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The perimeter of a flattened ellipse can be estimated accurately even from Maclaurin's series

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ABSTRACT

For the perimeter $P(a,b)$ of an ellipse with the semi-axes $a \geq b \geq 0$ a sequence $Q_n(a,b)$ is constructed such that the relative error of the approximation $P(a,b) \approx Q_n(a,b)$ satisfies the following inequalities

$$0 \leq -\frac{P(a,b)-Q_n(a,b)}{P(a,b)} \leq \frac{(1-q^2)^{n+1}}{(2n+1)^2} \leq \frac{1}{(2n+1)^2}\, e^{-q^2(n+1)},$$

true for $n \in \mathbb{N}$ and $q = \frac{b}{a} \in [0,1]$.

RESUMEN

Para el perímetro $P(a,b)$ de una elipse con semiejes $a \geq b \geq 0$, se construye una sucesión $Q_n(a,b)$ tal que el error relativo de la aproximación $P(a,b) \approx Q_n(a,b)$ satisface las siguientes desigualdades

$$0 \leq -\frac{P(a,b)-Q_n(a,b)}{P(a,b)} \leq \frac{(1-q^2)^{n+1}}{(2n+1)^2} \leq \frac{1}{(2n+1)^2}\, e^{-q^2(n+1)},$$

válidas para $n \in \mathbb{N}$ y $q = \frac{b}{a} \in [0,1]$.

Keywords and Phrases: approximation, elementary, ellipse, estimate, Maclaurin series, mathematical validity, perimeter, simple.

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1 Introduction

Injective parametric equations of the border of an ellipse having semi-axes of lengths a and $b \leq a$ are given as $x = x(t) = a\cos(t)$, $y = y(t) = b\sin(t)$, where $t \in [0, 2\pi)$. Its perimeter $P(a,b)$ is determined as

$$\begin{aligned} P(a,b) &= \int_0^{2\pi} \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}\,\mathrm{d}t = 4\int_0^{\frac{\pi}{2}} \sqrt{a^2\sin^2(t) + b^2\cos^2(t)}\,\mathrm{d}t \\ &= 4a\int_0^{\frac{\pi}{2}} \sqrt{1-\epsilon^2\cos^2(t)}\,\mathrm{d}t \underbrace{=}_{t=\pi/2-\tau} 4a\int_{\frac{\pi}{2}}^{0} \sqrt{1-\epsilon^2\sin^2(\tau)}(-\,\mathrm{d}\tau). \end{aligned}$$

Thus, the perimeter $P(a,b)$ of an ellipse having semi-axes of lengths a and $b \leq a$, is given as

$$P(a,b) = 4a\,E(\epsilon), \tag{1.1}$$

where

$$E(\epsilon) := \int_0^{\frac{\pi}{2}} \sqrt{1-\epsilon^2\sin^2(\tau)}\,\mathrm{d}\tau \tag{1.2}$$

is complete elliptic integral of the second kind and

$$\epsilon := \sqrt{1-\left(\tfrac{b}{a}\right)^2} = \sqrt{\tfrac{a^2-b^2}{a^2}} \in [0,1), \tag{1.3}$$

is the eccentricity of an ellipse.

For $b \approx 0$ it is intuitively evident that $P(a,b) > 2 \times 2a = 4a$. Moreover, since the functions $\epsilon \mapsto 1-\epsilon^2\sin^2(\tau)$ are decreasing on the interval $[0,1]$ for any $\tau \in [0,\pi/2]$, the function $E(\epsilon)$ is decreasing too. Therefore, we have

$$1 = \int_0^{\frac{\pi}{2}} \cos(\tau)\,\mathrm{d}\tau = E(1) \leq E(\epsilon) \leq E(0) = \frac{\pi}{2},$$

for $0 \leq \epsilon \leq 1$. Consequently, due to (1.1),

$$\inf_{0<b\leq a} P(a,b) = 4a < P(a,b) \leq P(a,a) = 2a\pi. \tag{1.4}$$

The first exact formula for an ellipse perimeter was presented 277 years ago by Collin Maclaurin [24], given as the sum of infinite series:

$$\begin{aligned} P(a,b) &= 2\pi a \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k}^2 (1-2k)\,\epsilon^{2k} \\ &= 2\pi a \sum_{k=0}^{\infty} \left(\frac{(2k)!}{(2^k\,k!)^2}\right)^2 \frac{\left(-\,\epsilon^{2k}\right)}{2k-1} \\ &= 2\pi a \left\{1 - \sum_{k=0}^{\infty} \left[\frac{1}{4^k}\binom{2k}{k}\right]^2 \frac{\epsilon^{2k}}{2k-1}\right\}, \end{aligned} \tag{1.5}$$

valid for $0 \le \epsilon \le 1$, where $\epsilon = \left(1 - b^2/a^2\right)^{1/2}$, called the eccentricity[1] of an ellipse. This series originates from the integral (1.2). Later, Ivory [13] discovered a faster converging series for the integral (1.2), which was later significantly improved by Gauss and Kummer. Additionally, Gauss developed very efficient, swiftly convergent method of arithmetic-geometric means for computation of the integral (1.2), see [1]. Subsequently, a lot of approximations of the ellipse perimeter have been found. For example, among them is very popular Ramanujan's "extraordinarily unusual and exotic" approximation [2]. Motivated by the Barnard–Pearce–Schovanec approximations [3] and Villarino's contribution on the accuracy of a Ramanujan's approximation [29] and his paper [28], we shall derive elementarily[2] an asymptotic estimate of the ellipse perimeter, based on the oldest Maclaurin series expansion. The result obtained surpasses most of the previous approximations.

2 Background

2.1 The binomial approximation

Using Taylor's formula (see for example [15, p. 111] with $x_0 = 0$, $h = x$ and $p = n$),

$$f(x) = f(0) + \sum_{i=1}^{n} \frac{f^{(i)}(0)}{i!} x^i + \frac{x^{n+1}}{n!} \int_0^1 (1-t)^n \, f^{(n+1)}(t\,x) \, \mathrm{d}t \,,$$

(true for $a, b \in \mathbb{R}$, $a < b$, $n \in \mathbb{N}$, $x \in [a, b]$ and $f \in C^{n+1}[a, b]$) for the function $f(x) \equiv (1+x)^{\frac{1}{2}}$, we obtain[3]

$$\begin{aligned}(1+x)^{\frac{1}{2}} =& 1 + \sum_{i=1}^{n} \binom{\frac{1}{2}}{i} x^i \\ &+ x^{n+1} \int_0^1 (1-t)^n \binom{\frac{1}{2}}{n+1} (n+1)(1+tx)^{\frac{1}{2}-n-1} \, \mathrm{d}t \,, \end{aligned} \tag{2.1}$$

valid for $x \in (-1, 1]$ and $n \in \mathbb{N}$.

Introducing w_i, called the i-th Wallis ratio, for[4] $i \ge 0$,

$$w_i := \prod_{j=1}^{i} \frac{2j-1}{2j} = \frac{(2i)!}{4^i (i!)^2} = \frac{1}{4^i} \binom{2i}{i}, \tag{2.2}$$

[1] We have $\epsilon = \sqrt{1-q^2}$, where $q := b/a$ is called the aspect ratio of an ellipse.

[2] not using complex analysis and absolute and uniform convergence of a series, as was used, for example, in [18]

[3] considering the identity $f^{(i)}(x) \equiv \binom{\frac{1}{2}}{i} (i!) (1+x)^{\frac{1}{2}-i}$

[4] $\prod_{j=m}^{n} x_j := 1$, for $m > n$; consequently $w_0 = 1$

we obtain

$$\binom{\frac{1}{2}}{i} = \frac{\prod_{j=0}^{i-1}(\frac{1}{2}-j)}{i!} = (-1)^{i-1}\frac{1}{2^i}\cdot\frac{\prod_{j=1}^{i-1}(2j-1)}{\prod_{j=1}^{i} j}$$

$$= (-1)^{i-1}\frac{1}{2i-1}\prod_{j=1}^{i}\frac{2j-1}{2j} = \underline{\underline{(-1)^{i-1}\frac{w_i}{2i-1}}}\,. \tag{2.3}$$

Thus, thanks to (2.1), replacing x by $-x$, we get

$$(1-x)^{\frac{1}{2}} = 1 - \sum_{i=1}^{n}\frac{w_i}{2i-1}x^i + r_n(x), \tag{2.4}$$

with the remainder

$$r_n(x) = -x^{n+1}\frac{w_{n+1}}{2n+1}(n+1)\int_0^1\left(\frac{1-t}{1-tx}\right)^n\frac{\mathrm{d}t}{(1-tx)^{\frac{1}{2}}}\,,$$

estimated, for $x\in(0,1)$, as

$$0 < -r_n(x) = \frac{x^{n+1}}{(1-x)^{\frac{1}{2}}}\cdot\frac{w_{n+1}}{2n+1}(n+1)\int_0^1\left(\frac{1-t}{1-tx}\right)^n\mathrm{d}t$$

$$< \frac{w_{n+1}}{(1-x)^{\frac{3}{2}}(2n+1)}x^{n+1}\,. \tag{2.5}$$

Indeed, using the substitution $\tau = \frac{1-t}{1-tx}$, i.e. $t = \frac{1-\tau}{1-\tau x}$ we have (considering $x\in(0,1)$)

$$\int_0^1\left(\frac{1-t}{1-tx}\right)^n\mathrm{d}t = \int_1^0\tau^n\left(-\frac{1-x}{(1-\tau x)^2}\right)\mathrm{d}\tau = \int_0^1\tau^n\cdot\frac{1-x}{(1-\tau x)^2}\,\mathrm{d}\tau$$

$$< \int_0^1\tau^n\cdot\frac{1-x}{(1-x)^2}\,\mathrm{d}\tau = \frac{1}{(1-x)(n+1)}\,.$$

2.2 Wallis ratios estimates

The integrals

$$I_n := \int_0^{\frac{\pi}{2}}\sin^n(x)\,\mathrm{d}x \qquad (n\ge 0), \tag{2.6}$$

satisfy the recurrence relation

$$I_n = \frac{n-1}{n}\,I_{n-2}, \qquad \text{for } n\ge 2,$$

where, obviously, we have $I_0 = \frac{\pi}{2}$ and $I_1 = 1$. Consequently, by induction we find

$$I_{2i} = \prod_{j=1}^{i}\frac{2j-1}{2j}\cdot\frac{\pi}{2} \;=\; w_i\cdot\frac{\pi}{2} \tag{2.7}$$

and

$$I_{2i+1} = \prod_{j=1}^{i} \frac{2j}{2j+1} = \frac{1}{(2i+1)w_i}\,. \tag{2.8}$$

Obviously, we estimate

$$0 < \sin^{2i+2}(x) < \sin^{2i+1}(x) < \sin^{2i}(x) < 1,$$

for $x \in \left(0, \frac{\pi}{2}\right)$ and $i \in \mathbb{N}$. Integrating, we obtain

$$0 \;<\; I_{2i+2} \;<\; I_{2i+1} \;<\; I_{2i} \;<\; 1,$$

for all $i \in \mathbb{N}$. Hence, thanks to (2.7)–(2.8), we get

$$\frac{2i+1}{2i+2} w_i \cdot \frac{\pi}{2} = w_{i+1} \cdot \frac{\pi}{2} \;<\; \frac{1}{(2i+1)w_i} \;<\; w_i \cdot \frac{\pi}{2}\,.$$

Consequently,

$$\frac{2}{\pi} \cdot \frac{1}{2i+1} < w_i^2 < \frac{2}{\pi} \cdot \frac{1}{2i-1} \qquad (i \in \mathbb{N}). \tag{2.9}$$

We remark that there exists a huge literature on useful, more accurate estimates for w_n, e.g. [4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16, 17, 19, 20, 21, 22, 23, 25, 26, 27, 31]. However, for our needs, there suffice rather rough estimates (2.9).

2.3 Some logarithmic formula expansion

For $p \geq 1$ and $-1 < t < 1$ we have

$$\begin{aligned} 2\sum_{j=0}^{p-1} t^{2j} &= \sum_{k=0}^{2(p-1)} \left(t^k + (-t)^k\right) = \sum_{k=0}^{2(p-1)} t^k + \sum_{k=0}^{2(p-1)} (-t)^k \\ &= \frac{1-t^{2p-1}}{1-t} + \frac{1-(-t)^{2p-1}}{1+t}\,. \end{aligned}$$

Consequently, integrating from 0 to $x \in (-1, 1)$, the first and the last part of these equalities, we obtain

$$\begin{aligned} 2\sum_{j=0}^{p-1} \frac{x^{2j+1}}{2j+1} &= \int_0^x \frac{1}{1-t}\,\mathrm{d}t - \int_0^x \frac{t^{2p-1}}{1-t}\,\mathrm{d}t + \int_0^x \frac{1}{1+t}\,\mathrm{d}t + \int_0^x \frac{t^{2p-1}}{1+t}\,\mathrm{d}t \\ &= -\ln(1-x) + \ln(1+x) - \underbrace{\int_0^x \left(\frac{1}{1-t} - \frac{1}{1+t}\right) t^{2p-1}\,\mathrm{d}t}_{=R_p^*(x)}. \end{aligned}$$

Thus,

$$\ln\left(\frac{1+x}{1-x}\right) = 2\sum_{i=1}^{p} \frac{x^{2i-1}}{2i-1} + R_p^*(x), \tag{2.10}$$

with the remainder $R_p^*(x)$,

$$R_p^*(x) := \int_0^x \frac{2t^{2p}}{1-t^2}\,\mathrm{d}t \geq \int_0^x 2t^{2p}\,\mathrm{d}t. \qquad (0<x<1),$$

estimated as

$$\frac{2x^{2p+1}}{2p+1} < R_p^*(x) < \frac{2x^{2p+1}}{(1-x^2)(2p+1)} \qquad (p\in\mathbb{N},\ 0<x<1) \tag{2.11}$$

From (2.10)–(2.11) we end up with the expansion

$$\ln\left(\frac{1+x}{1-x}\right) = 2\sum_{i=1}^{\infty}\frac{x^{2i-1}}{2i-1}, \tag{2.12}$$

true for $x\in(0,1)$ and, consequently, also for $x\in(-1,0]$.

3 The Maclaurin series

3.1 Derivation

Referring to (2.4)–(2.5) and (2.6)–(2.7), we have, for any $n\in\mathbb{N}$,

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sqrt{1-\underbrace{\epsilon^2\sin^2(\tau)}}\,\mathrm{d}\tau &= \frac{\pi}{2} - \sum_{i=1}^{n}\frac{w_i\,\epsilon^{2i}}{2i-1}\int_0^{\frac{\pi}{2}}\sin^{2i}(\tau)\,\mathrm{d}\tau + r_n^*(\epsilon)\\ &= \frac{\pi}{2} - \sum_{i=1}^{n}\frac{w_i\,\epsilon^{2i}}{2i-1}\left(w_i\frac{\pi}{2}\right) + r_n^*(\epsilon).\end{aligned}$$

Hence

$$\int_0^{\frac{\pi}{2}} \sqrt{1-\epsilon^2\sin^2(\tau)}\,\mathrm{d}\tau = \frac{\pi}{2}\left(1-\sum_{i=1}^{n}\frac{w_i^2}{2i-1}\,\epsilon^{2i}\right) + r_n^*(\epsilon), \tag{3.1}$$

where w_i is the i-th Wallis' ratio and the error term $r_n^*(\epsilon) := \int_0^{\pi/2} r_n\big(\epsilon^2\sin^2(\tau)\big)\,\mathrm{d}\tau$ is estimated, due to (2.5) and considering (2.6)–(2.7), as

$$\begin{aligned}0 \leq -r_n^*(\epsilon) &\leq \frac{\epsilon^{2n+2}}{1-\epsilon^2}\cdot\frac{w_{n+1}}{2n+1}\int_0^{\frac{\pi}{2}}\sin^{2n+2}(\tau)\,\mathrm{d}\tau\\ &= \frac{\epsilon^{2n+2}\,w_{n+1}}{(1-\epsilon^2)(2n+1)}\cdot w_{n+1}\,\frac{\pi}{2}\,.\end{aligned}$$

Thus, according to (2.9),

$$0 \leq -r_n(\epsilon) \leq \frac{\pi}{2}\cdot\frac{1}{1-\epsilon^2}\cdot\frac{w_{n+1}^2}{2n+1}\,\epsilon^{2n+2} \leq \frac{1}{1-\epsilon^2}\cdot\frac{\epsilon^{2n+2}}{(2n+1)^2}\ . \tag{3.2}$$

This estimate is not usable for $\epsilon\approx 1$, i.e. for $b\approx 0$ (for a very flattened ellipse).

As $w_n^2 \leq 1$, we have $\lim\limits_{n\to\infty} r_n(\epsilon) = 0$ for any $\epsilon < 1$, which is always true for ordinary ellipse, due to the equivalence $\epsilon = 1 \Leftrightarrow b = 0$. Hence, there holds the so-called Maclaurin series expansion[5]

$$\int_0^{\frac{\pi}{2}} \sqrt{1-\epsilon^2\sin^2(\tau)}\,\mathrm{d}\tau = \frac{\pi}{2}\left(1-\sum_{i=1}^{\infty}\frac{w_i^2}{2i-1}\,\epsilon^{2i}\right), \tag{3.3}$$

valid for $0 \leq \epsilon < 1$. In addition, the series on the right is convergent also for $\epsilon = 1$ due to (2.9). Indeed, we have $\frac{w_i^2}{2i-1} < \frac{1}{i^2}$, which implies the convergence of the series $\sum_{i=1}^{\infty} \frac{w_i^2}{2i-1}$.

Remark 3.1. *About fifty years after Maclaurin's book [24], including the series* (3.3)*, Ivory published article [13], where he presented the expansion*

$$\int_0^{\frac{\pi}{2}} \sqrt{1-\epsilon^2\sin^2(\tau)}\,\mathrm{d}\tau = \frac{\pi(a+b)}{4a}\left(1+\sum_{i=1}^{\infty}\frac{w_i^2}{(2i-1)^2}\,\lambda^{2i}\right) \qquad \left(\lambda = \frac{a-b}{a+b}\right),$$

where the series on the right converges slightly faster than the series in (3.3)*.*

Applying (2.9) for the partial sums

$$\mu_n(\epsilon) := \sum_{i=1}^{n}\frac{w_i^2}{2i-1}\,\epsilon^{2i} \qquad (n \in \mathbb{N}\cup\{\infty\}), \tag{3.4}$$

we shall estimate the series $\mu_\infty(\epsilon)$ figuring in (3.3).

3.2 Approximating $\mu_\infty(\epsilon)$

Using (2.9) we estimate,

$$\frac{2}{\pi(2i-1)(2i+1)} < \frac{w_i^2}{2i-1} < \frac{2}{\pi(2i-1)^2} \qquad (i \in \mathbb{N}). \tag{3.5}$$

Therefore

$$\mu_\infty(\epsilon) \approx \sum_{i=1}^{\infty}\frac{2\epsilon^{2i}}{\pi(2i-1)(2i+1)} \quad (0 \leq \epsilon < 1).$$

This idea produces the next theorem.

Theorem 3.2. *We have*

$$\mu_\infty(\epsilon) = M_n(\epsilon) + \delta_n(\epsilon), \tag{3.6}$$

where

$$M_n(\epsilon) = A(\epsilon) + B_n(\epsilon), \tag{3.7}$$

$$A(\epsilon) := \frac{1}{2\pi}\left[\left(\epsilon - \frac{1}{\epsilon}\right)\ln\left(\frac{1+\epsilon}{1-\epsilon}\right) + 2\right] \in \left(0, \tfrac{1}{\pi}\right), \tag{3.8}$$

[5] The coefficients of the original Maclaurin series [24] have a visually more complicated form.

$$B_n(\epsilon) := \sum_{i=1}^{n} \left(w_i^2 - \frac{2}{\pi(2i+1)} \right) \frac{\epsilon^{2i}}{2i-1}\,, \tag{3.9}$$

and

$$0 < \delta_n(\epsilon) < \delta_n^*(\epsilon) := \frac{2\epsilon^{2n+2}}{\pi(2n+1)^2}\,, \tag{3.10}$$

valid for any integer $n \geq 1$ *and every* $0 < \epsilon < 1$.

The basic function $A(\epsilon)$ *is strictly increasing having the range* $\left(0, \frac{1}{\pi}\right)$, *where* $\lim\limits_{\epsilon \downarrow 0} A(\epsilon) = 0$ *and* $\lim\limits_{\epsilon \uparrow 1} A(\epsilon) = \frac{1}{\pi}$. *Both sequences,* $n \mapsto B_n(\epsilon)$ *and* $n \mapsto \delta_n(\epsilon)$, *are strictly increasing, for every* $\epsilon \in (0,1)$.

The sequence $n \mapsto M_n(\epsilon)$ *converges strictly increasingly to* $\mu_\infty(\epsilon)$, *for any* $\epsilon \in (0,1)$. *Additionally, for every* $n \in \mathbb{N}$, *the functions* $\epsilon \mapsto M_n(\epsilon)$ *and* $\epsilon \mapsto \delta_n(\epsilon)$ *are strictly increasing on the interval* $(0,1)$.

Figure 1 shows, on the left, the graph[6] of the basic function $A(\epsilon)$, and, on the right, the graphs of the functions $M_1^*(\epsilon)$ and $\mu_\infty(\epsilon)$. As an example, we present $B_4^*(\epsilon)$ and $\delta_4^*(\epsilon)$ as follows:

$$\begin{aligned} B_4^*(\epsilon) &= \left(\tfrac{1}{4} - \tfrac{2}{3\pi}\right)\epsilon^2 + \frac{1}{3}\left(\tfrac{9}{64} - \tfrac{2}{5\pi}\right)\epsilon^4 + \frac{1}{5}\left(\tfrac{25}{256} - \tfrac{2}{7\pi}\right)\epsilon^6 + \frac{1}{7}\left(\tfrac{1225}{16384} - \tfrac{2}{9\pi}\right)\epsilon^8 \\ &\approx 0.037\,793\,409\,\epsilon^2 + 0.004\,433\,682\,\epsilon^4 + 0.001\,342\,114\,\epsilon^6 + 0.000\,576\,077\,\epsilon^8, \\ \delta_4^*(\epsilon) &\leq \frac{2\epsilon^{10}}{81\pi} \leq 0.00786\,\epsilon^{10} \qquad \left(\epsilon = \sqrt{1 - \left(\tfrac{b}{a}\right)^2}\right). \end{aligned}$$

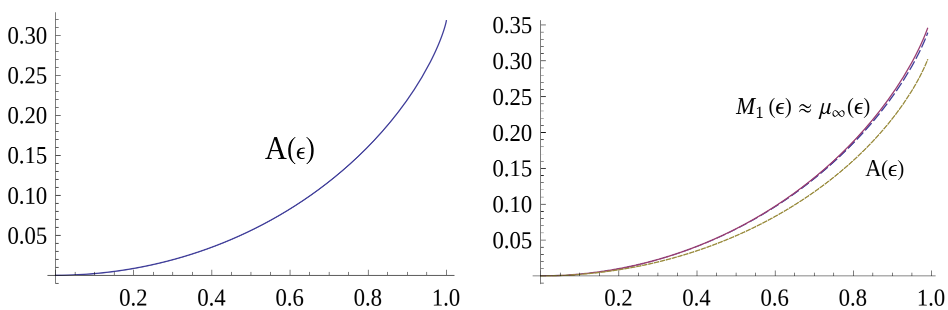

Figure 1: The graph of the basic function $A(\epsilon)$ (left) and the graphs of the functions $M_1(\epsilon)$, $\mu_\infty(\epsilon)$ and $A(\epsilon)$ (right).

Proof of Theorem 3.2. We have, for $0 < \epsilon < 1$,

$$\begin{aligned} \sum_{i=1}^{\infty} w_i^2 \frac{\epsilon^{2i}}{2i-1} &= \sum_{i=1}^{\infty} \frac{2\,\epsilon^{2i}}{\pi(2i-1)(2i+1)} \\ &\quad + \sum_{i=1}^{n} \left(\frac{w_i^2}{2i-1} - \frac{2}{\pi(2i-1)(2i+1)} \right) \epsilon^{2i} + \delta_n(\epsilon), \end{aligned} \tag{3.11}$$

[6]All the graphics and calculations in this paper are made using the Mathematica [30] computer system.

where

$$\delta_n(\epsilon) = \sum_{i=n+1}^{\infty} \left(w_i^2 - \frac{2}{\pi(2i+1)} \right) \frac{\epsilon^{2i}}{2i-1}. \tag{3.12}$$

Moreover, using (2.12), we have

$$\begin{aligned}
&\sum_{i=1}^{\infty} \frac{2}{\pi(2i-1)(2i+1)} \epsilon^{2i} \\
=&\frac{1}{\pi} \sum_{i=1}^{\infty} \left(\frac{1}{2i-1} - \frac{1}{2i+1} \right) \epsilon^{2i} \\
=&\frac{1}{\pi} \left(\frac{\epsilon}{2} \cdot 2 \sum_{i=1}^{\infty} \frac{\epsilon^{2i-1}}{2i-1} - \frac{1}{2\epsilon} \cdot 2 \sum_{i=1}^{\infty} \frac{\epsilon^{2i+1}}{2i+1} \right) \\
=&\frac{1}{\pi} \left[\frac{\epsilon}{2} \ln \left(\frac{1+\epsilon}{1-\epsilon} \right) - \frac{1}{2\epsilon} \left(\ln \left(\frac{1+\epsilon}{1-\epsilon} \right) - 2\epsilon \right) \right] \\
=&\frac{1}{2\pi} \left[\left(\epsilon - \frac{1}{\epsilon} \right) \ln \left(\frac{1+\epsilon}{1-\epsilon} \right) + 2 \right] = A(\epsilon).
\end{aligned}$$

Concerning $A(\epsilon) = \frac{1}{2\pi}\big(f(\epsilon)+2\big)$, the function $f(\epsilon) := \big(\epsilon - \frac{1}{\epsilon}\big) \ln \Big(\frac{1+\epsilon}{1-\epsilon}\Big)$ $(0 < \epsilon < 1)$ has the derivative $f'(\epsilon) = g(\epsilon)/\epsilon^2$, where $g(\epsilon) = (1+\epsilon^2) \ln \Big(\frac{1+\epsilon}{1-\epsilon}\Big) - 2\epsilon$, having the derivative

$$g'(\epsilon) = \frac{2\epsilon}{1-\epsilon^2} \left(2\epsilon + (1-\epsilon^2) \ln \left(\frac{1+\epsilon}{1-\epsilon} \right) \right) \ > \ 0 \qquad (0 < \epsilon < 1).$$

Thus, g is strictly increasing on $[0,1)$. Consequently, $g(\epsilon) > g(0) = 0$, i.e. $f'(\epsilon) > 0$, for $0 < \epsilon < 1$. Therefore, f is strictly increasing on $(0,1)$ too. Moreover, using (2.10)–(2.11) with $p = 1$, we have

$$f(\epsilon) = \tfrac{\epsilon^2-1}{\epsilon} \cdot 2 \left(\epsilon + \vartheta \cdot \tfrac{2\epsilon^3}{3(1-\epsilon^2)} \right) = 2(\epsilon^2 - 1) \left(1 + \vartheta \cdot \tfrac{2}{1-\epsilon^2} \cdot \tfrac{\epsilon^2}{3} \right),$$

for some $\vartheta = \vartheta(\epsilon) \in (0,1)$. Hence, $\lim\limits_{\epsilon\downarrow 0} f(\epsilon) = -2$, i.e. $\lim\limits_{\epsilon\downarrow 0} A(\epsilon) = \lim\limits_{\epsilon\uparrow 1} \frac{1}{2\pi}\big(f(\epsilon)+2\big) = 0$. In addition, $\lim\limits_{\epsilon\uparrow 1} f(\epsilon) = \lim\limits_{\epsilon\uparrow 1} \Big[\frac{\epsilon^2-1}{\epsilon} \cdot 2\ln(1+\epsilon)\Big] - \frac{1}{1} \cdot \lim\limits_{h\downarrow 0}\Big(-h\ln(h)\Big) = 0$, where $h = 1 - \epsilon^2$. Thus, $\lim\limits_{\epsilon\uparrow 1} A(\epsilon) = \lim\limits_{\epsilon\uparrow 1} \frac{1}{2\pi}\big(f(\epsilon)+2\big) = \frac{1}{\pi}$.

According to (3.5), all summands in $B_n(\epsilon)$ and $\delta_n(\epsilon)$ (see (3.12)) are positive, i.e. the sequences $n \mapsto B_n(\epsilon)$ and $n \mapsto \delta_n(\epsilon)$ are strictly increasing; consequently the sequence $n \mapsto M_n(\epsilon)$ is also strictly increasing, for every $\epsilon \in (0,1)$.

Since all coefficients of the power series $B_n(\epsilon)$ and $\delta_n(\epsilon)$ (see (3.9) and (3.12)) are positive, due to (3.5), the functions $\epsilon \mapsto M_n(\epsilon)$ and $\epsilon \mapsto \delta_n(\epsilon)$ are strictly increasing on the interval $(0,1)$, for any $n \in \mathbb{N}$.

According to (3.12) and (3.5), we estimate, for $\epsilon \in (0,1]$,

$$0 < \delta_n(\epsilon) < \sum_{i=n+1}^{\infty} \left(\frac{2}{\pi(2i-1)} - \frac{2}{\pi(2i+1)} \right) \frac{\epsilon^{2n+2}}{2n+1} = \frac{2\epsilon^{2n+2}}{\pi(2n+1)^2},$$

using the telescoping method of summation. $\square$

Example 3.3. *Theorem 3.2 is quite useful for an estimate of $\mu_\infty(\epsilon)$, consequently for an estimate of the perimeter of an ellipse. For example, for a very flattened ellipse with $q = 0.01$ we have $0.99994 < \epsilon(q) < 0.99995$ where $0.36315 < M_{20}(0.99995) < 0.36316\ldots$ and $\delta^*_{20}(0.99995) < 0.00038$. Therefore, $0.36315 < \mu_\infty(0.99995) < 0.36316 + 0.00038 = 0.36354$. Thus, to three decimal places, we have $\mu_\infty(0.99995) = 0.363\ldots$. Consequently, the perimeter $P(a,b)$ of the corresponding ellipse is given as $P(a,b) = 4a\cdot\frac{\pi}{2}\big(1-\mu_\infty(0.99995)\big) \approx 4a\cdot\frac{\pi}{2}\big(1-0.363\big) \approx 4.002\,a$ (compare with relations (1.4)).*

Remark 3.4. *Referring to Abel's theorem on the boundary behavior of a power series, if we continuously extend the domain of the function $A(\epsilon)$ to the closed interval $[0,1]$ by using limits, $A(0) := 0$ and $A(1) := \frac{1}{\pi}$, then (3.6), (3.7), (3.9) and (3.10) are all valid also for $\epsilon = 0$ and $\epsilon = 1$.*

Remark 3.5. *Alternatively, we can estimate the remainder $r^{**}_n(\epsilon) := \mu_\infty(\epsilon) - M_n(\epsilon)$ as follows:*

$$\begin{aligned} r^{**}_n(\epsilon) &\le \sum_{i=n+1}^{\infty} \frac{w_i^2\,\epsilon^{2i}}{2i-1} \le \frac{w_{n+1}^2\epsilon^{2n+2}}{2n+1}\sum_{j=0}^{\infty}\epsilon^{2j} \\ &= \frac{w_{n+1}^2\epsilon^{2n+2}}{(2n+1)(1-\epsilon^2)} \le \frac{1}{1-\epsilon^2}\cdot\frac{2\epsilon^{2n+2}}{\pi(2n+1)^2}\,. \end{aligned}$$

This simple method works quite well for ϵ, which "differs enough from 1", but it is useless for ϵ, which is close to 1.

4 The main result

Theorem 4.1. *For every $n \in \mathbb{N}$, the perimeter $P(a,b)$ of an ellipse having semi-major and semi-minor axes, a and b, the aspect ratio $q = q(a,b) := b/a$, and the eccentricity $\epsilon = \epsilon(a,b) := \sqrt{1-q^2}$, the n-th approximation $Q_n(a,b) \approx P(a,b)$,*

$$Q_n(a,b) := 2\pi a\Big(1 - M_n\big(\epsilon\big)\Big) = 2\pi a\Big(1 - A(\epsilon) - B_n\big(\epsilon\big)\Big), \tag{4.1}$$

has the relative error,

$$\frac{P(a,b) - Q_n(a,b)}{P(a,b)} =: \rho_n(q) \qquad \Big(q = q(a,b) = \big(\tfrac{b}{a}\big)^2\Big)\ ,$$

estimated as

$$-\frac{1}{(2n+1)^2}\,e^{-q^2(n+1)} \le -\frac{\big(1-q^2\big)^{n+1}}{(2n+1)^2} =: \rho^*_n(q) \le \rho_n(q) \le 0\,.$$

Here, $A(\epsilon)$ and $B_n\big(\epsilon\big)$ are defined in Theorem 3.2 and we have $B_{n+1}\big(\epsilon\big) = B_n\big(\epsilon\big) + \Big(w_{n+1}^2 - \frac{2}{\pi(2n+3)}\Big)\,\frac{\epsilon^{2n+2}}{2n+1}$, for $n \in \mathbb{N}$ and $0 \le \epsilon \le 1$.

Proof. Thanks to (1.1), (1.2), (1.4) and (3.3), we estimate

$$-\frac{P(a,b)-Q_n(a,b)}{P(a,b)} = -\frac{2\pi a\Big(1-M_n(\epsilon)-\delta_n(\epsilon)\Big)-2\pi a\Big(1-M_n(\epsilon)\Big)}{P(a,b)}$$
$$\overset{(1.4)}{<} \frac{2\pi a\,\delta_n(\epsilon)}{4a} \leq \frac{\pi\,\delta_n(\epsilon)}{2} < \frac{\pi}{2}\cdot\frac{2\epsilon^{2n+2}}{\pi(2n+1)^2} = \frac{\epsilon^{2n+2}}{(2n+1)^2},$$

where, considering the convexity of the exponential function or, referring to [16, (6a)] with $\varepsilon = q^2$ and $h = -q^2$, we have

$$\epsilon^{2n+2} = (1-q^2)^{n+1} \leq e^{-q^2(n+1)} \qquad (0 \leq q < 1). \qquad \square$$

Figures 2–3 show, for several values of n, the graphs of actual relative errors $-\rho_n(q) = \big[\mu_\infty\big(\epsilon(q)\big) - M_n\big(\epsilon(q)\big)\big]/\big[1-\mu_\infty\big(\epsilon(q)\big)\big]$ (left) together with their upper bounds $-\rho_n^*(q)$ (right).

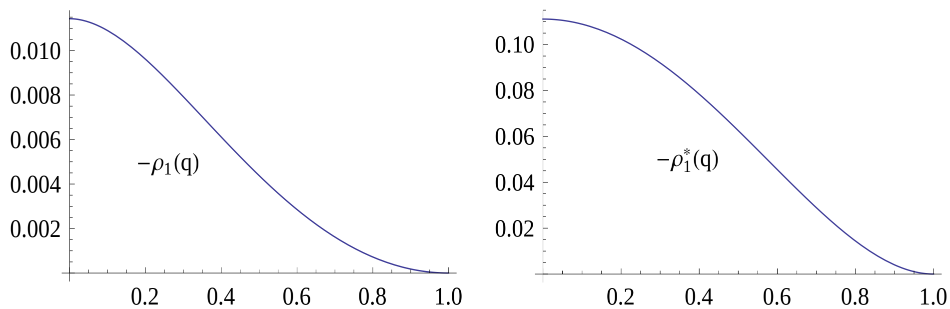

Figure 2: The graphs of the functions $q \mapsto -\rho_1(q)$ and $q \mapsto -\rho_1^*(q)$.

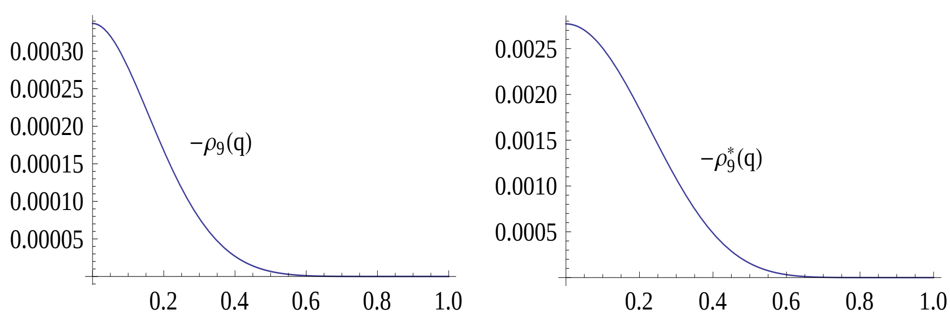

Figure 3: The graphs of the functions $q \mapsto -\rho_9(q)$ and $q \mapsto -\rho_9^*(q)$.

Table 1 additionally confirms the usefulness of the derived formula.

Conclusion. The article demonstrates that with the help of 277 years old Maclaurin series the perimeter of an ellipse can be accurately estimated, even if an ellipse flattens into a line segment. This is done only by elementary means, not using complex analysis or elliptical integral theory, neither arithmetic-geometric means nor hypergeometric functions.

q	0.00001	0.1	0.2	0.3	0,4	0,5
$-\rho_{20}(q)$	$< 8{\cdot}10^{-5}$	$< 6{\cdot}10^{-5}$	$< 2{\cdot}10^{-5}$	$< 5{\cdot}10^{-6}$	$< 6{\cdot}10^{-7}$	$< 4{\cdot}10^{-8}$
$-\rho^*_{20}(q)$	$< 6{\cdot}10^{-4}$	$< 5{\cdot}10^{-4}$	$< 3{\cdot}10^{-4}$	$< 9{\cdot}10^{-5}$	$< 2{\cdot}10^{-5}$	$< 2{\cdot}10^{-6}$

Table 1: The actual error $\rho_{20}(q)$ and the a priori estimated error $\rho^*_{20}(q)$.

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