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Generalized trace pseudo-spectrum of matrix pencils

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ABSTRACT

The objective of the study was to investigate a new notion of generalized trace pseudo-spectrum for an matrix pencils. In particular, we prove many new interesting properties of the generalized trace pseudo-spectrum. In addition, we show an analogue of the spectral mapping theorem for the generalized trace pseudo-spectrum in the matrix algebra.

RESUMEN

El objetivo de este estudio es investigar una nueva noción de pseudo-espectro traza generalizado para pinceles de matrices. En particular, demostramos variadas propiedades nuevas e interesantes del pseudo-espectro traza generalizado. Adicionalmente, mostramos un análogo del teorema espectral de aplicaciones para el pseudo-espectro traza generalizado en el álgebra de matrices.

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1 Introduction

Let $\mathcal{M}_n(\mathbb{C})$ ($\mathcal{M}_n(\mathbb{R})$) denote the algebra of all $n \times n$ complex (real) matrices, $\mathcal{I}$ denotes the $n \times n$ identity matrix and the conjugate transpose of $\mathcal{U}$ is denoted by $\mathcal{U}^*$. We denote by Tr, (resp. Det) the trace (resp. determinant) map on $\mathcal{M}_n(\mathbb{C})$. In the present paper, we study the problem of finding the eigenvalues of the generalized eigenvalue problem

$$\mathcal{U}x = \lambda\mathcal{V}x.$$

Next, let $\lambda \in \mathbb{C}$ and

$$s_n(\lambda\mathcal{V}-\mathcal{U}) \leq \ldots \leq s_2(\lambda\mathcal{V}-\mathcal{U}) \leq s_1(\lambda\mathcal{V}-\mathcal{U})$$

be the singular values of the matrix pencils $\lambda\mathcal{V}-\mathcal{U}$ where $s_1(\lambda\mathcal{V}-\mathcal{U})$ is the smallest and $s_n(\lambda\mathcal{V}-\mathcal{U})$ is largest singular values of the matrix pencil. Let $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$, then the set of all eigenvalues of the matrix pencils of the form $\lambda\mathcal{V}-\mathcal{U}$ is denoted by $\sigma(\mathcal{U}, \mathcal{V})$ and is defined as

$$\sigma(\mathcal{U}, \mathcal{V}) = \Big\{\lambda \in \mathbb{C} : \ \lambda\mathcal{V}-\mathcal{U} \text{ is not invertible}\Big\},$$

and its spectral radius by

$$r(\mathcal{U}, \mathcal{V}) = \sup\Big\{|\lambda| : \lambda \in \sigma(\mathcal{U}, \mathcal{V})\Big\}.$$

For an $n\times n$ complex matrices $\mathcal{U}$ and $\mathcal{V}$ and a non-negative real number ε, the pseudo-spectrum of the matrix pencils of the form $\lambda\mathcal{V}-\mathcal{U}$ is defined as the following closed set in the complex plane

$$\sigma_\varepsilon(\mathcal{U}, \mathcal{V}) = \Big\{\lambda \in \mathbb{C} : \ s_n(\lambda\mathcal{V}-\mathcal{U}) \leq \varepsilon\Big\}.$$

Let $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ and $0 < \varepsilon < 1$. The condition pseudo-spectrum of the matrix pencils $\lambda\mathcal{V}-\mathcal{U}$ is denoted by $\Sigma_\varepsilon(\mathcal{U}, \mathcal{V})$ and is defined as

$$\Sigma_\varepsilon(\mathcal{U}, \mathcal{V}) = \Big\{\lambda \in \mathbb{C} : s_n(\lambda\mathcal{V}-\mathcal{U}) \leq \varepsilon \ s_1(\lambda\mathcal{V}-\mathcal{U})\Big\}.$$

Let ε be a small positive number. For an operator $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$, recall that the determinant spectrum of matrix pencils of the form $\lambda\mathcal{V}-\mathcal{U}$ is the set $d_\varepsilon(\mathcal{U}, \mathcal{V})$ and is defined as

$$d_\varepsilon(\mathcal{U}, \mathcal{V}) = \Big\{\lambda \in \mathbb{C} : |\det(\lambda\mathcal{V}-\mathcal{U})| \leq \varepsilon\Big\}.$$

The analysis of eigenvalues and eigenvectors has had a great effect on mathematics, science, engineering, and many other fields. Then, there are countless applications for this type of analysis. The study of matrix pencils is by now a very thoughtful subject, with the notion of pseudospectrum playing a key role in the theory. However, matrix pencils play an important role in numerical linear algebra, perturbation theory, generalized eigenvalue problems. In this paper, we interest by a generalization of eigenvalues called generalized trace pseudo-spectrum for an element in the matrix algebra to give more information about the matrix pencils of the form $\lambda\mathcal{V}-\mathcal{U}$. For more information on various details on the above concepts, properties and applications of pseudo-spectrum [2, 3, 6, 7, 9], condition spectrum [1, 4, 5] and determinant spectrum [8]. Now, we introduce the new concept of the generalized trace pseudo-spectrum in the following definition.

Definition 1.1. *For $\varepsilon > 0$, the generalized trace pseudo-spectrum of the matrix pencils of the form $\lambda\mathcal{V} - \mathcal{U} \in \mathcal{M}_n(\mathbb{C})$ is denoted by $\mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$ and is defined as*

$$\mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}) = \sigma(\mathcal{U}, \mathcal{V}) \bigcup \Big\{\lambda \in \mathbb{C} : |\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U})| \leq \varepsilon\Big\}.$$

The generalized trace pseudoresolvent of the matrix pencils of the form $\lambda\mathcal{V} - \mathcal{U}$ is denoted by $\mathrm{Tr}\rho_\varepsilon(\mathcal{U}, \mathcal{V})$ and is defined as

$$\mathrm{Tr}\rho_\varepsilon(\mathcal{U}, \mathcal{V}) = \rho(\mathcal{U}, \mathcal{V}) \bigcap \Big\{\lambda \in \mathbb{C} : |\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U})| > \varepsilon\Big\}.$$

The singular values of a the matrix pencil are important not only for their role in diagonalization but also for their utility in a variety of applications. Since $\mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$ use all the singular values of $\lambda\mathcal{V} - \mathcal{U}$ to get defined, it is expected to give more information about $\mathcal{U}, \mathcal{V}$ than pseudo-spectrum and condition spectrum. Since the definition use idea of "Trace" the generalization of eigenvalues defined above is named as generalized trace pseudo-spectrum. It is easily seen that the map

$$\mathcal{U} \to \mathrm{Tr}(\mathcal{U})$$

is continuous linear functional. Here, some important properties of the trace of $\mathcal{U}, \mathcal{B} \in \mathcal{M}_n(\mathbb{C})$ are

$$\mathrm{Tr}(\mathcal{U}\mathcal{B}) = \mathrm{Tr}(\mathcal{B}\mathcal{U}),$$

$$\mathrm{Tr}(\alpha\mathcal{U}) = \alpha\mathrm{Tr}(\mathcal{U}) \quad \text{with } \alpha \in \mathbb{C},$$

$$\mathrm{Tr}(\mathcal{U} + \mathcal{B}) = \mathrm{Tr}(\mathcal{U}) + \mathrm{Tr}(\mathcal{B}).$$

An outline of this paper is the following. In Section 2, we focuses on a new description of the generalized trace pseudo-spectra. Not only do we give a characterization of the generalized trace pseudo-spectrum in the matrix algebra. but also we investigate the connection between generalized trace pseudo-spectrum and algebraic multiplicity of the eigenvalues. In Section 3, we give an analogue of the spectral mapping theorem for the generalized trace pseudo-spectrum in the matrix algebra.

2 Generalized trace pseudo-spectrum.

In this section, some relevant properties of the generalized trace pseudo-spectrum are discussed in detail. For $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ and $\varepsilon > 0$, the generalized trace pseudo-spectrum of the matrix pencils of the form $\lambda\mathcal{V} - \mathcal{U}$ is denoted by $\mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$ and is defined as

$$\mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}) = \sigma(\mathcal{U}, \mathcal{V}) \bigcup \big\{\lambda \in \mathbb{C} : |\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U})| \leq \varepsilon\big\}.$$

The generalized trace pseudo-resolvent of the matrix pencils of the form $\lambda\mathcal{V}-\mathcal{U}$ is denoted by $\mathrm{Tr}\rho_{\varepsilon}(\mathcal{U},\mathcal{V})$ and is defined as

$$\mathrm{Tr}\rho_{\varepsilon}(\mathcal{U},\mathcal{V})=\rho(\mathcal{U},\mathcal{V})\bigcap\{\lambda\in\mathbb{C}:|\mathrm{Tr}(\lambda\mathcal{V}-\mathcal{U})|>\varepsilon\}$$

while the generalized trace pseudo-spectral radius of the matrix pencils of the form $\lambda\mathcal{V}-\mathcal{U}$ is defined as

$$\mathrm{Trr}_{\varepsilon}(\mathcal{U},\mathcal{V}):=\sup\Big\{|\lambda|:\lambda\in\mathrm{Tr}_{\varepsilon}(\mathcal{U},\mathcal{V})\Big\}.$$

Remark 2.1. *Let $\mathcal{U},\mathcal{V}\in\mathcal{M}_n(\mathbb{C})$. Then, if $\mathcal{V}$ is nonsingular, then it is possible to reduce the generalized trace pseudo-spectrum to a standard trace pseudo-spectrum for the matrices $\mathcal{V}^{-1}\mathcal{U}$ or $\mathcal{U}\mathcal{V}^{-1}$. i.e.*

$$\mathrm{Tr}_{\varepsilon}(\mathcal{U},\mathcal{V})=\sigma(\mathcal{V}^{-1}\mathcal{U},\mathcal{I})\bigcup\Big\{\lambda\in\mathbb{C}:|\mathrm{Tr}(\lambda-\mathcal{V}^{-1}\mathcal{U})|\le\varepsilon\Big\},$$

or

$$\mathrm{Tr}_{\varepsilon}(\mathcal{U},\mathcal{V})=\sigma(\mathcal{U}\mathcal{V}^{-1},\mathcal{I})\bigcup\Big\{\lambda\in\mathbb{C}:|\mathrm{Tr}(\lambda-\mathcal{U}\mathcal{V}^{-1})|\le\varepsilon\Big\}.$$

The following theorem gives some properties of the generalized trace pseudo-spectrum that follow in a straightforward manner from the definition of the generalized trace pseudo-spectrum.

Theorem 2.1. *Let $\mathcal{U},\mathcal{V}\in\mathcal{M}_n(\mathbb{C})$ and $\varepsilon>0$. Then,*

(i) $\mathrm{Tr}_0(\mathcal{U},\mathcal{V})=\bigcap\limits_{\varepsilon>0}\mathrm{Tr}_{\varepsilon}(\mathcal{U},\mathcal{V})$.

(ii) *If* $0<\varepsilon_1<\varepsilon_2$, *then* $\mathrm{Tr}_{\varepsilon_1}(\mathcal{U},\mathcal{V})\subset\mathrm{Tr}_{\varepsilon_2}(\mathcal{U},\mathcal{V})$.

(iii) $\mathrm{Tr}_{\varepsilon}(\mathcal{U},\mathcal{V})$ *is a non-empty compact subset of* $\mathbb{C}$.

(iv) *If* $\alpha\in\mathbb{C}$ *and* $\beta\in\mathbb{C}\backslash\{0\}$, *then* $\mathrm{Tr}_{\varepsilon}(\beta\mathcal{U}+\alpha\mathcal{V},\mathcal{V})=\beta\mathrm{Tr}_{\frac{\varepsilon}{|\beta|}}(\mathcal{U},\mathcal{V})+\alpha$.

(v) $\mathrm{Tr}_{\varepsilon}(\alpha\mathcal{V},\mathcal{V})=\left\{\lambda\in\mathbb{C}:|\lambda-\alpha|\le\dfrac{\varepsilon}{|\mathrm{Tr}(\mathcal{V})|}\right\}$ *for all* $\lambda,\alpha\in\mathbb{C}$.

Proof. The proofs of items (i) and (ii) are clear from the definition of generalized trace pseudo-spectrum.

(iii) Using the continuity from $\mathbb{C}$ to $[0,\infty[$ of the map

$$\lambda\longrightarrow|\mathrm{Tr}(\lambda\mathcal{V}-\mathcal{U})|,$$

we get that $\mathrm{Tr}_{\varepsilon}(\mathcal{U},\mathcal{V})$ is a compact set in the complex plane containing the eigenvalues of the matrix pencils $\lambda\mathcal{V}-\mathcal{U}$.

(iv) In fact, it is well know

$$\begin{aligned}
\mathrm{Tr}_{\varepsilon}(\beta\mathcal{U}+\alpha\mathcal{V},\mathcal{V}) &= \Big\{\lambda\in\mathbb{C} : |\mathrm{Tr}(\lambda\mathcal{V}-\beta\mathcal{U}-\alpha\mathcal{V})|\leq\varepsilon\Big\}\\
&= \Big\{\lambda\in\mathbb{C} : |\beta|\Big|\mathrm{Tr}\Big(\frac{\lambda-\alpha}{\beta}\mathcal{V}-\mathcal{U}\Big)\Big|\leq\varepsilon\Big\}\\
&= \Big\{\lambda\in\mathbb{C} : \Big|\mathrm{Tr}\Big(\frac{\lambda-\alpha}{\beta}\mathcal{V}-\mathcal{U}\Big)\Big|\leq\frac{\varepsilon}{|\beta|}\Big\}.
\end{aligned}$$

Then, $\lambda\in\mathrm{Tr}_{\varepsilon}(\beta\mathcal{U}+\alpha\mathcal{V},\mathcal{V})$. Thus, $\dfrac{\lambda-\alpha}{\beta}\in\mathrm{Tr}_{\frac{\varepsilon}{|\beta|}}(\mathcal{U},\mathcal{V})$. Hence, $\lambda\in\beta\mathrm{Tr}_{\frac{\varepsilon}{|\beta|}}(\mathcal{U},\mathcal{V})+\alpha$.

(v) Let $\lambda\in\mathrm{Tr}_{\varepsilon}(\alpha\mathcal{V},\mathcal{V})$, then

$$|\mathrm{Tr}(\lambda\mathcal{V}-\alpha\mathcal{V})| = |\lambda-\alpha||\mathrm{Tr}(\mathcal{V})|\leq\varepsilon.$$

This means that $\mathrm{Tr}_{\varepsilon}(\alpha\mathcal{V},\mathcal{V})=\Big\{\lambda\in\mathbb{C} : |\lambda-\alpha|\leq\dfrac{\varepsilon}{|\mathrm{Tr}(\mathcal{V})|}\Big\}$ for all $\lambda,\alpha\in\mathbb{C}$. Q.E.D.

Theorem 2.2. *Let $\mathcal{U},\mathcal{V}\in\mathcal{M}_n(\mathbb{C})$ and $\varepsilon>0$. Then,*

(i) *If $\mathcal{U}=\mathcal{Z}\mathcal{B}\mathcal{Z}^{-1}$ and $\mathcal{Z}\mathcal{V}=\mathcal{V}\mathcal{Z}$ for all nonsingular matrix $\mathcal{Z}\in\mathcal{M}_n(\mathbb{C})$ we have,*

$$\mathrm{Tr}_{\varepsilon}(\mathcal{U},\mathcal{V})=\mathrm{Tr}_{\varepsilon}(\mathcal{B},\mathcal{V}).$$

(ii) *If $\mathcal{U}=\mathcal{Z}\mathcal{B}\mathcal{Z}^{-1}$ and $\mathcal{V}=\mathcal{Z}\mathcal{K}\mathcal{Z}^{-1}$ for all nonsingular matrix $\mathcal{Z}\in\mathcal{M}_n(\mathbb{C})$ we have,*

$$\mathrm{Tr}_{\varepsilon}(\mathcal{U},\mathcal{V})=\mathrm{Tr}_{\varepsilon}(\mathcal{B},\mathcal{K}).$$

(iii) *The map $T\to\mathrm{Tr}_{\varepsilon}(\mathcal{U},\mathcal{V})$ is an upper semi continuous function from $\mathcal{M}_n(\mathbb{C})$ to compact subsets of $\mathbb{C}$.* ♢

Proof. (i) Let $\lambda\in\mathrm{Tr}_{\varepsilon}(\mathcal{B},\mathcal{V})$, then

$$\begin{aligned}
|\mathrm{Tr}(\lambda\mathcal{V}-\mathcal{B})| &= |\mathrm{Tr}(\lambda\mathcal{V}-\mathcal{Z}^{-1}\mathcal{U}\mathcal{Z})|,\\
&= |\mathrm{Tr}(\lambda\mathcal{Z}^{-1}\mathcal{Z}\mathcal{V}-\mathcal{Z}^{-1}\mathcal{U}\mathcal{Z})|\\
&= |\mathrm{Tr}(\mathcal{Z}^{-1}(\lambda\mathcal{Z}\mathcal{V}-\mathcal{U}\mathcal{Z})|\\
&= |\mathrm{Tr}(\mathcal{Z}^{-1}(\lambda\mathcal{V}-\mathcal{U})\mathcal{Z}|=|\mathrm{Tr}(\lambda\mathcal{V}-\mathcal{U})|\leq\varepsilon.
\end{aligned}$$

It follows that, $\lambda\in\mathrm{Tr}_{\varepsilon}(\mathcal{U},\mathcal{V})$.

The proofs of items (ii) and (iii) follows immediately from Definition 1.1. Q.E.D.

The following example shows that the converse of the assertion (i) is not true.

Example 2.1. *Let* $\mathcal{U} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $\mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $\mathcal{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. *Then,* $\mathcal{U}$ *and* $\mathcal{B}$ *are not similar and for* $\varepsilon > 0$, *we have*

$$\mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V}) = \mathrm{Tr}_\varepsilon(\mathcal{B},\mathcal{V}) = \{\lambda \in \mathbb{C} : |\lambda - 2| \le \varepsilon\}.$$

In the following, we obtain additional results on $\mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V})$ that are useful in our analysis.

Theorem 2.3. *Let* $\mathcal{U},\mathcal{V} \in \mathcal{M}_n(\mathbb{C})$, $\lambda \in \mathbb{C}$, *and* $\varepsilon > 0$. *Then, there is* $\mathcal{D} \in \mathcal{M}_n(\mathbb{C})$ *such that* $|\mathrm{Tr}(\mathcal{D})| \le \varepsilon$ *and* $\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U} - \mathcal{D}) = 0$ *if, and only if,* $\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V})$. ♢

Proof. To see this, we suppose that there exists $\mathcal{D} \in \mathcal{M}_n(\mathbb{C})$ such that $|\mathrm{Tr}(\mathcal{D})| \le \varepsilon$ and

$$\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U} - \mathcal{D}) = 0.$$

Then,

$$|\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U})| = |\mathrm{Tr}(\mathcal{D})| \le \varepsilon.$$

Thus, $\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V})$. Conversely, let $\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V})$. Then, we will discuss these two cases:

$\underline{1^{st}\ case}$: If $\lambda \in \mathrm{Tr}_0(\mathcal{U},\mathcal{V})$, then it is sufficient to take $(\mathcal{D} = 0_{n\times n})$.

$\underline{2^{nd}\ case}$: $\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V}) \backslash \mathrm{Tr}_0(\mathcal{U},\mathcal{V})$. Then,

$$|\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U})| \le \varepsilon.$$

Now, we consider

$$\mathcal{D} = \frac{\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U})}{n}\,\mathcal{I}.$$

It is easy to verify that, $\mathcal{D} \in \mathcal{M}_n(\mathbb{C})$ and

$$|\mathrm{Tr}(\mathcal{D})| = \left|\mathrm{Tr}\left(\frac{\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U})}{n}\,\mathcal{I}\right)\right| = \frac{|\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U})|}{n}\,\mathrm{Tr}(\mathcal{I}) \le \varepsilon.$$

Also, we have

$$\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U} - \mathcal{D}) = \mathrm{Tr}\left(\lambda\mathcal{V} - \mathcal{U} - \frac{\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U})}{n}\,\mathcal{I}\right) = 0.$$

Q.E.D.

Theorem 2.4. *Let* $\mathcal{U},\mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ *and* $\varepsilon > 0$. *Then,*

$$\mathrm{Tr}_\delta(\mathcal{U},\mathcal{V}) + \Theta_\varepsilon \subseteq \mathrm{Tr}_{\varepsilon+\delta}(\mathcal{U},\mathcal{V}), \tag{1}$$

holds for $\delta, \varepsilon > 0$ *with* Θ_ε, *denoting the closed disk in the complex plane centered at the origin with radius* $\frac{\varepsilon}{|\mathrm{Tr}(\mathcal{V})|}$. *If we take* $\delta = 0$, *we obtain an inner bound for* $\mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V})$, *namely*

$$\mathrm{Tr}_0(\mathcal{U},\mathcal{V}) + \Theta_\varepsilon \subseteq \mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V}). \tag{2}$$

Proof. Let $\lambda \in \mathrm{Tr}_\delta(\mathcal{U},\mathcal{V}) + \Theta_\varepsilon$. Then, there exists there exists $\lambda_1 \in \mathrm{Tr}_\delta(\mathcal{U},\mathcal{V})$ and $\lambda_2 \in \Theta_\varepsilon$ such that $\lambda = \lambda_1 + \lambda_2$. Therefore,

$$|\mathrm{Tr}(\lambda_1\mathcal{V} - \mathcal{U})| \leq \delta$$

and

$$\begin{aligned} |\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U})| &= |\mathrm{Tr}((\lambda_1 + \lambda_2)\mathcal{V} - \mathcal{U})| \\ &= |\mathrm{Tr}(\lambda_2\mathcal{V}) + \mathrm{Tr}(\lambda_1\mathcal{V} - \mathcal{U})| \\ &\leq |\lambda_2||\mathrm{Tr}(\mathcal{V})| + |\mathrm{Tr}(\lambda_1\mathcal{V} - \mathcal{U})| \\ &\leq |\mathrm{Tr}(\mathcal{V})||\lambda_2| + |\mathrm{Tr}(\lambda_1\mathcal{V} - \mathcal{U})| \leq \varepsilon + \delta, \end{aligned}$$

so that (1) holds. Finally, let $\delta = 0$, then the desired inclusion (2) is obtained. Q.E.D.

Theorem 2.5. *Let $\mathcal{U},\mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ such that $\mathcal{U}\mathcal{B} = \mathcal{B}\mathcal{U}$ and $\varepsilon > 0$. If $\mathcal{U}$ is normal, then*

$$\mathrm{Tr}_\varepsilon(\mathcal{U} + \mathcal{B},\mathcal{V}) \subseteq \sigma(\mathcal{U},\mathcal{V}) + \mathrm{Tr}_\varepsilon(\mathcal{B},\mathcal{V}).$$

Proof. We assume that $\mathcal{U}$ is normal, so there exists a unitary matrix $\mathcal{Z} \in \mathcal{M}_n(\mathbb{C})$ such that

$$\mathcal{Z}^*\mathcal{U}\mathcal{Z} = \lambda_1\mathcal{I}_{n_1} \oplus \lambda_2\mathcal{I}_{n_2} \oplus \ldots \oplus \lambda_k\mathcal{I}_{n_k}.$$

The condition $\mathcal{U}\mathcal{B} = \mathcal{B}\mathcal{U}$ implies that

$$\mathcal{Z}^*\mathcal{B}\mathcal{Z} = \mathcal{U}_1 \oplus \mathcal{U}_2 \ldots \oplus \mathcal{U}_k$$

where, $\mathcal{U}_i \in \mathcal{M}_{n_k}(\mathbb{C})$, $i = 1, \ldots, k$. Then,

$$\begin{aligned} \mathrm{Tr}_\varepsilon(\mathcal{U} + \mathcal{B},\mathcal{V}) &= \mathrm{Tr}_\varepsilon(\mathcal{Z}^*\mathcal{U}\mathcal{Z} + \mathcal{Z}^*\mathcal{B}\mathcal{Z},\mathcal{V}) \\ &= \mathrm{Tr}_\varepsilon((\lambda_1\mathcal{I}_{n_1} + \mathcal{U}_1) \oplus \ldots \oplus (\lambda_k\mathcal{I}_{n_k} + \mathcal{U}_k),\mathcal{V}) \\ &= \bigcup_{i=1}^{k} \mathrm{Tr}_\varepsilon(\lambda_i\mathcal{I}_{n_i} + \mathcal{U}_i,\mathcal{V}) \\ &= \bigcup_{i=1}^{k} \lambda_i + \mathrm{Tr}_\varepsilon(\mathcal{U}_i,\mathcal{V}) \\ &\subseteq \sigma(\mathcal{U},\mathcal{V}) + \mathrm{Tr}_\varepsilon(\mathcal{B},\mathcal{V}). \end{aligned}$$

The proof is thus complete. Q.E.D.

Remark 2.2. *Let $\mathcal{U},\mathcal{B}$ and $\mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ and $\varepsilon > 0$. Then, using Theorem 2.5, we obtain the following inequality,*

$$\mathrm{Trr}_\varepsilon(\mathcal{U} + \mathcal{B},\mathcal{V}) \subseteq r(\mathcal{U},\mathcal{V}) + \mathrm{Trr}_\varepsilon(\mathcal{B},\mathcal{V}).$$

$\diamondsuit$

Theorem 2.6. *Let* $\mathcal{U}, \mathcal{B}$ *and* $\mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ *and* $\varepsilon > 0$. *Then,*

(i) $\mathrm{Tr}_\varepsilon(\mathcal{U}\mathcal{B}, \mathcal{V}) = \mathrm{Tr}_\varepsilon(\mathcal{B}\mathcal{U}, \mathcal{V})$.

(ii) $\mathrm{Tr}_{\frac{\varepsilon}{2}}(\mathcal{U}, \mathcal{V}) + \mathrm{Tr}_{\frac{\varepsilon}{2}}(\mathcal{B}, \mathcal{V}) \subseteq \mathrm{Tr}_\varepsilon(\mathcal{U} + \mathcal{B}, \mathcal{V})$.

Proof. (i) Let $\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U}\mathcal{B}, \mathcal{V})$, then

$$\begin{aligned} \varepsilon \geq |\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U}\mathcal{B})| &= |\mathrm{Tr}(\lambda\mathcal{V}) + \mathrm{Tr}(-\mathcal{U}\mathcal{B})| \\ &= |\mathrm{Tr}(\lambda\mathcal{V}) + \mathrm{Tr}(-\mathcal{B}\mathcal{U})| \\ &= |\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{B}\mathcal{U})|. \end{aligned}$$

Hence, $\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{B}\mathcal{U}, \mathcal{V})$. Thus,

$$\mathrm{Tr}_\varepsilon(\mathcal{U}\mathcal{B}, \mathcal{V}) \subseteq \mathrm{Tr}_\varepsilon(\mathcal{B}\mathcal{U}, \mathcal{V}).$$

The conclusion can be obtained similarly to the first inclusion, then we deduce that

$$\mathrm{Tr}_\varepsilon(\mathcal{B}\mathcal{U}, \mathcal{V}) = \mathrm{Tr}_\varepsilon(\mathcal{U}\mathcal{B}, \mathcal{V}).$$

(ii) Let $\lambda \in \mathrm{Tr}_{\frac{\varepsilon}{2}}(\mathcal{U}, \mathcal{V}) + \mathrm{Tr}_{\frac{\varepsilon}{2}}(\mathcal{B}, \mathcal{V})$. Then, there exists

$$\lambda_1 \in \mathrm{Tr}_{\frac{\varepsilon}{2}}(\mathcal{U}, \mathcal{V}) \text{ and } \lambda_1 \in \mathrm{Tr}_{\frac{\varepsilon}{2}}(\mathcal{B}, \mathcal{V})$$

such that $\lambda = \lambda_1 + \lambda_2$. Therefore,

$$\mathrm{Tr}(\lambda_1\mathcal{V} - \mathcal{U}) \leq \frac{\varepsilon}{2} \quad \text{and} \quad \mathrm{Tr}(\lambda_2\mathcal{V} - \mathcal{B}) \leq \frac{\varepsilon}{2}.$$

On the other hand,

$$\begin{aligned} |\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U} - \mathcal{B})| &= |\mathrm{Tr}(\lambda_1\mathcal{V} - \mathcal{U} + \lambda_2\mathcal{V} - \mathcal{B})| \\ &\leq |\mathrm{Tr}(\lambda_1\mathcal{V} - \mathcal{U})| + |\mathrm{Tr}(\lambda_2\mathcal{V} - \mathcal{B})| \\ &\leq \varepsilon \end{aligned}$$

Then, $\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U} + \mathcal{B}, \mathcal{V})$. Q.E.D.

Theorem 2.7. *Let* $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ *and* $\mathcal{N} \in \mathcal{M}_n(\mathbb{C})$ *is a nilpotent matrix and* $\varepsilon > 0$. *Then,*

$$\mathrm{Tr}_\varepsilon(\mathcal{U} + \mathcal{N}, \mathcal{V}) = \mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}).$$

◊

Proof. " $\subseteq$ " Let $\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U} + \mathcal{N}, \mathcal{V})$, then $|\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U} - \mathcal{N})| \leq \varepsilon$. Since

$$|\mathrm{Tr}(\lambda\mathcal{V} - \mathcal{U}) - \mathrm{Tr}(\mathcal{N})| \leq \varepsilon.$$

Using the fact that the matrix trace vanishes on nilpotent matrices, therefore

$$\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}).$$

Hence,

$$\mathrm{Tr}_\varepsilon(\mathcal{U}+\mathcal{N}, \mathcal{V}) \subseteq \mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}).$$

" $\supseteq$ " Let $\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$, then $|\mathrm{Tr}(\lambda\mathcal{V}-\mathcal{U})| \leq \varepsilon$. Now, we can write for any $\lambda \in \mathbb{C}$

$$|\mathrm{Tr}(\lambda\mathcal{V}-\mathcal{U})| = |\mathrm{Tr}(\lambda\mathcal{V}-\mathcal{U}-\mathcal{N}+\mathcal{N})| = |\mathrm{Tr}(\lambda\mathcal{V}-\mathcal{U}-\mathcal{N}) + \mathrm{Tr}(\mathcal{N})|.$$

Because, $\mathrm{Tr}(\mathcal{N}) = 0$, it follows that $|\mathrm{Tr}(\lambda\mathcal{V}-\mathcal{U}-\mathcal{N})| \leq \varepsilon$. Consequently,

$$\mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}) \subseteq \mathrm{Tr}_\varepsilon(\mathcal{U}+\mathcal{N}, \mathcal{V}).$$

Q.E.D.

3 Trace pseudospectral mapping Theorem

Let $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ and f be an analytic function defined on D, an open set containing $\mathrm{Tr}_0(\mathcal{U}, \mathcal{V})$. For each $\varepsilon > 0$, we define

$$\varphi(\varepsilon) = \sup_{\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V})} |\mathrm{Tr}\big(f(\lambda)\mathcal{V} - f(\mathcal{U})\big)|$$

and suppose there exists $\varepsilon_0 > 0$ such that $\mathrm{Tr}_{\varepsilon_0}(f(\mathcal{U}), \mathcal{V}) \subseteq f(D)$. Then, for $0 < \varepsilon < \varepsilon_0$ we define

$$\phi(\varepsilon) = \sup_{\mu \in f^{-1}(\mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V})) \cap D} |\mathrm{Tr}(\mu\mathcal{V}-\mathcal{U})|.$$

Lemma 3.1. *Let $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ and $\varepsilon > 0$, then $\varphi(\varepsilon)$ and $\phi(\varepsilon)$ are well defined, $\lim\limits_{\varepsilon\to 0} \varphi(\varepsilon) = 0$ and $\lim\limits_{\varepsilon\to 0} \phi(\varepsilon) = 0$.*

Proof. In the order to prove that $\varphi(\varepsilon)$ is well defined, we define $h : \mathbb{C} \to \mathbb{R}_+$

$$h(\lambda) = |\mathrm{Tr}\big(f(\lambda)\mathcal{V} - f(\mathcal{U})\big)|$$

Since $h(\lambda)$ is continuous and $\mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$ is a compact subset of $\mathbb{C}$, then it is clear that

$$\varphi(\varepsilon) = \sup\big\{h(\lambda) : \lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})\big\}.$$

We conclude, $\varphi(\varepsilon)$ is well defined. Now, let assume that there exists $\varepsilon_0 > 0$ such that

$$\mathrm{Tr}_{\varepsilon_0}(f(\mathcal{U}), \mathcal{V}) \subseteq f(D).$$

We show that for $0 < \varepsilon < \varepsilon_0$, $\phi(\varepsilon)$ is well defined. Define $g : \mathbb{C} \to \mathbb{R}_+$,

$$g(\mu) = |\mathrm{Tr}(\mu\mathcal{V} - \mathcal{U})|.$$

Since g is continuous for all $\mu \in \mathbb{C}$, then $\phi(\varepsilon)$ is well defined. It is also clear that $\varphi(\varepsilon)$ and $\phi(\varepsilon)$ are a monotonically non-decreasing function, $\varphi(\varepsilon)$ and $\phi(\varepsilon)$ goes to zero as ε goes to zero. Q.E.D.

Theorem 3.1. *Let $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ and let f be an analytic function defined on D, an open set containing $\mathrm{Tr}_0(\mathcal{U}, \mathcal{V})$. Then, for each*

$$f(\mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})) \subseteq \mathrm{Tr}_{\varphi(\varepsilon)}(f(\mathcal{U}), \mathcal{V}),$$

where $\varphi(\varepsilon)$ defined above.

Proof. Let $\lambda \in \mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})$. Then, using Lemma 3.1 we obtain that $\varphi(\varepsilon)$ is well defined and $\lim\limits_{\varepsilon \to 0} \varphi(\varepsilon) = 0$. Therefore, $h(\lambda) \leq \varphi(\varepsilon)$. Hence

$$|\mathrm{Tr}\big(f(\lambda)\mathcal{V} - f(\mathcal{U})\big)| := h(\lambda) \leq \varphi(\varepsilon).$$

Thus, $f(\lambda) \in \mathrm{Tr}_{\varphi(\varepsilon)}(f(\mathcal{U}), \mathcal{V})$. This means that

$$f(\mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V})) \subseteq \mathrm{Tr}_{\varphi(\varepsilon)}(f(\mathcal{U}), \mathcal{V}).$$

Q.E.D.

Theorem 3.2. *Let $\mathcal{U}, \mathcal{V} \in \mathcal{M}_n(\mathbb{C})$ and let f be an analytic function defined on D, an open set containing $\mathrm{Tr}_0(\mathcal{U}, \mathcal{V})$. Then, for each*

$$\mathrm{Tr}_\varepsilon(f(\mathcal{U}), \mathcal{V}) \subseteq f(\mathrm{Tr}_{\phi(\varepsilon)}(\mathcal{U}, \mathcal{V})).$$

where $\phi(\varepsilon)$ defined above.

Proof. Let $\lambda \in \mathrm{Tr}_\varepsilon(f(\mathcal{U}), \mathcal{V})$. Then, using Lemma 3.1 we obtain the existence of $\varepsilon_0 > 0$ such that

$$\mathrm{Tr}_\varepsilon(f(\mathcal{U}), \mathcal{V}) \subseteq \mathrm{Tr}_{\varepsilon_0}(f(\mathcal{U}), \mathcal{V}) \subseteq f(D).$$

Consider $\mu \in D$ such that $\lambda = f(\mu)$. Then $\mu \in f^{-1}(\mathrm{Tr}_\varepsilon(\mathcal{U}, \mathcal{V}))$, hence

$$g(\mu) \leq \phi(\varepsilon).$$

Therefore,

$$|\mathrm{Tr}\big(\mu\mathcal{V} - \mathcal{U}\big)| := g(\mu) \leq \phi(\varepsilon)$$

Thus, $\mu \in \mathrm{Tr}_{\phi(\varepsilon)}(\mathcal{U}, \mathcal{V})$. Then, $\lambda = f(\mu) \in f(\mathrm{Tr}_{\phi(\varepsilon)}(\mathcal{U}, \mathcal{V}))$. This means that

$$\mathrm{Tr}_\varepsilon(f(\mathcal{U}), \mathcal{V}) \subseteq f(\mathrm{Tr}_{\phi(\varepsilon)}(\mathcal{U}, \mathcal{V})).$$

Q.E.D.

Corollary 3.1. *Combining the two inclusions in Theorems 3.1 and 3.2, we get*

$$f(\mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V})) \subseteq \mathrm{Tr}_{\varphi(\varepsilon)}(f(\mathcal{U}),\mathcal{V}) \subseteq f(\mathrm{Tr}_{\phi(\varphi(\varepsilon))}(\mathcal{U},\mathcal{V})$$

and

$$\mathrm{Tr}_\varepsilon(f(\mathcal{U}),\mathcal{V}) \subseteq f(\mathrm{Tr}_{\phi(\varepsilon)}(\mathcal{U},\mathcal{V})) \subseteq \mathrm{Tr}_{\varphi(\phi(\varepsilon))}(f(\mathcal{U}),\mathcal{V}).$$

Here are some remarks.

Remark 3.1. (i) *It will be clear from the proofs of Theorems 3.1 and 3.2 that the the functions φ and ϕ measure the sizes of the trace pseudo-spectra are optimal.*

(ii) *From the definitions of φ and ϕ, the set inclusions are sharp in the sense that the functions cannot be replaced by smaller functions.*

(iii) *In general, the spectral mapping theorem is not true for generalized trace pseudo-spectrum.*

Example 3.1. *Let $\alpha,\beta \in \mathbb{C}$ with $\alpha \neq \beta \neq 0$ and let $\mathcal{U} = \begin{pmatrix} \alpha & 1 \\ 0 & \beta \end{pmatrix}$, $\mathcal{V} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and $f(\lambda) = \lambda^2$. Then $f(\mathcal{U}) = \begin{pmatrix} \alpha^2 & \alpha+\beta \\ 0 & \beta^2 \end{pmatrix}$. A direct computation shows that*

$$\mathrm{Tr}_\varepsilon(f(\mathcal{U}),\mathcal{V}) = \left\{\lambda \in \mathbb{C} : |2\lambda - \alpha^2| \leq \varepsilon - \beta^2\right\},$$

$$f(\mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V})) = \left\{\lambda^2 \in \mathbb{C} : |2\lambda - \alpha^2| \leq \varepsilon - \beta^2\right\}.$$

We can see for all $\varepsilon > 0$ that $\mathrm{Tr}_\varepsilon(f(\mathcal{U}),\mathcal{V}) \neq f(\mathrm{Tr}_\varepsilon(\mathcal{U},\mathcal{V}))$.

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