

## Contractive mapping theorems in Partially ordered metric spaces

N.SESHAGIRI RAO<sup>1</sup>, K.KALYANI<sup>2</sup> AND KEJAL KHATRI<sup>3</sup>

<sup>1</sup>*Department of Applied Mathematics, School of Applied Natural Sciences  
Adama Science and Technology University  
Post Box No.1888, Adama, Ethiopia.*

<sup>2</sup>*Department of Mathematics  
Vignan's Foundation for Science, Technology & Research  
Vadlamudi-522213, Andhra Pradesh, India.*

<sup>3</sup>*Department of Mathematics  
Government College Simalwara  
Dungarpur 314403, Rajasthan, India.*

*seshu.namana@gmail.com, kalyani.namana@gmail.com, kejal0909@gmail.com*

### ABSTRACT

The purpose of this paper is to establish some coincidence, common fixed point theorems for monotone  $f$ -non decreasing self mappings satisfying certain rational type contraction in the context of a metric spaces endowed with partial order. Also, the results involving an integral type of such classes of mappings are discussed in application point of view. These results generalize and extend well known existing results in the literature.

### RESUMEN

El propósito de este artículo es establecer teoremas de coincidencia y de punto fijo común para auto mapeos monótonos  $f$ -no decrecientes satisfaciendo ciertas contracciones de tipo racional en el contexto de espacios métricos dotados de un orden parcial. Adicionalmente, resultados que involucran clases de mapeos de tipo integral son discutidos desde un punto de vista de las aplicaciones. Estos resultados generalizan y extienden resultados bien conocidos, existentes en la literatura.

**Keywords and Phrases:** Partially ordered metric spaces; Rational contractions; Compatible mappings; Weakly compatible mappings.

**2020 AMS Mathematics Subject Classification:** 47H10; 54H25; 26A42; 46T99.



## 1 Introduction

Ever since in Fixed point theory and Approximation theory, the classical Banach contraction principle plays a vital role to obtain an unique solution of the results. Of course, it is very important and popular tool in different fields of mathematics to solve the existing problems in nonlinear analysis. Since then a lot of variety of generalizations of this Banach contraction principle [1] have been taken place in a metric fixed point theory by improving the underlying contraction condition [2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. Thereafter vigorous research work has been obtained by weakening its hypotheses on various spaces such as rectangular metric spaces, pseudo metric spaces, fuzzy metric spaces, quasi metric spaces, quasi semi-metric spaces, probabilistic metric spaces,  $D$ -metric spaces,  $G$ -metric spaces,  $F$ -metric spaces, cone metric spaces, and so on to prove the existing results. Prominent work on the existence and uniqueness of a fixed point and common fixed point theorems involving monotone mappings on cone metric spaces, partially ordered metric spaces and others spaces [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30] generate natural interest to establish usable fixed point results.

The aim of this paper is to prove some coincidence, common fixed point results in the frame work of partially ordered metric spaces for a pair of self-mappings satisfying a generalized contractive condition of rational type. These results generalize and extend the results of Harjani et al.[19] and Chandok [28] in ordered metric space. Also the applications of these results are presented on taking integral type contractions in the same space.

## 2 Preliminaries

The following definitions are frequently used in results given in upcoming sections.

**Definition 1.** *The triple  $(X, d, \preceq)$  is called a partially ordered metric space, if  $(X, \preceq)$  is a partially ordered set together with  $(X, d)$  is a metric space.*

**Definition 2.** *If  $(X, d)$  is a complete metric space, then the triple  $(X, d, \preceq)$  is called a partially ordered complete metric space.*

**Definition 3.** *Let  $(X, \preceq)$  be a partially ordered set. A self-mapping  $f : X \rightarrow X$  is said to be strictly increasing, if  $f(x) \prec f(y)$ , for all  $x, y \in X$  with  $x \prec y$  and is also said to be strictly decreasing, if  $f(x) \succ f(y)$ , for all  $x, y \in X$  with  $x \prec y$ .*

**Definition 4.** *A point  $x \in A$ , where  $A$  is a non-empty subset of a metric space  $(X, d)$  is called a common fixed (coincidence) point of two self-mappings  $f$  and  $T$  if  $fx = Tx = x(fx = Tx)$ .*

**Definition 5.** *The two self-mappings  $f$  and  $T$  defined over a subset  $A$  of a metric space  $(X, d)$  are called commuting if  $fTx = Tf x$  for all  $x \in A$ .*

**Definition 6.** Two self-mappings  $f$  and  $T$  defined over  $A \subset X$  are compatible, if for any sequence  $\{x_n\}$  with  $\lim_{n \rightarrow +\infty} f x_n = \lim_{n \rightarrow +\infty} T x_n = \mu$ , for some  $\mu \in A$  then  $\lim_{n \rightarrow +\infty} d(T f x_n, f T x_n) = 0$ .

**Definition 7.** Two self-mappings  $f$  and  $T$  defined over  $A \subset X$  are said to be weakly compatible, if they commute at their coincidence points. i.e., if  $f x = T x$  then  $f T x = T f x$ .

**Definition 8.** Let  $f$  and  $T$  be two self-mappings defined over a partially ordered set  $(X, \preceq)$ . A mapping  $T$  is called a monotone  $f$  non-decreasing if

$$f x \preceq f y \text{ implies } T x \preceq T y, \text{ for all } x, y \in X.$$

**Definition 9.** Let  $A$  be a non-empty subset of a partially ordered set  $(X, \preceq)$ . If every two elements of  $A$  are comparable then it is called well ordered set.

**Definition 10.** A partially ordered metric space  $(X, d, \preceq)$  is called an ordered complete, if for each convergent sequence  $\{x_n\}_{n=0}^{+\infty} \subset X$ , one of the following condition holds

- if  $\{x_n\}$  is a nondecreasing sequence in  $X$  such that  $x_n \rightarrow x$  implies  $x_n \preceq x$ , for all  $n \in \mathbb{N}$  that is,  $x = \sup\{x_n\}$  or
- if  $\{x_n\}$  is a nonincreasing sequence in  $X$  such that  $x_n \rightarrow x$  implies  $x \preceq x_n$ , for all  $n \in \mathbb{N}$  that is,  $x = \inf\{x_n\}$ .

### 3 Main Results

In this section, we prove some coincidence, common fixed point theorems in the context of ordered metric space.

**Theorem 1.** Let  $(X, d, \preceq)$  be a complete partially ordered metric space. Suppose that the self-mappings  $f$  and  $T$  on  $X$  are continuous,  $T$  is a monotone  $f$ -nondecreasing,  $T(X) \subseteq f(X)$  and satisfying the following condition

$$d(Tx, Ty) \leq \alpha \frac{d(fx, Tx) d(fy, Ty)}{d(fx, fy)} + \beta [d(fx, Tx) + d(fy, Ty)] + \gamma d(fx, fy) \quad (3.1)$$

for all  $x, y$  in  $X$  with  $f(x) \neq f(y)$  are comparable, where  $\alpha, \beta, \gamma \in [0, 1)$  with  $0 \leq \alpha + 2\beta + \gamma < 1$ . If there exists a point  $x_0 \in X$  such that  $f(x_0) \preceq T(x_0)$  and the mappings  $T$  and  $f$  are compatible, then  $T$  and  $f$  have a coincidence point in  $X$ .

*Proof.* Let  $x_0 \in X$  such that  $f(x_0) \preceq T(x_0)$ . Since from hypotheses, we have  $T(X) \subseteq f(X)$  then, we can choose a point  $x_1 \in X$  such that  $f x_1 = T x_0$ . But  $T x_1 \in f(X)$  then, again there exists another point  $x_2 \in X$  such that  $f x_2 = T x_1$ . By continuing the same way, we can construct a sequence  $\{x_n\}$  in  $X$  such that  $f x_{n+1} = T x_n$ , for all  $n$ .

Again, by hypotheses, we have  $f(x_0) \preceq T(x_0) = f(x_1)$  and  $T$  is a monotone  $f$ -nondecreasing mapping then, we get  $T(x_0) \preceq T(x_1)$ . Similarly, we obtain  $T(x_1) \preceq T(x_2)$ , since  $f(x_1) \preceq f(x_2)$  and then by continuing the same procedure, we obtain that

$$T(x_0) \preceq T(x_1) \preceq T(x_2) \preceq \dots\dots\dots T(x_n) \preceq T(x_{n+1}) \preceq \dots\dots\dots$$

The equality  $T(x_{n+1}) = T(x_n)$  is impossible because  $f(x_{n+2}) \neq f(x_{n+1})$  for all  $n \in \mathbb{N}$ . Thus  $d(T(x_n), T(x_{n+1})) > 0$  for all  $n \geq 0$  therefore, from contraction condition (3.1), we have

$$d(Tx_{n+1}, Tx_n) \leq \alpha \frac{d(fx_{n+1}, Tx_{n+1}) d(fx_n, Tx_n)}{d(fx_{n+1}, fx_n)} + \beta [d(fx_{n+1}, Tx_{n+1}) + d(fx_n, Tx_n)] + \gamma d(fx_{n+1}, fx_n)$$

which intern implies that

$$d(Tx_{n+1}, Tx_n) \leq \alpha d(Tx_n, Tx_{n+1}) + \beta [d(Tx_n, Tx_{n+1}) + d(Tx_{n-1}, Tx_n)] + \gamma d(Tx_n, Tx_{n-1})$$

Finally, we arrive at

$$d(Tx_{n+1}, Tx_n) \leq \left( \frac{\beta + \gamma}{1 - \alpha - \beta} \right) d(Tx_n, Tx_{n-1})$$

Continuing the same process up to  $(n - 1)$  times, we get

$$d(Tx_{n+1}, Tx_n) \leq \left( \frac{\beta + \gamma}{1 - \alpha - \beta} \right)^n d(Tx_1, Tx_0)$$

Let  $k = \frac{\beta + \gamma}{1 - \alpha - \beta} \in [0, 1)$ , then from triangular inequality for  $m \geq n$ , we have

$$\begin{aligned} d(Tx_m, Tx_n) &\leq d(Tx_m, Tx_{m-1}) + d(Tx_{m-1}, Tx_{m-2}) + \dots\dots\dots + d(Tx_{n+1}, Tx_n) \\ &\leq (k^{m-1} + k^{m-2} + \dots\dots\dots + k^n) d(Tx_1, Tx_0) \\ &\leq \frac{k^n}{1 - k} d(Tx_1, Tx_0) \end{aligned}$$

as  $m, n \rightarrow +\infty$ ,  $d(Tx_m, Tx_n) \rightarrow 0$ , which shows that the sequence  $\{Tx_n\}$  is a Cauchy sequence in  $X$ . So, by the completeness of  $X$ , there exists a point  $\mu \in X$  such that  $Tx_n \rightarrow \mu$  as  $n \rightarrow +\infty$ .

Again, by the continuity of  $T$ , we have

$$\lim_{n \rightarrow +\infty} T(Tx_n) = T \left( \lim_{n \rightarrow +\infty} Tx_n \right) = T\mu.$$

But  $fx_{n+1} = Tx_n$ , then  $fx_{n+1} \rightarrow \mu$  as  $n \rightarrow +\infty$  and from the compatibility for  $T$  and  $f$ , we have

$$\lim_{n \rightarrow +\infty} d(T(fx_n), f(Tx_n)) = 0.$$

Further by triangular inequality, we have

$$d(T\mu, f\mu) = d(T\mu, T(fx_n)) + d(T(fx_n), f(Tx_n)) + d(f(Tx_n), f\mu)$$

On taking limit as  $n \rightarrow +\infty$  in both sides of the above equation and using the fact that  $T$  and  $f$  are continuous then, we get  $d(T\mu, f\mu) = 0$ . Thus,  $T\mu = f\mu$ . Hence,  $\mu$  is a coincidence point of  $T$  and  $f$  in  $X$ . ■

**Corollary 1.** *Let  $(X, d, \preceq)$  be a complete partially ordered metric space. Suppose that the self-mappings  $f$  and  $T$  on  $X$  are continuous,  $T$  is a monotone  $f$ -nondecreasing,  $T(X) \subseteq f(X)$  and satisfying the following condition*

$$d(Tx, Ty) \leq \alpha \frac{d(fx, Tx) d(fy, Ty)}{d(fx, fy)} + \beta [d(fx, Tx) + d(fy, Ty)]$$

for all  $x, y$  in  $X$  with  $f(x) \neq f(y)$  are comparable and for some  $\alpha, \beta \in [0, 1)$  with  $0 \leq \alpha + 2\beta < 1$ . If there exists a point  $x_0 \in X$  such that  $f(x_0) \preceq T(x_0)$  and the mappings  $T$  and  $f$  are compatible, then  $T$  and  $f$  have a coincidence point in  $X$ .

*Proof.* Set  $\gamma = 0$  in Theorem 1. ■

**Corollary 2.** *Let  $(X, d, \preceq)$  be a complete partially ordered metric space. Suppose that the self-mappings  $f$  and  $T$  on  $X$  are continuous,  $T$  is a monotone  $f$ -nondecreasing,  $T(X) \subseteq f(X)$  and satisfying the following condition*

$$d(Tx, Ty) \leq \beta [d(fx, Tx) + d(fy, Ty)] + \gamma d(fx, fy)$$

for all  $x, y$  in  $X$  with  $f(x) \neq f(y)$  are comparable and for some  $\beta, \gamma \in [0, 1)$  with  $0 \leq 2\beta + \gamma < 1$ . If there exists a point  $x_0 \in X$  such that  $f(x_0) \preceq T(x_0)$  and the mappings  $T$  and  $f$  are compatible, then  $T$  and  $f$  have a coincidence point in  $X$ .

*Proof.* The proof can be obtained by setting  $\alpha = 0$  in Theorem 1. ■

We may remove the continuity criteria of  $T$  in Theorem 1 is still valid by assuming the following hypothesis in  $X$ :

If  $\{x_n\}$  is a nondecreasing sequence in  $X$  such that  $x_n \rightarrow x$ , then  $x_n \preceq x$  for all  $n \in \mathbb{N}$ .

**Theorem 2.** *Let  $(X, d, \preceq)$  be a complete partially ordered metric space. Suppose that  $f$  and  $T$  are self-mappings on  $X$ ,  $T$  is a monotone  $f$ -nondecreasing,  $T(X) \subseteq f(X)$  and satisfying*

$$d(Tx, Ty) \leq \alpha \frac{d(fx, Tx) d(fy, Ty)}{d(fx, fy)} + \beta [d(fx, Tx) + d(fy, Ty)] + \gamma d(fx, fy) \quad (3.2)$$

for all  $x, y$  in  $X$  with  $f(x) \neq f(y)$  are comparable and for some  $\alpha, \beta, \gamma \in [0, 1)$  with  $0 \leq \alpha + 2\beta + \gamma < 1$ . If there exists a point  $x_0 \in X$  such that  $f(x_0) \preceq T(x_0)$  and  $\{x_n\}$  is a nondecreasing sequence in  $X$  such that  $x_n \rightarrow x$ , then  $x_n \preceq x$  for all  $n \in \mathbb{N}$ .

If  $f(X)$  is a complete subset of  $X$ , then  $T$  and  $f$  have a coincidence point in  $X$ . Further, if  $T$  and  $f$  are weakly compatible, then  $T$  and  $f$  have a common fixed point in  $X$ . Moreover, the set

of common fixed points of  $T$  and  $f$  is well ordered if and only if  $T$  and  $f$  have one and only one common fixed point in  $X$ .

*Proof.* Suppose  $f(X)$  is a complete subset of  $X$ . As we know from the proof of Theorem 1, the sequence  $\{Tx_n\}$  is a Cauchy sequence and hence  $\{fx_n\}$  is also a Cauchy sequence in  $(f(X), d)$  as  $fx_{n+1} = Tx_n$  and  $T(X) \subseteq f(X)$ . Since  $f(X)$  is complete then there exists some  $fu \in f(X)$  such that

$$\lim_{n \rightarrow +\infty} T(x_n) = \lim_{n \rightarrow +\infty} f(x_n) = f(u).$$

Also note that the sequences  $\{Tx_n\}$  and  $\{fx_n\}$  are nondecreasing and from hypotheses, we have  $T(x_n) \preceq f(u)$  and  $f(x_n) \preceq f(u)$  for all  $n \in \mathbb{N}$ . But  $T$  is a monotone  $f$ -nondecreasing then, we get  $T(x_n) \preceq T(\mu)$  for all  $n$ . Letting  $n \rightarrow +\infty$ , we obtain that  $f(u) \preceq T(u)$ .

Suppose that  $f(u) \prec T(u)$  then define a sequence  $\{u_n\}$  by  $u_0 = u$  and  $fu_{n+1} = Tu_n$  for all  $n \in \mathbb{N}$ . An argument similar to that in the proof of Theorem 1 yields that  $\{fu_n\}$  is a nondecreasing sequence and  $\lim_{n \rightarrow +\infty} f(u_n) = \lim_{n \rightarrow +\infty} T(u_n) = f(v)$  for some  $v \in X$ . So from hypotheses, it is clear that  $\sup f(u_n) \preceq f(v)$  and  $\sup T(u_n) \preceq f(v)$ , for all  $n \in \mathbb{N}$ . Notice that

$$f(x_n) \preceq f(u) \preceq f(u_1) \preceq \dots \preceq f(u_n) \preceq \dots \preceq f(v).$$

**Case:1** Suppose if there exists some  $n_0 \geq 1$  such that  $f(x_{n_0}) = f(u_{n_0})$  then, we have

$$f(x_{n_0}) = f(u) = f(u_{n_0}) = f(u_1) = T(u).$$

Hence,  $u$  is a coincidence point of  $T$  and  $f$  in  $X$ .

**Case:2** Suppose that  $f(x_{n_0}) \neq f(u_{n_0})$  for all  $n$  then, from (3.2), we have

$$\begin{aligned} d(fx_{n+1}, fu_{n+1}) &= d(Tx_n, Tu_n) \\ &\leq \alpha \frac{d(fx_n, Tx_n) d(fu_n, Tu_n)}{d(fx_n, fu_n)} + \beta [d(fx_n, Tx_n) + d(fu_n, Tu_n)] \\ &\quad + \gamma d(fx_n, fu_n) \end{aligned}$$

Taking limit as  $n \rightarrow +\infty$  on both sides of the above inequality, we get

$$\begin{aligned} d(fu, fv) &\leq \gamma d(fu, fv) \\ &< d(fu, fv), \text{ since } \gamma < 1. \end{aligned}$$

Thus, we have

$$f(u) = f(v) = f(u_1) = T(u).$$

Hence, we conclude that  $u$  is a coincidence point of  $T$  and  $f$  in  $X$ .

Now, suppose that  $T$  and  $f$  are weakly compatible. Let  $w$  be a coincidence point then,

$$T(w) = T(f(z)) = f(T(z)) = f(w), \text{ since } w = T(z) = f(z), \text{ for some } z \in X.$$

Now by contraction condition, we have

$$\begin{aligned} d(T(z), T(w)) &\leq \alpha \frac{d(fz, Tz) d(fw, Tw)}{d(fz, fw)} + \beta [d(fz, Tz) + d(fw, Tw)] + \gamma d(fz, fw) \\ &\leq \gamma d(T(z), T(w)) \end{aligned}$$

as  $\gamma < 1$ , then  $d(T(z), T(w)) = 0$ . Therefore,  $T(z) = T(w) = f(w) = w$ . Hence,  $w$  is a common fixed point of  $T$  and  $f$  in  $X$ .

Now suppose that the set of common fixed points of  $T$  and  $f$  is well ordered, we have to show that the common fixed point of  $T$  and  $f$  is unique. Let  $u$  and  $v$  be two common fixed points of  $T$  and  $f$  such that  $u \neq v$  then from (3.2), we have

$$\begin{aligned} d(u, v) &\leq \alpha \frac{d(fu, Tu) d(fv, Tv)}{d(fu, fv)} + \beta [d(fu, Tu) + d(fv, Tv)] + \gamma d(fu, fv) \\ &\leq \gamma d(u, v) \\ &< d(u, v), \text{ since } \gamma < 1, \end{aligned}$$

which is a contradiction. Thus,  $u = v$ . Conversely, suppose  $T$  and  $f$  have only one common fixed point then the set of common fixed points of  $T$  and  $f$  being a singleton is well ordered. This completes the proof. ■

**Corollary 3.** *Let  $(X, d, \preceq)$  be a complete partially ordered metric space. Suppose that  $f$  and  $T$  are self-mappings on  $X$ ,  $T$  is a monotone  $f$ -nondecreasing,  $T(X) \subseteq f(X)$  and satisfying*

$$d(Tx, Ty) \leq \alpha \frac{d(fx, Tx) d(fy, Ty)}{d(fx, fy)} + \beta [d(fx, Tx) + d(fy, Ty)]$$

for all  $x, y$  in  $X$  with  $f(x) \neq f(y)$  are comparable and for some  $\alpha, \beta \in [0, 1)$  with  $0 \leq \alpha + 2\beta < 1$ . If there exists a point  $x_0 \in X$  such that  $f(x_0) \preceq T(x_0)$  and  $\{x_n\}$  is a nondecreasing sequence in  $X$  such that  $x_n \rightarrow x$ , then  $x_n \preceq x$  for all  $n \in \mathbb{N}$ .

If  $f(X)$  is a complete subset of  $X$ , then  $T$  and  $f$  have a coincidence point in  $X$ . Further, if  $T$  and  $f$  are weakly compatible, then  $T$  and  $f$  have a common fixed point in  $X$ . Moreover, the set of common fixed points of  $T$  and  $f$  is well ordered if and only if  $T$  and  $f$  have one and only one common fixed point in  $X$ .

*Proof.* Set  $\gamma = 0$  in Theorem 2. ■

**Corollary 4.** *Let  $(X, d, \preceq)$  be a complete partially ordered metric space. Suppose that  $f$  and  $T$  are self-mappings on  $X$ ,  $T$  is a monotone  $f$ -nondecreasing,  $T(X) \subseteq f(X)$  and satisfying*

$$d(Tx, Ty) \leq \beta [d(fx, Tx) + d(fy, Ty)] + \gamma d(fx, fy)$$

for all  $x, y$  in  $X$  for which  $f(x) \neq f(y)$  are comparable and for some  $\beta, \gamma \in [0, 1)$  with  $0 \leq 2\beta + \gamma < 1$ . If there exists a point  $x_0 \in X$  such that  $f(x_0) \preceq T(x_0)$  and  $\{x_n\}$  is a nondecreasing sequence in  $X$  such that  $x_n \rightarrow x$ , then  $x_n \preceq x$  for all  $n \in \mathbb{N}$ .

If  $f(X)$  is a complete subset of  $X$ , then  $T$  and  $f$  have a coincidence point in  $X$ . Further, if  $T$  and  $f$  are weakly compatible, then  $T$  and  $f$  have a common fixed point in  $X$ . Moreover, the set of common fixed points of  $T$  and  $f$  is well ordered if and only if  $T$  and  $f$  have one and only one common fixed point in  $X$ .

*Proof.* Set  $\alpha = 0$  in Theorem 2. ■

**Remark 1.** (i). If  $\beta = 0$ , in Theorem 1 and Theorem 2, we obtain Theorem 2.1 and Theorem 2.3 of Chandok [28].

(ii). If  $f = I$  and  $\beta = 0$ , in Theorem 1 and Theorem 2, then we get Theorem 2.1 and Theorem 2.3 of Harjani et al. [19].

## 4 Applications

In this section, we state some applications of the main result for a self mapping involving the integral type contractions.

Let us denote  $\tau$ , a set of all functions  $\varphi$  defined on  $[0, +\infty)$  satisfying the following conditions:

- (1) each  $\varphi$  is Lebesgue integrable mapping on each compact subset of  $[0, +\infty)$  and
- (2) for any  $\epsilon > 0$ , we have  $\int_0^\epsilon \varphi(t) dt > 0$ .

**Theorem 3.** Let  $(X, d, \preceq)$  be a complete partially ordered metric space. Suppose that the self-mappings  $f$  and  $T$  on  $X$  are continuous,  $T$  is a monotone  $f$ -nondecreasing,  $T(X) \subseteq f(X)$  and satisfying the following condition

$$\int_0^{d(Tx, Ty)} \varphi(t) dt \leq \alpha \int_0^{\frac{d(fx, Tx) \cdot d(fy, Ty)}{d(fx, fy)}} \varphi(t) dt + \beta \int_0^{d(fx, Tx) + d(fy, Ty)} \varphi(t) dt + \gamma \int_0^{d(fx, fy)} \varphi(t) dt \quad (4.1)$$

for all  $x, y$  in  $X$  with  $f(x) \neq f(y)$  are comparable,  $\varphi(t) \in \tau$  and for some  $\alpha, \beta, \gamma \in [0, 1)$  such that  $0 \leq \alpha + 2\beta + \gamma < 1$ . If there exists a point  $x_0 \in X$  such that  $f(x_0) \preceq T(x_0)$  and the mappings  $T$  and  $f$  are compatible, then  $T$  and  $f$  have a coincidence point in  $X$ .

Similarly, we can obtain the following results in complete partially ordered metric space, by putting  $\gamma = 0$  and  $\alpha = 0$  in an integral contraction of Theorem 3.

**Theorem 4.** Let  $(X, d, \preceq)$  be a complete partially ordered metric space. Suppose that the self-mappings  $f$  and  $T$  on  $X$  are continuous,  $T$  is a monotone  $f$ -nondecreasing,  $T(X) \subseteq f(X)$  and satisfying the following condition

$$\int_0^{d(Tx, Ty)} \varphi(t) dt \leq \alpha \int_0^{\frac{d(fx, Tx) \cdot d(fy, Ty)}{d(fx, fy)}} \varphi(t) dt + \beta \int_0^{d(fx, Tx) + d(fy, Ty)} \varphi(t) dt \quad (4.2)$$

for all  $x, y$  in  $X$  with  $f(x) \neq f(y)$  are comparable,  $\varphi(t) \in \tau$  and where  $\alpha, \beta \in [0, 1)$  such that  $0 \leq \alpha + 2\beta < 1$ . If there exists a point  $x_0 \in X$  such that  $f(x_0) \preceq T(x_0)$  and the mappings  $T$  and  $f$  are compatible, then  $T$  and  $f$  have a coincidence point in  $X$ .

**Theorem 5.** Let  $(X, d, \preceq)$  be a complete partially ordered metric space. Suppose that the self-mappings  $f$  and  $T$  on  $X$  are continuous,  $T$  is a monotone  $f$ -nondecreasing,  $T(X) \subseteq f(X)$  and satisfying the following condition

$$\int_0^{d(Tx, Ty)} \varphi(t) dt \leq \beta \int_0^{d(fx, Tx) + d(fy, Ty)} \varphi(t) dt + \gamma \int_0^{d(fx, fy)} \varphi(t) dt \quad (4.3)$$

for all  $x, y$  in  $X$  with  $f(x) \neq f(y)$  are comparable,  $\varphi(t) \in \tau$  and for some  $\beta, \gamma \in [0, 1)$  such that  $0 \leq 2\beta + \gamma < 1$ . If there exists a point  $x_0 \in X$  such that  $f(x_0) \preceq T(x_0)$  and the mappings  $T$  and  $f$  are compatible, then  $T$  and  $f$  have a coincidence point in  $X$ .

**Corollary 5.** By replacing  $\beta = 0$  in Theorem 3, we obtain the Corollary 2.5 of Chandok [28].

We illustrate the usefulness of the obtained results for the existence of the coincidence point in the space.

**Example 1.** Define a metric  $d : X \times X \rightarrow [0, +\infty)$  by  $d(x, y) = |x - y|$ , where  $X = [0, 1]$  with usual order  $\leq$ . Suppose that  $T$  and  $f$  be two self mappings on  $X$  such that  $Tx = \frac{x^2}{2}$  and  $fx = \frac{2x^2}{1+x}$ , then  $T$  and  $f$  have a coincidence in point  $X$ .

*Proof.* By definition of a metric  $d$ , it is clear that  $(X, d)$  is a complete metric space. Obviously,  $(X, d, \leq)$  is a partially ordered complete metric space with usual order. Let  $x_0 = 0 \in X$  then  $f(x_0) \leq T(x_0)$  and also by definition;  $T, f$  are continuous,  $T$  is a monotone  $f$ -nondecreasing and  $T(X) \subseteq f(X)$ .

Now for any distinct  $x, y$  in  $X$ , we have

$$\begin{aligned} d(Tx, Ty) &= \frac{1}{2}|x^2 - y^2| = \frac{1}{2}(x + y)|x - y| \\ &< \frac{\alpha}{4} \frac{x^2 y^2}{(x + y + xy)} \frac{|x - 3||y - 3|}{|x - y|} + \frac{\beta}{2} \frac{x^2(1 + y)|x - 3| + y^2(1 + x)|y - 3|}{(1 + x)(1 + y)} \\ &\quad + \gamma \frac{2(x + y + xy)}{(1 + x)(1 + y)} |x - y| \end{aligned}$$

$$\begin{aligned}
 &< \alpha \frac{\frac{x^2|x-3|}{2(1+x)} \cdot \frac{y^2|y-3|}{2(1+y)}}{2|x-y| \frac{x+y+xy}{(1+x)(1+y)}} + \beta \left[ \frac{x^2|x-3|}{2(1+x)} + \frac{y^2|y-3|}{2(1+y)} \right] + \gamma \frac{2(x+y+xy)}{(1+x)(1+y)} |x-y| \\
 &< \alpha \frac{d(fx, Tx) d(fy, Ty)}{d(fx, fy)} + \beta [d(fx, Tx) + d(fy, Ty)] + \gamma d(fx, fy)
 \end{aligned}$$

Then, the contraction condition in Theorem 1 holds by selecting proper values of  $\alpha, \beta, \gamma$  in  $[0, 1)$  such that  $0 \leq \alpha + 2\beta + \gamma < 1$ . Therefore  $T, f$  have a coincidence point  $0 \in X$ . ■

Similarly the following is one more example of main Theorem 1.

**Example 2.** A distance function  $d : X \times X \rightarrow [0, +\infty)$  by  $d(x, y) = |x - y|$ , where  $X = [0, 1]$  with usual order  $\leq$ . Define two self mappings  $T$  and  $f$  on  $X$  by  $Tx = x^2$  and  $fx = x^3$ , then  $T$  and  $f$  have two coincidence points  $0, 1$  in  $X$  with  $x_0 = \frac{1}{2}$ .

## Acknowledgments

The authors would like to thank the editor and referees for their precise remarks which improve the presentation of the paper.

## References

- [1] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations untegrales, *Fund. Math.* **3** (1922), 133–181.
- [2] B. K. Dass and S. Gupta, An extension of Banach contraction principle through rational expression, *Indian J. Pure Appl. Math.* **6** (1975), no. 12, 1455–1458.
- [3] S.K. Chetterjee, Fixed point theorems, *C.R. Acad. Bulgara Sci.* **25** (1972), 727–730.
- [4] M. Edelstein, On fixed points and periodic points under contraction mappings, *J. Lond. Math. Soc.* **37** (1962), 74–79.
- [5] G.C. Hardy, T.Rogers, A generalization of fixed point theorem of S. Reich, *Can. Math. Bull.* **16** (1973), 201–206.
- [6] D.S. Jaggi, Some unique fixed point theorems, *Indian J. Pure Appl. Math.* **8**, (1977) 223–230.
- [7] R. Kannan, Some results on fixed points. II, *Amer. Math. Monthly* **76** (1969), 405–408.
- [8] S. Reich, Some remarks concerning contraction mappings, *Canad. Math. Bull.* **14** (1971), 121–124.
- [9] M.R. Singh, A.K. Chatterjee, Fixed point theorems, *Commun. Fac. Sci. Univ. Ank. Series A1* **37** (1988), 1–4.
- [10] D. R. Smart, *Fixed point theorems*, Cambridge University Press, London, 1974.
- [11] C. S. Wong, Common fixed points of two mappings, *Pacific J. Math.* **48** (1973), 299–312.
- [12] R. P. Agarwal, M. A. El-Gebeily and D. O'Regan, Generalized contractions in partially ordered metric spaces, *Appl. Anal.* **87** (2008), no. 1, 109–116.
- [13] I. Altun, B. Damjanović and D. Djorić, Fixed point and common fixed point theorems on ordered cone metric spaces, *Appl. Math. Lett.* **23** (2010), no. 3, 310–316.
- [14] A. Amini-Harandi and H. Emami, A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations, *Nonlinear Anal.* **72** (2010), no. 5, 2238–2242.
- [15] A. Chanda, B. Damjanović and L. K. Dey, Fixed point results on  $\theta$ -metric spaces via simulation functions, *Filomat* **31** (2017), no. 11, 3365–3375.
- [16] M. Arshad, A. Azam and P. Vetro, Some common fixed point results in cone metric spaces, *Fixed Point Theory Appl.* **2009**, Art. ID 493965, 11 pp.

- [17] M. Arshad, J. Ahmad and E. Karapınar, Some common fixed point results in rectangular metric spaces, *Int. J. Anal.* **2013**, Art. ID 307234, 7 pp.
- [18] T.G. Bhaskar, V. Lakshmikantham, Fixed point theory in partially ordered metric spaces and applications, *Nonlinear Anal. Theory Methods Appl.* **65** (2006), 1379–1393.
- [19] J. Harjani, B. López and K. Sadarangani, A fixed point theorem for mappings satisfying a contractive condition of rational type on a partially ordered metric space, *Abstr. Appl. Anal.* **2010**, Art. ID 190701, 8 pp.
- [20] S. Hong, Fixed points of multivalued operators in ordered metric spaces with applications, *Nonlinear Anal.* **72** (2010), no. 11, 3929–3942.
- [21] I.X. Liu, M. Zhou, B. Damjanovic, Nonlinear Operators in Fixed point theory with Applications to Fractional Differential and Integral Equations, *Journal of Function spaces* 2018, Article ID 9863267, 11 pages (2018), Doi:10.1155/2018/9063267.
- [22] M. Zhou, X. Liu, B. Damjanovic, Arslan Hojat Ansari, Fixed point theorems for several types of Meir-Keeler contraction mappings in  $M_s$ -metric spaces, *J. Comput. Anal. Appl.* **25** (2018), no. 7, 1337–1353.
- [23] J.J. Nieto, R.R. López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, *Order* **22** (2005), 223–239.
- [24] J.J. Nieto, R.R. López: Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equation, *Acta Math. Sin. Engl. Ser.* **23**(12), 2205–2212 (2007).
- [25] M. Öztürk and M. Başarır, On some common fixed point theorems with rational expressions on cone metric spaces over a Banach algebra, *Hacet. J. Math. Stat.* **41** (2012), no. 2, 211–222.
- [26] A. C. M. Ran and M. C. B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, *Proc. Amer. Math. Soc.* **132** (2004), no. 5, 1435–1443.
- [27] S. Chandok, Some common fixed point results for generalized weak contractive mappings in partially ordered metric spaces, *Journal of Nonlinear Anal. Opt.* **4** (2013), 45–52 .
- [28] S. Chandok, Some common fixed point results for rational type contraction mappings in partially ordered metric spaces, *Math. Bohem.* **138** (2013), no. 4, 407–413.
- [29] E. S. Wolk, Continuous convergence in partially ordered sets, *General Topology and Appl.* **5** (1975), no. 3, 221–234.
- [30] X. Zhang, Fixed point theorems of multivalued monotone mappings in ordered metric spaces, *Appl. Math. Lett.* **23** (2010), no. 3, 235–240.