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The K-theory ranks for crossed products of C^*-algebras by the group of integers

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ABSTRACT

We study the K-theory ranks for crossed products of C^*-algebras by the group of integers. As an application, we obtain certain estimates for the K-theory ranks of the group C^*-algebras of torsion free, finitely generated, nilpotent or solvable discrete groups, written as successive semi-direct products.

RESUMEN

Estudiamos los rangos de K-teoría para productos cruzados de C^*-álgebras por el grupo de los enteros. Como aplicación, obtenemos ciertas estimaciones para los rangos de K-teoría de las C^*-álgebras de grupos libres de torsión, finitamente generados, nilpotentes o solubles, escritos como productos semidirectos sucesivos.

Keywords and Phrases: K-theory, C*-algebra, crossed product, Betti number, discrete group.

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1 Introduction

In this paper we study the (free or $\mathbb{Z}$) ranks of the K-theory groups for crossed products of C^*-algebras by $\mathbb{Z}$ the group of integers. Such C^*-algebras and their K-theory play fundamental roles in the theory of C^*-algebras and K-theory (cf. Blackadar [1], Pedersen [2], Tomiyama [10], Wegge-Olsen [11]). By using the Pimsner-Voiculescu six-term exact sequence (PV) of the K-theory groups of the crossed product C^*-algebra $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ of a C^*-algebra $\mathfrak{A}$ by an action α of $\mathbb{Z}$ by automorphisms (Pimsner and Voiculescu [3], cf. [1]), in Section 2 we estimate the K-theory group ranks of $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ in terms of those of $\mathfrak{A}$. This simple result should be new in some insight and interesting in some sense, as another introductory step in this developed research area. As an easy, direct application of PV, in Section 3 we obtain certain estimates for the K-theory ranks of the group C^*-algebras of torsion free, finitely generated, nilpotent or solvable discrete groups, written as successive semi-direct products by torsion free, abelian groups. There may be more other applications left to be considered, but not so many probably. May as well refer to [5], [6], [7], [8], [9] for some related details. In particular, in [5], [7], and [9], the K-theory groups of the C^*-algebras of the generalized Heisenberg discrete nilpotent groups as typical examples of non-type I discrete amenable groups are computed by some methods of determining K-theory class generators as projections or unitaries, of the K-theory groups, but it seems that still, the K-theory groups of the C^*-algebras of general (torsion free, finitely generated) nilpotent (or solvable) discrete groups are not yet done completely, because of some difficulties involving successive unknown group actions. However, this time, without determining their K-theory groups as groups, the K-theory group rank estimates are obtained by us in such a way mentioned above, as the motivated examples, as given in Section 3.

2 The K-theory ranks for crossed product C^*-algebras by $\mathbb{Z}$

Let $\mathfrak{A}$ be a C^*-algebra. We denote by $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ the crossed product C^*-algebra of $\mathfrak{A}$ by an action α of $\mathbb{Z}$ on $\mathfrak{A}$ by automorphisms, where $\alpha_n = \alpha^n = \alpha \circ \cdots \circ \alpha$ as the n-fold composition of $\alpha = \alpha_1 : \mathfrak{A} \to \mathfrak{A}$ for $n \in \mathbb{Z}$ (cf. Blackadar [1], Pedersen [2], Tomiyama [10]). There is the following **Pimsner-Voiculescu** six-term exact sequence of the K-theory abelian groups (K_0 additive and K_1 multiplicative) (Pimsener and Voiculescu [3], cf. [1]):

$$\begin{tikzcd}
K_0(\mathfrak{A}) \arrow[r, "(\mathrm{id}-\alpha)_*"] & K_0(\mathfrak{A}) \arrow[r, "i_*"] & K_0(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \arrow[d, "\exp"', "\partial"] \\
K_1(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \arrow[u, "\partial", "\mathrm{ind}"'] & K_1(\mathfrak{A}) \arrow[l, "i_*"'] & K_1(\mathfrak{A}), \arrow[l, "(\mathrm{id}-\alpha)_*"']
\end{tikzcd}$$

where $\mathrm{id} : \mathfrak{A} \to \mathfrak{A}$ is the identity map and $i : \mathfrak{A} \to \mathfrak{A} \rtimes_\alpha \mathbb{Z}$ is the canonical inclusion map and the K-theory group maps $(\mathrm{id} - \alpha)_*$ and i_* are induced by $\mathrm{id} - \alpha$ and i, respectively, and the upward and downward arrows as the boundary maps ∂ are the index map as ind and the exponential map as exp, respectively.

It follows from exactness of the PV diagram above that

Lemma 2.1. *For any* C^**-algebra* $\mathfrak{A}$ *and any* $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$*, we have the following short exact sequences: for* $j = 0, 1$*,*

$$\begin{aligned} 0 \longrightarrow K_j(\mathfrak{A})/(\mathrm{id} - \alpha)_* K_j(\mathfrak{A}) = K_j(\mathfrak{A})/\ker(i_*) \\ \xrightarrow{i_*} K_j(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \xrightarrow{\partial} \mathrm{im}(\partial) = \ker(\mathrm{id} - \alpha)_* \to 0 \end{aligned}$$

with $(\mathrm{id}-\alpha)_* K_j(\mathfrak{A}) = \ker(i_*) \subset K_j(\mathfrak{A})$*, where* $(\mathrm{id}-\alpha)_* K_j(\mathfrak{A})$ *is the image of* $K_j(\mathfrak{A})$ *under* $(\mathrm{id}-\alpha)_*$ *and* $\ker(\mathrm{id} - \alpha)_*$ *is the kernel of* $(\mathrm{id} - \alpha)_*$ *on* K_0 *or* K_1*, and* $\mathrm{im}(\partial)$ *is the image of the boundary map* ∂ *equal to* exp *or* ind*.*

Let G be an abelian group. We denote by $\mathrm{rank}_\mathbb{Z}\, G$ the $\mathbb{Z}$-rank (or free rank) of G, which is also called the **Betti** number of G, denoted as $b(G)$. For a C^*-algebra $\mathfrak{A}$, set $b_j(\mathfrak{A}) = b(K_j(\mathfrak{A}))$ for $j = 0, 1$, each of which we call the j-th **Betti** number of $\mathfrak{A}$ (cf. [6]). We denote by $t(G)$ the **torsion** rank of G, which is defined to be the number of direct sum components of indecomposable, finite cyclic groups in G. Set $t_j(\mathfrak{A}) = t(K_j(\mathfrak{A}))$ for $j = 0, 1$, each of which we may call the j-th **torsion** rank of $\mathfrak{A}$.

Recall as a fundament fact in group theory that a finitely generated abelian group H has the following direct product decomposition:

$$H \cong \mathbb{Z}^{b(H)} \times \mathbb{Z}_{p_1^{n_1}} \times \cdots \mathbb{Z}_{p_{t(H)}^{n_{t(H)}}},$$

where $p_1, \cdots p_{t(H)}$ are primes and $n_1, \cdots, n_{t(H)}$ are some positive integers and each $\mathbb{Z}_{p_j^{n_j}} = \mathbb{Z}/p_j^{n_j}\mathbb{Z}$ for $1 \le j \le t(H)$ is the finite cyclic group of order $p_j^{n_j}$, that is indecomposable, and these powers of primes are distinct.

Lemma 2.2. *For a short exact sequence* $1 \to H \to G \to G/H \to 1$ *of finitely generated, abelian groups, we have* $b(H) \le b(G)$ *and* $b(G/H) \le b(G)$ *and* $b(G) = b(H) + b(G/H)$*.*

Proof. Note that there is no homomorphism from a finite torsion group to a torsion free group. Hence $b(H) \le b(G)$, and $b(G/H) = b(G) - b(H) \le b(G)$. □

Proposition 1. *For any* $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$*, we have that for* $j = 0, 1$*,*

$$b_j(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \le b_0(\mathfrak{A}) + b_1(\mathfrak{A})$$

and $b(K_j(\mathfrak{A})/(\mathrm{id} - \alpha)_* K_j(\mathfrak{A})) \le b_j(\mathfrak{A} \rtimes_\alpha \mathbb{Z})$*.*

Proof. By using the Lemmas 2.1 and 2.2 above, we obtain

$$\begin{aligned} b_j(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) &= b_j(K_j(\mathfrak{A})/\ker(i_*)) + b_{j+1}(\ker(\mathrm{id}-\alpha)_*) \\ &\leq b_j(K_j(\mathfrak{A})) + b_{j+1}(K_{j+1}(\mathfrak{A})) \end{aligned}$$

for $j = 0, 1$ and $j + 1 \pmod 2$, and $b_j(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \geq b_j(K_j(\mathfrak{A})/\ker(i_*))$. □

Let G be an abelian group. Let G_f and G_t denote the free and torsion parts of G respectively, so that $G \cong G_f \times G_t$ with $b(G) = b(G_f)$ and $t(G) = t(G_t)$.

Lemma 2.3. *Let G be a finitely generated, abelian group and H a subgroup. Then there is the following short exact sequence of groups, preserving the free and torsion parts of H and G :*

$$0 \to H = H_f \times H_t \to G = G_f \times G_t \to G/H = (G_f/H_f) \times (G_t/H_t) \to 0$$

with $G_t \cong H_t \times (G_t/H_t)$ and $(G_f/H_f)_t \times (G_t/H_t) \cong (G/H)_t$ and $(G_f/H_f)_f = (G/H)_f$. It then follows that

$$t(H) \leq t(G) \leq t(H) + t(G/H).$$

Proof. Note that there are injective maps from $\mathbb{Z}$ to $\mathbb{Z}$ and from $\mathbb{Z}^k$ to $\mathbb{Z}^l$ with $k \leq l$, but there is no injective map from $\mathbb{Z}$ to a finite cyclic group. It follows that an injective map from H to G preserves their free and torsion parts. Note also that G_t/H_t is a torsion group, but G_f/H_f may have its free part $(G_f/H_f)_f$ and torsion part $(G_f/H_f)_t$. □

Remark. The inequality $t(G/H) \leq t(G)$ does not hold in general. For instance, there is a quotient map from $G = \mathbb{Z}$ to $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, with $H = 2\mathbb{Z}$, so that $t(H) = t(G) = 0 < 1 = t(G/H) = t(H) + t(G/H)$.

Proposition 2. *It then follows that for $j = 0, 1 \in \mathbb{Z}_2$,*

$$t_j(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \leq t(K_j(\mathfrak{A})/(\mathrm{id}-\alpha)_* K_j(\mathfrak{A})) + t(\ker(\mathrm{id}-\alpha)_*)$$

with $\ker(\mathrm{id}-\alpha)_* \subset K_{j+1}(\mathfrak{A})$ *as a subgroup, and*

$$t(K_j(\mathfrak{A})/(\mathrm{id}-\alpha)_* K_j(\mathfrak{A})) \leq t_j(\mathfrak{A} \rtimes_\alpha \mathbb{Z}).$$

Remark. Let $\mathfrak{A}$ be a C^*-algebra. Set $\chi(\mathfrak{A}) = b_0(\mathfrak{A}) - b_1(\mathfrak{A})$, which is called the **Euler** characteristic of $\mathfrak{A}$, where we assume that it is defined to be an integer or $\pm\infty$ (or formally $\infty - \infty$). If $\chi(\mathfrak{A})$ and $\chi(\mathfrak{A} \rtimes_\alpha \mathbb{Z})$ are finite, then it holds that $\chi(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) = 0$ by using the PV diagram (see [6] or [8]).

Let $\mathfrak{A}$ be a C^*-algebra. We denote by $\mathfrak{A} \rtimes_{\alpha(1)} \mathbb{Z} \cdots \rtimes_{\alpha(n)} \mathbb{Z}$ the n-fold successive crossed product C^*-algebra of $\mathfrak{A}$ by successive actions $\alpha(j)$ of $\mathbb{Z}$ $(1 \leq j \leq n)$. It then follows that

Theorem 2.1. *For such an* n*-fold successive crossed product* C^**-algebra of a* C^**-algebra* $\mathfrak{A}$ *by* n *successive actions of* $\mathbb{Z}$ *as above or below, we have*

$$b_j(\mathfrak{A} \rtimes_{\alpha(1)} \mathbb{Z} \cdots \rtimes_{\alpha(n)} \mathbb{Z}) \leq 2^{n-1}(b_0(\mathfrak{A}) + b_1(\mathfrak{A}))$$

for $j = 0, 1$.

Proof. When $n = 2$, we have

$$\begin{aligned} b_j(\mathfrak{A} \rtimes_{\alpha(1)} \mathbb{Z} \rtimes_{\alpha(2)} \mathbb{Z}) &\leq b_0(\mathfrak{A} \rtimes_{\alpha(1)} \mathbb{Z}) + b_1(\mathfrak{A} \rtimes_{\alpha(1)} \mathbb{Z}) \\ &\leq 2(b_0(\mathfrak{A}) + b_1(\mathfrak{A})). \end{aligned}$$

When $n = 3$, we have

$$\begin{aligned} b_j(\mathfrak{A} \rtimes_{\alpha(1)} \mathbb{Z} \rtimes_{\alpha(2)} \mathbb{Z} \rtimes_{\alpha(3)} \mathbb{Z}) &\leq b_0(\mathfrak{A} \rtimes_{\alpha(1)} \mathbb{Z} \rtimes_{\alpha(2)} \mathbb{Z}) + b_1(\mathfrak{A} \rtimes_{\alpha(1)} \mathbb{Z} \rtimes_{\alpha(2)} \mathbb{Z}) \\ &\leq 2[b_0(\mathfrak{A} \rtimes_{\alpha(1)} \mathbb{Z}) + b_1(\mathfrak{A} \rtimes_{\alpha(1)} \mathbb{Z})] \\ &\leq 2^2(b_0(\mathfrak{A}) + b_1(\mathfrak{A})). \end{aligned}$$

The general case follows by induction with respect to n. □

3 Examples and more

Example 1. Let $C(\mathbb{T}^n)$ be the C^*-algebra of all continuous, complex-valued functions on the n-dimensional torus $\mathbb{T}^n$, which is also the univesal C^*-algebra generated by mutually commuting n unitaries. The C^*-algebra is regarded as the successive crossed product C^*-algebra of $\mathbb{C}$ by trivial actions id of $\mathbb{Z}$:

$$C(\mathbb{T}^n) \cong C^*(\mathbb{Z}^n) \cong \mathbb{C} \rtimes_{\alpha(1)} \mathbb{Z} \cdots \rtimes_{\alpha(n)} \mathbb{Z}$$

with $\alpha(j) = \mathrm{id}$ for $1 \leq j \leq n$, via the Fourier transform from $C^*(\mathbb{Z}^n)$ to $C(\mathbb{T}^n)$, with $\mathbb{T}^n$ as the dual group of $\mathbb{Z}^n$. It then follows that

$$b_j(C(\mathbb{T}^n)) \leq 2^{n-1}(b_0(\mathbb{C}) + b_1(\mathbb{C})) = 2^{n-1}(1 + 0) = 2^{n-1}$$

for $j = 0, 1$. Moreover, the estimate equality holds. Because $K_j(C(\mathbb{T}^n)) \cong \mathbb{Z}^{2^{n-1}}$ (cf. [11]), which is also deduced by using the Pimsner-Voiculescu six-term exact sequence repeatedly.

Example 2. Let $\mathbb{T}^n_\Theta$ denote the n-dimensional noncommutative torus, which is the C^*-algebra generated by n unitaries u_j such that $u_j u_k = e^{2\pi i \theta_{j,k}} u_k u_j$ for $1 \leq j, k \leq n$, where $i = \sqrt{-1}$ and $\Theta = (\theta_{j,k})$ is a $n \times n$ skew adjoint matrix over $\mathbb{R}$ of reals so that $-\Theta = \Theta^t$ the transpose of Θ (cf. [1], [11]). The C^*-algebra is regarded as the successive crossed product C^*-algebra of $\mathbb{C}$ by id of $\mathbb{Z}$:

$$\mathbb{T}^n_\Theta \cong \mathbb{C} \rtimes_{\mathrm{id}} \mathbb{Z} \rtimes_{\alpha(2)} \mathbb{Z} \cdots \rtimes_{\alpha(n)} \mathbb{Z}$$

and by successive actions $\alpha(j)$ for $2 \leq j \leq n$ given by

$$\alpha(j)u_k = \mathrm{Ad}(u_j)u_k = u_j u_k u_j^* = e^{2\pi i\theta_{j,k}} u_k$$

for $1 \leq k \leq j-1$. It then follows that

$$b_j(\mathbb{T}^n_\Theta) \leq 2^{n-1}(b_0(\mathbb{C}) + b_1(\mathbb{C})) = 2^{n-1}(1+0) = 2^{n-1}$$

for $j = 0, 1$. Moreover, the estimate equality holds. b Because $K_j(\mathbb{T}^n_\Theta) \cong \mathbb{Z}^{2^{n-1}}$, which is deduced by using the Pimsner-Voiculescu six-term exact sequence repeatedly. Note that Example 3.1 is just the case where Θ is the zero matrix.

Example 3. Let H_{2n+1} be the discrete Heisenberg nilpotent group of rank $2n+1$, consisting of the following $(n+2) \times (n+2)$ invertible matrices:

$$H_{2n+1} = \left\{ \begin{pmatrix} 1 & a & c \\ 0_{n,1} & 1_n & b^t \\ 0 & 0_{1,n} & 1 \end{pmatrix} \in GL_{n+2}(\mathbb{R}) \,|\, a, b \in \mathbb{Z}^n, c \in \mathbb{Z} \right\}$$

where 1_n is the $n \times n$ identity matrix and $0_{j,k}$ is the $j \times k$ zero matrix, and with $a, b \in \mathbb{Z}^n$ as row vectors and b^t the transpose of b. The group H_{2n+1} is viewed as the semi-direct product $\mathbb{Z}^{n+1} \rtimes_\alpha \mathbb{Z}^n$ of tuples (c, b, a) identified with the matrices above, where the action α is defined by matrix multiplication as

$$\alpha_a(c, b) = a(c, b)a^{-1} = (c + \sum_{j=1}^n a_j b_j, b) \in \mathbb{Z}^{n+1},$$

where $a = (a_1, \cdots, a_n) = (0, 0_n, a)$ and $(c, b) = (c, b_1, \cdots, b_n) = (c, b, 0_n)$, with $0_n = (0, \cdots, 0)$ the zero of $\mathbb{Z}^n$. Then the group C^*-algebra $C^*(H_{2n+1}) = C^*(\mathbb{Z}^{n+1} \rtimes_\alpha \mathbb{Z}^n)$ is regarded as the crossed product C^*-algebra $C^*(\mathbb{Z}^{n+1}) \rtimes_\alpha \mathbb{Z}^n$, where the action α of the semi-direct product group is extended and identified with that of the crossed product C^*-algebra, by the same symbol as α (also in what follows). Note that each element of an amenable (such as nilpotent or solvable) discrete group Γ is identified with the corresponding unitary under the left regular representation λ on $l^2(\Gamma)$ the Hilbert space of all square summable, complex-valued functions on Γ (cf. [2]). Let e_j $(1 \leq j \leq 2n+1)$ be the canonical basis for $\mathbb{Z}^{n+1}$ and $\mathbb{Z}^n$ in $\mathbb{Z}^{n+1} \rtimes_\alpha \mathbb{Z}^n$ and let $u_j = \lambda_{e_j}$ $(1 \leq j \leq 2n+1)$ be the corresponding unitaries in $C^*(\mathbb{Z}^{n+1} \rtimes_\alpha \mathbb{Z}^n)$. Then we have that

$$\alpha_a(u_1) = \lambda_{\alpha_a(e_1)} = \lambda_{e_1} = u_1,$$
$$\alpha_a(u_j) = \lambda_{\alpha_a(e_j)} = \lambda_{a_{j-1}e_1 + e_j} = u_1^{a_{j-1}} u_j$$

for $2 \leq j \leq n+1$. It then follows that

$$b_j(C^*(H_{2n+1})) \leq 2^{n-1}(b_0(C(\mathbb{T}^{n+1})) + b_1(C(\mathbb{T}^{n+1})) = 2^{n-1}(2^n + 2^n) = 2^{2n}$$

for $j = 0, 1$. In fact, it is computed in [9, Theorem 4.7] that $K_j(C^*(H_{2n+1})) \cong \mathbb{Z}^{2^n(2^n-1)+1}$ for $j = 0, 1$, with $2^n(2^n - 1) + 1 \leq 2^{2n}$ for $n \geq 1$ (cf. [5], [7]).

Theorem 3.1. *Let* G *be a successive semi-direct product of torsion free, finitely generated discrete group, written as* $G = \mathbb{Z}^{n_0} \rtimes_{\alpha(1)} \mathbb{Z}^{n_1} \cdots \rtimes_{\alpha(k)} \mathbb{Z}^{n_k}$ *for some* $n_0, \cdots, n_k \geq 1$, $k \geq 1$. *Let* $C^*(G)$ *be the group* C^**-algebra of* G. *Then* $b_j(C^*(G)) \leq 2^{n_0+n_1+\cdots+n_k-1}$ *for* $j = 0, 1$.

Proof. Note that

$$C^*(G) \cong C^*(\mathbb{Z}^{n_0}) \rtimes_{\alpha(1)} \mathbb{Z}^{n_1} \cdots \rtimes_{\alpha(k)} \mathbb{Z}^{n_k}$$

with $C^*(\mathbb{Z}^{n_0}) \cong C(\mathbb{T}^{n_0})$, where the right hand side above is viewed as an $n_1+\cdots+n_k$ fold, crossed product C^*-algebra by the successive actions of $\mathbb{Z}$. □

Theorem 3.2. *Let* G *be a torsion free, finitely generated nilpotent discrete group, with* $b(G) = n$. *Then* $b_j(C^*(G)) \leq 2^{n-1}$ *for* $j = 0, 1$.

Proof. It is well known that such a nilpotent discrete group can be written as such a successive semi-direct product as in the theorem above. □

Remark. These theorems above partially answer to a question as given in the Remark of [9, Theorem 4.7]. Note that any torsion free, finitely generated solvable discrete group may be not be written as such a successive semi-direct product as above, in the sense as neither always being split nor being supper-solvable with such a normal series (cf. [4]).

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