - g(H, \phi \tilde{\nabla}_X X) + g(H, \sigma(X, PX)).$$

Using (2.4) and (2.18), the first and last term of right hand side of the above equation are identically zero and hence by (2.3), the second term gives

$$g(\nabla^{\perp}_X H, FX) = g(\phi H, \tilde{\nabla}_X X).$$

Again by using (2.6) and (2.18), we obtain

$$g(\nabla^{\perp}_X H, FX) = g(\phi H, H)\|X\|^2 = 0.$$

This means that

$$\nabla^{\perp}_X H \in \Gamma(\mu). \tag{3.11}$$

Now, taking the inner product in (3.10) with FY for any $Y \in \Gamma(TM)$, we get

$$g(\nabla^{\perp}_X \phi H, FY) = -g(FA_H X, FY) + g(\phi \nabla^{\perp}_X H, FY).$$

Using (3.11) and from (3.3) and (3.9), we obtain

$$g(X, Y)\|H\|^2 = \sin^2\theta\{g(A_H X, Y) - \eta(Y)g(A_H X, \xi)\}. \tag{3.12}$$

Hence by (2.8) and (2.18), the above equation reduces to

$$g(X, Y)\|H\|^2 = \sin^2\theta\{g(X, Y)\|H\|^2 - \eta(Y)g(\sigma(X, Y), H)\}. \tag{3.13}$$

Since for a para-Kenmotsu manifold $\tilde{M}$, $\sigma(X, \xi) = 0$ for any X tangent to $\tilde{M}$, thus we obtain

$$g(X, Y)\|H\|^2 = \sin^2\theta\{g(X, Y)\|H\|^2.$$

Therefore, the above equation can be written as

$$\cos^2\theta g(X, Y)\|H\|^2 = 0. \tag{3.14}$$

Since M is proper slant submanifold, thus from (3.14) we conclude that $H = 0$, i.e., M is totally geodesic in $\tilde{M}$. This completes the proof. $\square$

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