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Weak solutions to Neumann discrete nonlinear system of Kirchhoff type

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ABSTRACT

We prove the existence of weak solutions for discrete nonlinear system of Kirchhoff type. We build some Hilbert spaces with suitable norms. We define the notion of weak solution corresponding to the problem (1.1). The proof of the main result is based on a minimization method of an energy functional J.

RESUMEN

Probamos la existencia de soluciones débiles para sistemas discretos no-lineales de tipo Kirchhoff. Construimos algunos espacios de Hilbert con normas apropiadas. Definimos la noción de solución débil correspondiente al problema (1.1). La demostración del resultado principal se basa en un método de minimización de un funcional de energía J.

Keywords and Phrases: Nonlinear difference equations, anisotropic nonlinear discrete systems, minimization methods, weak solutions.

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1 Introduction

In this paper, we are going to investigate the existence of weak solutions for the following anisotropic nonlinear discrete system.

For $\quad i = 1, \cdots, n$

$$\begin{cases} -M\left(A(k-1, \Delta u_i(k-1))\right)\Delta(a(k-1, \Delta u_i(k-1))) = f_i(k, u(k)), \ \ k \in \mathbb{Z}[1, T] \\ \\ \qquad \Delta u_i(0) = \Delta u_i(T) = 0 \end{cases} \tag{1.1}$$

where $\Delta u_i(k) = u_i(k+1) - u_i(k)$ is the forward difference operator for any $\ i = 1, \cdots, n$; $\mathbb{Z}[1, T] = \{1, \ldots, T\}$ for $T \geq 2$ and a, f_i are functions to be defined later.

In the last few years, great attention has been paid to the study of fourth-order nonlinear difference equations. These equations have been widely used to study discrete models in many fields such as computer science, economics, neural network, ecology, cybernetics, etc. For background and recent results, we refer the reader to [2]-[12], [14] and the references therein.
Note that in recent years, much attention has been paid to problems not local since they appear in physical phenomena like the theory of nonlinear elasticity, heat diffusion, etc. Among this problems, we find Kirchhoff type problems, which are known by the presence of the term $M(\int_\Omega |\nabla u|^2)\Delta u$ in the continuous case. As far as we know, the first study which deals with anisotropic discrete boundary value problems of $p(.)$-Kirchhoff type difference equation was done by Yucedag (see [11]). The function $M(A(k-1, \Delta u(k-1)))$ which appear in the left-hand side of problem (1.1) is more general.
The main operator $\Delta(a(k-1, \Delta u(k-1)))$ in problem (1.1) can be seen as a discrete counterpart of the anisotropic operator $\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a\left(x, \frac{\partial}{\partial x_i} u\right)$. The functional a derives from a potential with $a(k, \xi) = \frac{\partial}{\partial \xi} A(k, \xi)$.
Our goal is to use a minimization method in order to establish some existence results of solutions of (1.1). The idea of the proof is to transfer the problem of the existence of solutions for (1.1) into the problem of existence of a minimizer for some associated energy functional. This method was successfully used by Bonanno et al. [1] for the study of an eigenvalue nonhomogeneous Neumann problem, where, under an appropriate oscillating behaviour of the nonlinear term, they proved the existence of a determined open interval of positive parameters for which the problem considered admits infinitely many weak solutions that strongly converge to zero, in an appropriate Orlicz Sobolev space.
Motivated by the work of [13] where J. Zhao proved the existence of positive solutions, the approach presented in this article is different than the one given in the papers mentioned above. To the best of

our knowledge , results on existence of weak solutions of system (1.1), using minimization method, have not been found in the literature.
The remaining part of this paper is organized as follows. Section 2 is devoted to mathematical preliminaries. The main existence result is proved in Section 3. In the Section 4, we give an extension of our system.

2 Mathematical background

In the T-dimensional Hilbert space

$$H = \left\{u : \mathbb{Z}[0, T+1] \longrightarrow \mathbb{R}^n \quad \text{such that} \quad \Delta u(0) = \Delta u(T) = 0\right\},$$

with the inner product

$$\langle u, v\rangle = \sum_{i=1}^{n} \sum_{k=1}^{T+1} \Delta u_i(k-1)\Delta v_i(k-1), \qquad \forall\, u, v \in H,$$

we consider the norm

$$\|u\| = \left(\sum_{i=1}^{n} \sum_{k=1}^{T+1} |\Delta u_i(k-1)|^2\right)^{\frac{1}{2}}. \tag{2.1}$$

We denote

$$H_i = \left\{u_i : \mathbb{Z}[0, T+1] \longrightarrow \mathbb{R} \quad \text{such that} \quad \Delta u_i(0) = \Delta u_i(T) = 0\right\}, \quad \text{for} \quad i = 1, \cdots, n$$

with the norm

$$|u_i|_h = \left(\sum_{k=1}^{T+1} |\Delta u_i(k-1)|^2\right)^{\frac{1}{2}} \quad \forall\, u_i \in H_i \quad \text{for} \quad i = 1, \cdots, n. \tag{2.2}$$

Moreover, we may consider H_i with the following norm

$$|u_i|_m = \left(\sum_{k=1}^{T} |u_i(k)|^m\right)^{\frac{1}{m}} \quad \forall\, u_i \in H_i, \quad m \geq 2 \quad \text{for} \quad i = 1, \cdots, n. \tag{2.3}$$

We have the following inequalities (see [2])

$$T^{(2-m)/(2m)}|u_i|_2 \leq |u_i|_m \leq T^{1/m}|u_i|_2, \quad \forall\, u_i \in H_i, \quad m \geq 2 \quad \text{for} \quad i = 1, \cdots, n. \tag{2.4}$$

Let the function

$$p : \mathbb{Z}[0, T] \longrightarrow (2, +\infty) \tag{2.5}$$

denoted by

$$p^- = \min_{k\in\mathbb{Z}[0,T]} p(k) \qquad \text{and} \qquad p^+ = \max_{k\in\mathbb{Z}[0,T]} p(k).$$

For the data a and f_i, we assume the following.

$$(H_1).\quad \begin{cases} a(k,.):\mathbb{R}\to\mathbb{R},\quad k\in\mathbb{Z}[0,T] \text{ and there exists } A(.,.):\mathbb{Z}[0,T]\times\mathbb{R}\to\mathbb{R} \\ \text{which satisfies } a(k,\xi)=\frac{\partial}{\partial\xi}A(k,\xi) \text{ and } A(k,0)=0, \quad \text{for all } k\in\mathbb{Z}[0,T]. \end{cases}$$

(H_2). For all $k\in\mathbb{Z}[0,T]$ and $\xi\neq\eta$

$$(a(k,\xi)-a(k,\eta)).(\xi-\eta)>0. \tag{2.6}$$

(H_3). For any $k\in\mathbb{Z}[0,T]$, $\xi\in\mathbb{R}$, we have

$$A(k,\xi)\geq\frac{1}{p(k)}|\xi|^{p(k)}. \tag{2.7}$$

(H_4). For each $k\in\mathbb{Z}[0,T]$, the function $f_i(k,.):\mathbb{R}^n\longrightarrow\mathbb{R}$ is jointly continuous and there exists $(\alpha_i(.))_{1\leq i\leq n}:\mathbb{Z}[0,T]\longrightarrow(0,+\infty)$ and a function $(r_i(.))_{1\leq i\leq n}:\mathbb{Z}[0,T]\longrightarrow[2,+\infty)$ such that

$$|f_i(k,u)|\leq\alpha_i(k)\left(1+|u_i(k)|^{r_i(k)-1}\right) \tag{2.8}$$

where $\qquad 2\leq r_i(k)<p^-\quad$ for $\quad i=1,\cdots,n.$
In what follows, we denote by :

$$r^-=\min_{\{(k,i)\in\mathbb{Z}[0,T]\times\mathbb{Z}[1,n]\}}r_i(k)\qquad\text{and}\qquad r^+=\max_{\{(k,i)\in\mathbb{Z}[0,T]\times\mathbb{Z}[1,n]\}}r_i(k).$$

For each $i=1,\cdots,n$, there exists $h_i\in\mathbb{R}^n$ such that

$$\nabla F_i(k,u)(h_i)=f_i(k,u)\quad\forall u\in\ H\quad\text{for}\quad i=1,\cdots,n. \tag{2.9}$$

By (2.8) there exists $(\beta_i(.))_{1\leq i\leq n}:\mathbb{Z}[0,T]\longrightarrow(0,+\infty)$ such that

$$|F_i(k,u)|\leq\beta_i(k)\left(1+|u_i(k)|^{r_i(k)}\right)\quad\text{for}\quad i=1,\cdots,n \tag{2.10}$$

where

$$0<\underline{\beta}=\inf_{\{(k,i)\in\ \mathbb{Z}[0,T]\times\mathbb{Z}[1,n]\}}\beta_i(k)\leq\sup_{\{(k,i)\in\ \mathbb{Z}[0,T]\times\mathbb{Z}[1,n]\}}\beta_i(k)=\overline{\beta}<+\infty. \tag{2.11}$$

(H_5). We also assume that the function $M:(0,+\infty)\longrightarrow(0,+\infty)$ is continuous and non-decreasing and there exist positive numbers B_1, B_2 with $B_1\leq B_2$ and $\alpha>1$ such that

$$B_1t^{\alpha-1}\leq M(t)\leq B_2t^{\alpha-1}\quad\text{for}\quad t>t^*>0. \tag{2.12}$$

Example 2.1.
There are many functions satisfying both $(H_1)-(H_4)$. *Let us mention the following.*

- $A(k,\xi) = \dfrac{1}{p(k)}\left(\left(1+|\xi|^2\right)^{p(k)/2} - 1\right)$, *where* $a(k,\xi) = \left(1+|\xi|^2\right)^{(p(k)-2)/2}\xi$, $\forall\ k \in \mathbb{Z}[0,T],\ \xi \in \mathbb{R}$,
- $f_i(k,\xi) = 1 + \left|\xi_i\right|^{p(k)-1},\ \ \forall\ (k,i) \in \mathbb{Z}[0,T] \times \mathbb{Z}[1,n]$ *and* $\xi = (\xi_1, \cdots, \xi_n)$,
- $M(t) = 1,\ \ \forall\ t \in (0,+\infty)$.

Moreover, we may consider H with the following norm

$$\|u\|_m = \sum_{i=1}^{n}\left(\sum_{k=1}^{T}|u_i(k)|^m\right)^{\frac{1}{m}}, \quad \forall\ u \in H \quad \text{and} \quad m \geq 2. \tag{2.13}$$

Using the relation (2.4) we can prove the following lemma.

Lemma 2.2. *We have the following inequalities*

$$T^{(2-m)/(2m)}\|u\|_2 \leq \|u\|_m \leq T^{1/m}\|u\|_2, \quad \forall\ u \in H \quad \textit{and} \quad m \geq 2. \tag{2.14}$$

We need the following auxiliary results throughout our paper.

Lemma 2.3.

(1) *There exist two positive constant* $C_1,\ C_2$ *such that*

$$\sum_{i=1}^{n}\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^{p(k-1)} \geq C_1\left(\sum_{i=1}^{n}\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^2\right)^{\frac{p^-}{2}} - C_2, \tag{2.15}$$

for all $u \in H$ *with* $|u_i|_h > 1$.

(2) *For any* $m \geq 2$ *there exists a positive constant* c_m *such that*

$$\sum_{i=1}^{n}\sum_{k=1}^{T}|u_i(k)|^m \leq c_m \sum_{i=1}^{n}\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^m, \quad \forall u \in H. \tag{2.16}$$

Indeed,

(1) By [6], there exists the positive constants λ_i and μ_i for $i = 1, \cdots n$

$$\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^{p(k-1)} \geq \lambda_i\left(\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^2\right)^{\frac{p^-}{2}} - \mu_i \ \ \forall\ u_i \in H_i \ \text{and}\ |u_i|_h > 1.$$

$$\sum_{i=1}^{n}\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^{p(k-1)} \geq \min_{1\leq i\leq n}(\lambda_i)\sum_{i=1}^{n}\left(\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^2\right)^{\frac{p^-}{2}} - \max_{1\leq i\leq n}(\mu_i)\, n.$$

Since the function $x \longmapsto x^{\frac{p^-}{2}}$ is convex because $p^- > 2$, then we have

$$\sum_{i=1}^{n}\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^{p(k-1)} \geq \min_{1\leq i\leq n}(\lambda_i)\left(\sum_{i=1}^{n}\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^2\right)^{\frac{p^-}{2}} - \max_{1\leq i\leq n}(\mu_i)\,n.$$

We deduce that

$$\sum_{i=1}^{n}\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^{p(k-1)} \geq C_1\left(\sum_{i=1}^{n}\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^2\right)^{\frac{p^-}{2}} - C_2.$$

(2) By [8], for any $m \geq 2$ there exists a positive constant c_m such that for $i = 1, \cdots, n$

$$\sum_{k=1}^{T}|u_i(k)|^m \leq c_m \sum_{k=1}^{T+1}|\Delta u_i(k-1)|^m \quad \forall\, u_i \in H_i.$$

Therefore

$$\sum_{i=1}^{n}\sum_{k=1}^{T}|u_i(k)|^m \leq c_m \sum_{i=1}^{n}\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^m \quad \forall\, u \in H.$$

3 Existence of weak solutions

In this section, we study the existence of weak solution of problem (1.1).

Definition 3.1. *A weak solutions of problem (1.1) is* $u \in H$ *such that*

$$\begin{aligned}&\sum_{i=1}^{n}\left[M\left(\sum_{k=1}^{T+1}A(k-1,\Delta u_i(k-1))\right)\sum_{k=1}^{T+1}a(k-1,\Delta u_i(k-1))\Delta v_i(k-1)\right]\\&=\sum_{i=1}^{n}\sum_{k=1}^{T}f_i(k,u(k))v_i(k)\end{aligned} \tag{3.1}$$

for all $v \in H$.

Note that, since H is a finite dimensional space, the weak solutions coincide with the classical solution the problem (1.1).

Theorem 3.2. *Assume that* $(H_1)-(H_5)$ *holds. Then, there exists a weak solution of the problem (1.1).*

To prove this, we define the energy functional $J : H \longrightarrow \mathbb{R}$ by

$$J(u) = \sum_{i=1}^{n} \widehat{M}\left(\sum_{k=1}^{T+1} A(k-1, \Delta u_i(k-1))\right) - \sum_{i=1}^{n}\sum_{k=1}^{T} F_i\big(k, u(k)\big) \tag{3.2}$$

where $\widehat{M(t)} = \int_0^t M(s)ds.$

Lemma 3.3. *The functional* J *is well defined on* H *and is of class* $C^1\big(H, \mathbb{R}\big)$ *with the derivative given by*

$$\begin{aligned} \langle J'(u), v\rangle \;=& \sum_{i=1}^{n}\left[M\left(\sum_{k=1}^{T+1} A(k-1, \Delta u_i(k-1)\right)\sum_{k=1}^{T+1} a(k-1, \Delta u_i(k-1))\Delta v_i(k-1)\right] \\ -& \sum_{i=1}^{n}\sum_{k=1}^{T} f_i(k, u(k))v_i(k), \end{aligned} \tag{3.3}$$

for all $u, v \in H$.

Indeed, let's

$$I(u) = \sum_{i=1}^{n} \widehat{M}\left(\sum_{k=1}^{T+1} A(k-1, \Delta u_i(k-1))\right) \text{ and } \Lambda(u) = \sum_{i=1}^{n}\sum_{k=1}^{T} F_i\big(k, u(k)\big).$$

Since $\widehat{M}(.)$, $A(k, .)$ and $F(k, .)$ are continuous for all $k \in \mathbb{Z}[0, T]$, then

$$|I(u)| = \left|\sum_{i=1}^{n} \widehat{M}\left(\sum_{k=1}^{T+1} A(k-1, \Delta u_i(k-1))\right)\right| < +\infty,$$

$$|\Lambda(u)| \;=\; \left|\sum_{i=1}^{n}\sum_{k=1}^{T} F_i\big(k, u(k)\big)\right| < +\infty.$$

The energy functional J is well defined on H.

It is not difficult to see that the functional I derivative are give by

$$\langle I'(u), v\rangle = \sum_{i=1}^{n}\left[M\left(\sum_{k=1}^{T+1} A(k-1, \Delta u_i(k-1)\right)\sum_{k=1}^{T+1} a(k-1, \Delta u_i(k-1))\Delta v_i(k-1)\right] \tag{3.4}$$

On the other hand, for all $u, v \in H$, there exists $h_i \in \mathbb{R}^n$ such that

$$\begin{aligned}
\langle \Lambda'(u), v\rangle &= \lim_{t\to 0^+} \frac{\Lambda(u+tv)-\Lambda(u)}{t} \\
&= \lim_{t\to 0^+} \sum_{i=1}^{n}\sum_{k=1}^{T} \frac{F_i(k,u(k)+tv(k))-F_i(k,u(k))}{t} \\
&= \sum_{i=1}^{n}\sum_{k=1}^{T} \lim_{t\to 0^+} \frac{F_i(k,u(k)+tv(k))-F_i(k,u(k))}{t} \\
&= \sum_{i=1}^{n}\sum_{k=1}^{T} \nabla F_i(k,u(k))(h_i)v_i(k) \\
&= \sum_{i=1}^{n}\sum_{k=1}^{T} f_i(k,u(k))v_i(k).
\end{aligned}$$

The functional J is clearly of class C^1 □

Lemma 3.4. *The functional* J *is lower semi-continuous.*

Indeed since the functional Λ is completely continuous and weakly lower semi-continuous, we have to prove the semi-continuity of I.

A is convex with respect to the second variable according (H_1) and (H_2). With the assumption (H_5) we conclude that I is convex. Thus, it is enough to show that I is lower semi-continuous. For this, we fix $u \in H$ and $\varepsilon > 0$. Since I is convex, we deduce that, for any $v \in H$.

$$\begin{aligned}
I(v) &\geq I(u) + \langle I'(u), v-u\rangle \\
&\geq I(u) - \sum_{i=1}^{n}\Bigg[M\left(\sum_{k=1}^{T+1} A(k-1,\Delta u_i(k-1))\right) \\
&\qquad\qquad \times \sum_{k=1}^{T+1} |a(k-1,\Delta u_i(k-1))||\Delta v_i(k-1)-\Delta u_i(k-1)|\Bigg] \\
&\geq I(u) - C_M\left(\sum_{i=1}^{n}\sum_{k=1}^{T+1} |a(k-1,\Delta u_i(k-1))||\Delta v_i(k-1)-\Delta u_i(k-1)|\right),
\end{aligned}$$

where $\quad C_M = \left(\sum_{i=1}^{n} M\left(\sum_{k=1}^{T+1} A(k-1,\Delta u_i(k-1)\right)\right)$

By using Schwartz inequality, we get :

$$\begin{aligned} I(v) &\geq I(u) - C_M \sum_{i=1}^{n} \left[\left(\sum_{k=1}^{T+1} |a(k-1, \Delta u_i(k-1))|^2 \right)^{\frac{1}{2}} \right. \\ &\qquad\qquad \left. \times \left(\sum_{k=1}^{T+1} |\Delta v_i(k-1) - \Delta u_i(k-1)|^2 \right)^{\frac{1}{2}} \right] \\ &\geq I(u) - C_M \left[\sum_{i=1}^{n} \left(\sum_{k=1}^{T+1} |a(k-1, \Delta u_i(k-1))|^2 \right)^{\frac{1}{2}} \right] \\ &\qquad\qquad \times \left[\sum_{i=1}^{n} \left(\sum_{k=1}^{T+1} |\Delta v_i(k-1) - \Delta u_i(k-1)|^2 \right)^{\frac{1}{2}} \right] \end{aligned}$$

By (2.2)

$$I(v) \geq I(u) - C_M \left[\sum_{i=1}^{n} \left(\sum_{k=1}^{T+1} |a(k-1, \Delta u_i(k-1))|^2 \right)^{\frac{1}{2}} \right] \left[\sum_{i=1}^{n} |v_i - u_i|_h \right].$$

Since H_i is finite dimensional, there exist the positive constants θ_i for $i = 1, \cdots, n$ such that

$$|v_i|_h \leq \theta_i |v_i|_2 \quad \forall\, v_i \in H_i. \tag{3.5}$$

Then,

$$\begin{aligned} I(v) &\geq I(u) - C_M \left[\sum_{i=1}^{n} \left(\sum_{k=1}^{T+1} |a(k-1, \Delta u_i(k-1))|^2 \right)^{\frac{1}{2}} \right] \left[\sum_{i=1}^{n} \theta_i |v_i - u_i|_2 \right] \\ &\geq I(u) - \max_{1 \leq i \leq n} (\theta_i)\, C_M \left[\sum_{i=1}^{n} \left(\sum_{k=1}^{T+1} |a(k-1, \Delta u_i(k-1))|^2 \right)^{\frac{1}{2}} \right] \left[\sum_{i=1}^{n} |v_i - u_i|_2 \right]. \end{aligned}$$

Also, the space H is finite dimensional, there exists a positive constant γ such that:

$$\|u\|_2 \leq \gamma \|u\| \quad \forall\, u \in H.$$

From this, we have

$$\begin{aligned} I(v) &\geq I(u) - \gamma \max_{1 \leq i \leq n} (\theta_i)\, C_M \left[\sum_{i=1}^{n} \left(\sum_{k=1}^{T+1} |a(k-1, \Delta u_i(k-1))|^2 \right)^{\frac{1}{2}} \right] \|v - u\| \\ &\geq I(u) - \left[1 + \gamma \max_{1 \leq i \leq n} (\theta_i)\, C_M \sum_{i=1}^{n} \left(\sum_{k=1}^{T+1} |a(k-1, \Delta u_i(k-1))|^2 \right)^{\frac{1}{2}} \right] \|v - u\| \end{aligned}$$

Finally

$$I(v) \geq I(u) - S(T,u)\|v-u\| \geq I(u) - \varepsilon, \tag{3.6}$$

for all $v \in H$ with $\|v-u\| < \delta = \frac{\varepsilon}{S(T,u)}$, where

$$S(T,u) = 1 + \gamma \max_{1\leq i\leq n} (\theta_i)\, C_M \sum_{i=1}^{n} \left(\sum_{k=1}^{T+1} |a(k-1, \Delta u_i(k-1))|^2 \right)^{\frac{1}{2}}.$$

We conclude that J is weakly lower semi-continuous.

Proposition 3.5. *The functional* J *is coercive and bounded from below.*

Indeed, according to (2.7), (2.10)-(2.12) we have

$$\begin{aligned}
J(u) &= \sum_{i=1}^{n} \widehat{M}\left(\sum_{k=1}^{T+1} A(k-1, \Delta u_i(k-1)) \right) - \sum_{i=1}^{n}\sum_{k=1}^{T} F_i\big(k, u(k)\big) \\
&\geq \frac{B_1}{\alpha(p^+)^\alpha} \left[\sum_{i=1}^{n} \left(\sum_{k=1}^{T+1} |\Delta u_i(k-1)|^{p(k-1)} \right)^{\alpha} \right] - \sum_{i=1}^{n}\sum_{k=1}^{T} F_i\big(k, u(k)\big) \\
&\geq \frac{B_1}{\alpha(p^+)^\alpha} \left[\sum_{i=1}^{n} \left(\sum_{k=1}^{T+1} |\Delta u_i(k-1)|^{p(k-1)} \right)^{\alpha} \right] - \sum_{i=1}^{n}\sum_{k=1}^{T} \beta_i(k) \left(1 + |u_i(k)|^{r_i(k)} \right) \\
&\geq \frac{B_1}{\alpha(p^+)^\alpha} \left[\sum_{i=1}^{n} \left(\sum_{k=1}^{T+1} |\Delta u_i(k-1)|^{p(k-1)} \right)^{\alpha} \right] - \overline{\beta} \sum_{i=1}^{n}\sum_{k=1}^{T} \left(1 + |u_i(k)|^{r_i(k)} \right) \\
&\geq \frac{B_1}{\alpha(p^+)^\alpha} \left[\sum_{i=1}^{n} \left(\sum_{k=1}^{T+1} |\Delta u_i(k-1)|^{p(k-1)} \right)^{\alpha} \right] - \overline{\beta} \sum_{i=1}^{n}\sum_{k=1}^{T} |u_i(k)|^{r_i(k)} - \overline{\beta} nT.
\end{aligned}$$

There exist η_i and ν_i such that

$$\begin{aligned}
J(u) \geq\ & \frac{B_1}{\alpha(p^+)^\alpha} \left[\min_{1\leq i\leq n} (\eta_i) \left(\sum_{i=1}^{n}\sum_{k=1}^{T+1} |\Delta u_i(k-1)|^{p(k-1)} \right)^{\alpha} - \max_{1\leq i\leq n} (\nu_i) \right] \\
-\ & \overline{\beta} \sum_{i=1}^{n}\sum_{k=1}^{T} |u_i(k)|^{r_i(k)} - \overline{\beta} nT.
\end{aligned} \tag{3.7}$$

To prove the coerciveness of the functional J, we may assume that $\|u\| > 1$ and we deduce from the above inequality (2.15) that

$$\begin{aligned} J(u) &\geq \frac{B_1}{\alpha(p^+)^\alpha}\left[\min_{1\leq i\leq n}(\eta_i)\left(C_1\left(\sum_{i=1}^{n}\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^2\right)^{\frac{p^-}{2}}-C_2\right)^{\alpha}-\max_{1\leq i\leq n}(\nu_i)\right] \\ &\quad -\overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T}|u_i(k)|^{r_i(k)}-\overline{\beta}nT. \end{aligned}$$

There exist a function $K(\alpha, C)$ such that

$$\begin{aligned} J(u) &\geq \frac{B_1}{\alpha(p^+)^\alpha}\left(\min_{1\leq i\leq n}(\eta_i)C_1^\alpha\|u\|^{\alpha p^-}-\min_{1\leq i\leq n}(\eta_i)K(\alpha,C)C_2^\alpha-\max_{1\leq i\leq n}(\nu_i)\right) \\ &- \quad \overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T}|u_i(k)|^{r_i(k)}-\overline{\beta}nT. \end{aligned}$$

Namely

$$J(u)\geq A_1\|u\|^{\alpha p^-}-\overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T}|u_i(k)|^{r_i(k)}-A_2,$$

where

$$A_1=\frac{B_1}{\alpha(p^+)^\alpha}\min_{1\leq i\leq n}(\eta_i)C_1^\alpha$$

and

$$A_2=\frac{B_1}{\alpha(p^+)^\alpha}\left(\min_{1\leq i\leq n}(\eta_i)K(\alpha,C)C_2^\alpha+\max_{1\leq i\leq n}(\nu_i)\right)+\overline{\beta}nT.$$

So

$$\begin{aligned} J(u) &\geq A_1\|u\|^{\alpha p^-}-\overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T}|u_i(k)|^{r_i(k)}-A_2 \\ &\geq A_1\|u\|^{\alpha p^-}-\overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T}|u_i(k)|^{r^+}-\overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T}|u_i(k)|^{r^-}-A_2. \end{aligned}$$

Using (2.16)

$$J(u)\geq A_1\|u\|^{\alpha p^-}-(C_{r^-})\overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T}|\Delta u_i(k)|^{r^-}-(C_{r^+})\overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T}|\Delta u_i(k)|^{r^+}-A_2$$

By using (2.4) there exists the positive constants K_1 and K_2 such that

$$J(u)\geq A_1\|u\|^{\alpha p^-}-K_1\sum_{i=1}^{n}\left(\sum_{k=1}^{T}|\Delta u_i(k)|^2\right)^{\frac{r^-}{2}}-K_2\sum_{i=1}^{n}\left(\sum_{k=1}^{T}|\Delta u_i(k)|^2\right)^{\frac{r^+}{2}}-A_2.$$

There exist the positive constants A_3, A_4, A_5 and A_6 such that

$$\begin{aligned} J(u) &\geq A_1\|u\|^{\alpha p^-} - K_1 A_3 \left(\sum_{i=1}^{n}\sum_{k=1}^{T}|\Delta u_i(k)|^2\right)^{\frac{r^-}{2}} \\ &- K_1\, A_4 - A_5\, K_2 \left(\sum_{i=1}^{n}\sum_{k=1}^{T}|\Delta u_i(k)|^2\right)^{\frac{r^+}{2}} - K_2\, A_6 - A_2. \end{aligned}$$

Consequently, there exist the positive constants A_7 , A_8 and A_9 such that

$$J(u) \geq A_1\|u\|^{\alpha p^-} - A_7\|u\|^{r^-} - A_8\|u\|^{r^+} - A_9. \tag{3.8}$$

Recall that $p^- > \dfrac{r^+}{\alpha} \geq \dfrac{r^-}{\alpha}$. Then J is coercive.

Besides, for $\|u\| \leq 1$, we have with (3.7)

$$\begin{aligned} J(u) &\geq \frac{B_1}{\alpha(p^+)^\alpha}\left[\min_{1\leq i\leq n}(\eta_i)\left(\sum_{i=1}^{n}\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^{p(k-1)}\right)^{\alpha} - \max_{1\leq i\leq n}(\nu_i)\right] \\ &- \overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T}|u_i(k)|^{r_i(k)} - \overline{\beta}nT \\ &\geq -\frac{B_1}{\alpha(p^+)^\alpha}\max_{1\leq i\leq n}(\nu_i) - \overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T}|u_i(k)|^{r_i(k)} - \overline{\beta}nT \\ &\geq -\frac{B_1}{\alpha(p^+)^\alpha}\max_{1\leq i\leq n}(\nu_i) - \overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T}|u_i(k)|^{r^-} - \overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T}|u_i(k)|^{r^+} - \overline{\beta}nT. \end{aligned}$$

Using (2.16)

$$J(u) \geq -\tfrac{B_1}{\alpha(p^+)^\alpha}\max_{1\leq i\leq n}(\nu_i) - (K_{r^-})\overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T}|\Delta u_i(k)|^{r^-} - (K_{r^+})\overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T}|\Delta u_i(k)|^{r^+} - \overline{\beta}nT.$$

By using (2.14) there exists the positives constants K_1' and K_2' such that

$$J(u) \geq -\frac{B_1}{\alpha(p^+)^\alpha}\max_{1\leq i\leq n}(\nu_i) - K_1'\sum_{i=1}^{n}\left(\sum_{k=1}^{T}|\Delta u_i(k)|^2\right)^{\frac{r^-}{2}} - K_2'\sum_{i=1}^{n}\left(\sum_{k=1}^{T}|\Delta u_i(k)|^2\right)^{\frac{r^+}{2}} - \overline{\beta}nT.$$

There exist the positive constants C'_3, C'_4, C'_5 and C'_6 such that

$$\begin{aligned} J(u) &\geq -\frac{B_1}{\alpha(p^+)^\alpha}\max_{1\leq i\leq n}(\nu_i) - K'_1 C'_3\left(\sum_{i=1}^{n}\sum_{k=1}^{T}|\Delta u_i(k)|^2\right)^{\frac{r^-}{2}} \\ &- K'_1\ C'_4 - C'_5\ K'_2\left(\sum_{i=1}^{n}\sum_{k=1}^{T}|\Delta u_i(k)|^2\right)^{\frac{r^+}{2}} - K'_2\ C'_6 - \overline{\beta} nT. \end{aligned}$$

Consequently, there exist the positive constants C'_7 and C'_8 such that

$$\begin{aligned} J(u) &\geq -\frac{B_1}{\alpha(p^+)^\alpha}\max_{1\leq i\leq n}(\nu_i) - C'_7\|u\|^{r^-} - K'_1\ C'_4 - C'_8\|u\|^{r^+} - K'_2\ C'_6 - \overline{\beta} nT \\ &\geq -\frac{B_1}{\alpha(p^+)^\alpha}\max_{1\leq i\leq n}(\nu_i) - C'_7 - K'_1\ C'_4 - C'_8 - K'_2\ C'_6 - \overline{\beta} nT. \end{aligned}$$

Thus, J is bounded from below □

Since J is weakly lower semi-continuous, bounded from below and coercive on H, using the relation between critical points of J and problem (1.1), we deduce that J has a minimizer which is a weak solution to problem (1.1).

4 An extension

In this section we are going to show that the existence result obtained for system (1.1) can be extended. Let's consider the following system.

For $i = 1, \cdots, n$

$$\begin{cases} -M\left(A(k-1,\Delta u_i(k-1))\right)\Delta(a(k-1,\Delta u_i(k-1))) + \sigma_i(k)\phi(k,u_i(k)) \\ \qquad\qquad\qquad\qquad = \delta_i(k) f_i(k,u(k)),\ \forall\ k\in\mathbb{Z}[1,T] \\ \Delta u_i(0) = \Delta u_i(T) = 0, \end{cases} \tag{4.1}$$

where $T \geq 2$ is a fixed integer, and we shall use the following assumption.

(H_6). $\sigma_i : \mathbb{Z}[1,T] \longrightarrow \mathbb{R}$ and $\delta_i : \mathbb{Z}[1,T] \longrightarrow \mathbb{R}$ are such that $\sigma_i(k) \geq \sigma_0 > 0$ for

$(k,i) \in \mathbb{Z}[1,T]\times\mathbb{Z}[1,n]$ and $0 < \delta_i(k) \leq \sup_{\{(k,i)\in\mathbb{Z}[1,T]\times\mathbb{Z}[1,n]\}} |\delta_i(k)| = \delta_0.$

(H_7). $\phi(k,t) = |t|^{p(k)-2}t$ for $(k,t)\in\mathbb{Z}[0,T]\times\mathbb{R}$.

In the $T-$dimensional Hilbert space H with the inner product

$$\langle u,v\rangle=\sum_{i=1}^{n}\sum_{k=1}^{T+1}\Delta u_i(k-1)\Delta v_i(k-1)+\sum_{i=1}^{n}\sum_{k=1}^{T+1}u_i(k)v_i(k),$$

we consider the norm

$$\|u\|=\sqrt{\langle u,u\rangle}=\left(\sum_{i=1}^{n}\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^2+\sum_{i=1}^{n}\sum_{k=1}^{T}|u_i(k)|^2\right)^{\frac{1}{2}}.$$

Definition 4.1. *A weak solution of problem (4.1) is a function* $u\in H$ *such that*

$$\sum_{i=1}^{n}\left[M\left(\sum_{k=1}^{T+1}A(k-1,\Delta u_i(k-1)\right)\sum_{k=1}^{T+1}a(k-1,\Delta u_i(k-1))\Delta v_i(k-1)\right]$$
$$+\sum_{i=1}^{n}\sum_{k=1}^{T}\sigma_i(k)|u_i(k)|^{p(k)-2}u_i(k)v_i(k)=\sum_{i=1}^{n}\sum_{k=1}^{T}\delta_i(k)f_i(k,u(k))v_i(k).$$

for all $v\in H$.

Theorem 4.2. *Under the assumptions* (H_1)- (H_6) *the problem (4.1) has a least weak solution in* H.

Indeed, for $u\in H$ we define the energy functional corresponding to system (4.1) by

$$J(u)=\sum_{i=1}^{n}\widehat{M}\left(\sum_{k=1}^{T+1}A(k-1,\Delta u_i(k-1))\right)+\sum_{i=1}^{n}\sum_{k=1}^{T}\frac{\sigma_i(k)}{p(k)}|u_i(k)|^{p(k)}-\sum_{i=1}^{n}\sum_{k=1}^{T}\delta_i(k)F_i\big(k,u(k)\big).$$

Obviously, J is class $C^1(H,\mathbb{R})$ and is weakly lower semicontinuous, and we show that

$$\begin{aligned}\langle J'(u),v\rangle &= \sum_{i=1}^{n}\left[M\left(\sum_{k=1}^{T+1}A(k-1,\Delta u_i(k-1)\right)\sum_{k=1}^{T+1}a(k-1,\Delta u_i(k-1))\Delta v_i(k-1)\right]\\ &+ \sum_{i=1}^{n}\sum_{k=1}^{T}\sigma_i(k)|u_i(k)|^{p(k)-2}u_i(k)v_i(k)-\sum_{i=1}^{n}\sum_{k=1}^{T}\delta_i(k)f_i\big(k,u(k)\big)v_i(k).\end{aligned}$$

for all $u,v\in H$.
This implies that the weak solution of system(4.1) coincides with the critical points of the functional J. It suffices to prove that J is bounded below and coercive in order to complete the proof.

$$\begin{aligned}
J(u) &= \sum_{i=1}^{n} \widehat{M}\left(\sum_{k=1}^{T+1} A(k-1, \Delta u_i(k-1))\right) + \sum_{i=1}^{n}\sum_{k=1}^{T} \frac{\sigma_i(k)}{p(k)}|u_i(k)|^{p(k)} - \sum_{i=1}^{n}\sum_{k=1}^{T} \delta_i(k) F_i\big(k, u(k)\big) \\
&\geq \sum_{i=1}^{n} \widehat{M}\left(\sum_{k=1}^{T+1} A(k-1, \Delta u_i(k-1))\right) - \sum_{i=1}^{n}\sum_{k=1}^{T} \delta_i(k) F_i\big(k, u(k)\big) \\
&\geq \sum_{i=1}^{n} \widehat{M}\left(\sum_{k=1}^{T+1} A(k-1, \Delta u_i(k-1))\right) - \delta_0 \sum_{i=1}^{n}\sum_{k=1}^{T} F_i\big(k, u(k)\big).
\end{aligned}$$

We obtain

$$\begin{aligned}
J(u) \quad &\geq \quad \tfrac{B_1}{\alpha(p^+)^\alpha}\left[\min_{1\leq i\leq n}(\eta_i)\left(\sum_{i=1}^{n}\sum_{k=1}^{T+1}|\Delta u_i(k-1)|^{p(k-1)}\right)^{\alpha} - \max_{1\leq i\leq n}(\nu_i)\right] \\
&- \quad \delta_0\overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T}|u_i(k)|^{r_i(k)} - \delta_0\overline{\beta}nT.
\end{aligned} \tag{4.2}$$

For $\|u\| > 1$, by the same procedure, we prove that

$$J(u) \geq A_1'\|u\|^{\alpha p^-} - A_7'\|u\|^{r^-} - A_8'\|u\|^{r^+} - A_9',$$

where $A_1',\ A_7',\ A_8'$ and A_9' are the positive constants.

Hence $p^- > \dfrac{r^+}{\alpha} \geq \dfrac{r^-}{\alpha}$, J is coercive.

If $\|u\| \leq 1$ by (4.2) we have

$$J(u) \geq -\frac{B_1}{\alpha(p^+)^\alpha}\max_{1\leq i\leq n}(\nu_i) - \delta_0\overline{\beta}\sum_{i=1}^{n}\sum_{k=1}^{T}|u_i(k)|^{r_i(k)} - \delta_0\overline{\beta}nT.$$

By the same reasoning

$$J(u) \geq -D_1 - \delta_0\overline{\beta}nT$$

where $D_1 > 0$.

Thus, J is bounded from below □

Since J is weakly lower semi-continuous, bounded from below and coercive on H, using the relation between critical points of J and problem (4.1), we deduce that J has a minimizer which is a weak solution to problem (4.1).

Competing interests

The authors declare that there is no conflict of interest regarding the publication of the paper.

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