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Anisotropic problem with non-local boundary conditions and measure data

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ABSTRACT

We study a nonlinear anisotropic elliptic problem with non-local boundary conditions and measure data. We prove an existence and uniqueness result of entropy solution.

RESUMEN

Estudiamos un problema elíptico nolineal anisotrópico con condiciones de borde no-locales y data de medida. Probamos un resultado de existencia y unicidad de la solución de entropía.

Keywords and Phrases: Entropy solution, non-local boundary conditions, Leray-Lions operator, bounded Radon diffuse measure, Marcinkiewicz spaces.

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1 Introduction and assumptions

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ such that $\partial\Omega$ is Lipschitz and $\partial\Omega = \Gamma_D \cup \Gamma_{Ne}$ with $\Gamma_D \cap \Gamma_{Ne} = \emptyset$. Our aim is to study the following problem.

$$P(\rho,\mu,d) \begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial}{\partial x_i} u\right) + |u|^{p_M(x)-2}u = \mu & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \left.\begin{array}{l} \rho(u) + \displaystyle\sum_{i=1}^{N} \int_{\Gamma_{Ne}} a_i\left(x, \frac{\partial}{\partial x_i} u\right) \eta_i = d \\ u \equiv \mathit{constant} \end{array}\right\} & \text{on } \Gamma_{Ne}, \end{cases} \tag{1.1}$$

where the right-hand side μ is a bounded Radon diffuse measure (that is μ does not charge the sets of zero $p_m(.)$-capacity), $\rho : \mathbb{R} \to \mathbb{R}$ a surjective, continuous and non-decreasing function, with $\rho(0) = 0$, $d \in \mathbb{R}$ and $\eta_i,\ i \in \{1, ..., N\}$ are the components of the outer normal unit vector.
For any $\Omega \subset \mathbb{R}^N$, we set

$$C_+(\bar{\Omega}) = \{h \in C(\bar{\Omega}) : \inf_{x \in \Omega} h(x) > 1\} \tag{1.2}$$

and we denote

$$h^+ = \sup_{x\in\Omega} h(x), \qquad h^- = \inf_{x\in\Omega} h(x). \tag{1.3}$$

For the exponents, $\vec{p}(.) : \bar{\Omega} \to \mathbb{R}^N$, $\vec{p}(.) = (p_1(.), ..., p_N(.))$ with $p_i \in C_+(\bar{\Omega})$ for every $i \in \{1, ..., N\}$ and for all $x \in \bar{\Omega}$. We put $p_M(x) = \max\{p_1(x), ..., p_N(x)\}$ and $p_m(x) = \min\{p_1(x), ..., p_N(x)\}$.
We assume that for $i = 1, ..., N$, the function $a_i : \Omega \times \mathbb{R} \to \mathbb{R}$ is Carathéodory and satisfies the following conditions.

- (H_1): $a_i(x,\xi)$ is the continuous derivative with respect to ξ of the mapping $A_i = A_i(x,\xi)$, that is, $a_i(x,\xi) = \frac{\partial}{\partial \xi} A_i(x,\xi)$ such that the following equality holds.

$$A_i(x, 0) = 0, \tag{1.4}$$

  for almost every $x \in \Omega$.

- (H_2) : There exists a positive constant C_1 such that

$$|a_i(x,\xi)| \leq C_1(j_i(x) + |\xi|^{p_i(x)-1}), \tag{1.5}$$

  for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}$, where j_i is a non-negative function in $L^{p_i'(.)}(\Omega)$, with $\frac{1}{p_i(x)} + \frac{1}{p_i'(x)} = 1$.

- (H_3) : there exists a positive constant C_2 such that

$$(a_i(x,\xi)-a_i(x,\eta)).(\xi-\eta) \geq \begin{cases} C_2|\xi-\eta|^{p_i(x)} & \text{if } |\xi-\eta|\geq 1, \\ C_2|\xi-\eta|^{p_i^-} & \text{if } |\xi-\eta|<1, \end{cases} \tag{1.6}$$

for almost every $x \in \Omega$ and for every ξ, $\eta \in \mathbb{R}$, with $\xi \neq \eta$.

- (H_4) : For almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}$,

$$|\xi|^{p_i(x)} \leq a_i(x,\xi).\xi \leq p_i(x)A_i(x,\xi). \tag{1.7}$$

- (H_5) : The variable exponents $p_i(.) : \bar{\Omega} \to [2,N)$ are continuous functions for all $i=1,...,N$ such that

$$\frac{\bar{p}(N-1)}{N(\bar{p}-1)} < p_i^- < \frac{\bar{p}(N-1)}{N-\bar{p}}, \ \sum_{i=1}^{N}\frac{1}{p_i^-} > 1 \text{ and } \frac{p_i^+ - p_i^- - 1}{p_i^-} < \frac{\bar{p}-N}{\bar{p}(N-1)}, \tag{1.8}$$

where $\dfrac{1}{\bar{p}} = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N}\frac{1}{p_i^-}$.

As examples under assumptions (H_1) -(H_5), we can give the following.

(1) Set $A_i(x,\xi) = (\frac{1}{p_i(x)})|\xi|^{p_i(x)}$ and $a_i(x,\xi) = |\xi|^{p_i(x)-2}\xi$, where $2 \leq p_i(x) < N$.

(2) $A_i(x,\xi) = (\frac{1}{p_i(x)})((1+|\xi|^2)^{\frac{p_i(x)}{2}}-1)$ and $a_i(x,\xi) = (1+|\xi|^2)^{\frac{p_i(x)-2}{2}}\xi$, where $2 \leq p_i(x) < N$.

We put for all $x \in \partial\Omega$,

$$p^{\partial}(x) = \begin{cases} \dfrac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

We introduce the numbers

$$q = \frac{N(\bar{p}-1)}{N-1}, \ q^* = \frac{Nq}{N-q} = \frac{N(\bar{p}-1)}{N-\bar{p}}. \tag{1.9}$$

We denote by $\mathcal{M}_b(\Omega)$ the space of bounded Radon measure in Ω, equipped with its standard norm $\|.\|_{\mathcal{M}_b(\Omega)}$. Note that, if u belongs to $\mathcal{M}_b(\Omega)$, then $|\mu|(\Omega)$ (the total variation of μ) is a bounded positive measure on Ω.
Given $\mu \in \mathcal{M}_b(\Omega)$, we say that μ is diffuse with respect to the capacity $W_0^{1,p(.)}(\Omega)$ ($p(.)$-capacity for short) if $\mu(A) = 0$, for every set A such that $\mathrm{Cap}_{p(.)}(A,\Omega) = 0$.
For every $A \subset \Omega$, we denote

$$S_{p(.)}(A) = \{u \in W_0^{1,p(.)}(\Omega) \cap C_0(\Omega) : u = 1 \text{ on } A, u \geq 0 \text{ on } \Omega\}.$$

The $p(.)$-capacity of every subset A with respect to Ω is defined by

$$\mathrm{Cap}_{p(.)}(A,\Omega) = \inf_{u\in S_{p(.)}(A)}\{\int_\Omega |\nabla u|^{p(x)}dx\}.$$

In the case $S_{p(.)}(A)=\emptyset$, we set $\mathrm{Cap}_{p(.)}(A,\Omega)=\infty$.
The set of bounded Radon diffuse measure in the variable exponent setting is denoted by $\mathcal{M}_b^{p(.)}(\Omega)$. We use the following result of decomposition of bounded Radon diffuse measure proved by Nyanquini et *al.* (see [31]).

Theorem 1.1. *Let $p(.) : \bar{\Omega} \to (1,\infty)$ be a continuous function and $\mu \in \mathcal{M}_b(\Omega)$. Then $\mu \in \mathcal{M}_b^{p(.)}(\Omega)$ if and only if $\mu \in L^1(\Omega) + W^{-1,p'(.)}(\Omega)$.*

Remark 1.2. *Since $\mu \in \mathcal{M}_b^{p_m(.)}(\Omega)$, the Theorem 1.1 implies that there exist $f \in L^1(\Omega)$ and $F \in (L^{p'_m(.)}(\Omega))^N$ such that*

$$\mu = f - divF, \tag{1.10}$$

where $\dfrac{1}{p_m(x)} + \dfrac{1}{p'_m(x)} = 1,\ \forall x \in \Omega.$

The study of nonlinear elliptic equations involving the p-Laplace operator is based on the theory of standard Sobolev spaces $W^{m,p}(\Omega)$ in order to find weak solutions. For the nonhomogeneous $p(.)$-Laplace operators, the natural setting for this approach is the use of the variable exponent Lebesgue and Sobolev spaces $L^{p(.)}(\Omega)$ and $W^{m,p(.)}(\Omega)$.
Variable exponent Lebesgue spaces appeared in the literature for the first time in a article by Orlicz in 1931. In the 1950's, this study was carred on by Nakano who made the first systematic study of spaces with variable exponent (called modular spaces). Nakano explicitly mentioned variable exponent Lebesgue spaces as an example of more general spaces he considered (see [30], p. 284). Later, the polish mathematicians investigated the modular function spaces (see [29]). Note also that H. Hudzik [18] investigated the variable exponent Sobolev spaces. Variable exponent Lebesgue spaces on the real line have been independently developed by Russian researchers, notably Sharapudinov [40] and Tsenov [42]. The next major step in the investigation of variable exponent Lebesgue and Sobolev spaces was the comprehensive paper by O. Kovacik and J. Rakosnik in the early 90's [23]. This paper established many of basic properties of Lebesgue and Sobolev spaces with variables exponent. Variable Sobolev spaces have been used in the last decades to model various phenomena. In [9], Chen, Levine and Rao proposed a framework for image restoration based on a Laplacian variable exponent. Another application which uses nonhomogeneous Laplace operators is related to the modelling of electrorheological fluids see [38]. The first major discovery in electrorheological fluids was due to Winslow in 1949 (cf. [43]). These fluids have the interesting property that their viscosity depends on the electric field in the fluid. They can raise the viscosity by as much as five orders of magnitude. This phenomenon is known as the Winslow effect. For some technical applications, we refer the readers to the work by Pfeiffer et *al* [33]. Electrorheological fluids have been used in robotics and space technology. The experimental research has been done mainly in

the USA, for instance in NASA laboratories. For more information on properties, modelling and the application of variable exponent spaces to these fluids, we refer to Diening [11], Rajagopal and Ruzicka [35], and Ruzicka [36]. In this paper, the operator involved in (1.1) is more general than the $p(.)$-Laplace operator. Thus, the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is not adequate to study nonlinear problems of this type. This leads us to seek entropy solutions for problems (1.1) in a more general variable exponent Sobolev space which was introduced for the first time by Mihaïlescu et *al.* [28], see also [34, 26, 27].

The need for such theory comes naturally every time we want to consider materials with inhomogeneities that have different behavior on different space directions. Non-local boundary value problems of various kinds for partial differential equations are of great interest by now in several fields of application. In a typical non-local problem, the partial differential equation (resp. boundary conditions) for an unknown function u at any point in a domain Ω involves not only the local behavior of u in a neighborhood of that point but also the non-local behavior of u elsewhere in Ω. For example, at any point in Ω the partial differential equation and/or the boundary conditions may contains integrals of the unknown u over parts of Ω, values of u elsewhere in D or, generally speaking, some non-local operator on u. Beside the mathematical interest of nonlocal conditions, it seems that this type of boundary condition appears in petroleum engineering model for well modeling in a $3D$ stratified petroleum reservoir with arbitrary geometry (see [12] and [15]). A lot of papers (see [34], [24], [25], [2], [19], [1]) on problems like (1.1) considered cases of generally boundary value condition. In [6], Bonzi et al. studied the following problems.

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial}{\partial x_i} u\right) + |u|^{p_M(x)-2}u = f & \text{in } \Omega \\ \displaystyle\sum_{i=1}^{N} a_i\left(x, \frac{\partial}{\partial x_i} u\right)\eta_i = -|u|^{r(x)-2}u & \text{on } \partial\Omega, \end{cases} \tag{1.11}$$

which correspond to the Robin type boundary condition. The authors used minimization techniques used in [8] to prove the existence and uniqueness of entropy solution. By the same techniques, Koné and *al.* proved the existence and uniqueness of entropy solution for the following problem.

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial}{\partial x_i} u\right) + |u|^{p_M(x)-2}u = f & \text{in } \Omega \\ \displaystyle\sum_{i=1}^{N} a_i\left(x, \frac{\partial}{\partial x_i} u\right)\eta_i + \lambda u = g & \text{on } \partial\Omega, \end{cases} \tag{1.12}$$

which correspond to the Fourier type boundary condition.

In a recent paper we studied a nonlinear elliptic anisotropic problem involving non- local conditions. We also considered variable exponent and general maximal monotone graph datum at the boundary and proved existence and uniqueness of weak solution to the following problem.

$$S(\rho,\mu,d)\begin{cases} -\displaystyle\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i\left(x,\frac{\partial}{\partial x_i}u\right)+|u|^{p_M(x)-2}u=f & \text{in } \Omega\\ u=0 & \text{on } \Gamma_D\\ \left.\begin{array}{l}\rho(u)+\displaystyle\sum_{i=1}^{N}\int_{\Gamma_{Ne}}a_i\left(x,\frac{\partial}{\partial x_i}u\right)\eta_i\ni d\\ u\equiv constant\end{array}\right\} & \text{on } \Gamma_{Ne},\end{cases}$$

where the right-hand side $f\in L^{\infty}(\Omega)$ and ρ a maximal monotone graph on $\mathbb{R}$ such that $D(\rho)=Im(\rho)=\mathbb{R}$ and $0\in\rho(0)$, $d\in\mathbb{R}$, by using the technique of monotone operators in Banach spaces (see [21]) and approximation methods. There are two difficulties associated with the study of problem $P(\rho,\mu,d)$. The first is to give a sense to the partial derivative of u which appear in the term $a_i\left(x,\frac{\partial}{\partial x_i}u\right)$. As μ is a measure (even if μ is a integrable function), then we cannot take the partial derivative of u in the usual distribution sense. The idea consists in considering troncatures of the solution u (see [5]). The second difficulty appears with the question of uniqueness of solutons. We obtain existence and uniqueness of a special class of solutions of problem $P(\rho,\mu,d)$ that satisfy an extra condition that we call the entropy condition (see formula (2.9)). An alternative notion of solution which can leads to existence and uniqueness of solution to problem $P(\rho,\mu,d)$ is the notion of renormalized solution. But in this work, we consider the notion of entropy solution.

The paper is organized as follows. Section 2 is devoted to mathematical preliminaries including, among other things, a brief discussion on variable exponent Lebesgue, Sobolev, anisotropic and Marcinkiewicz spaces. In Section 3, we study an approximated problem and in Section 4, we prove by using the results of the Section 3, the existence and uniqueness of entropy solution of problem $P(\rho,\mu,d)$.

2 Preliminary

This part is related to anisotropic Lebesgue and Sobolev spaces with variable exponent and some of their properties.

Given a measurable function $p(.):\Omega\to[1,\infty)$. We define the Lebesgue space with variable exponent $L^{p(.)}(\Omega)$ as the set of all measurable functions $u:\Omega\to\mathbb{R}$ for which the convex modular

$$\rho_{p(.)}(u):=\int_{\Omega}|u|^{p(x)}dx$$

is finite.

If the exponent is bounded, i.e, if $p_+<\infty$, then the expression

$$|u|_{p(.)}:=\inf\left\{\lambda>0:\rho_{p(.)}(\frac{u}{\lambda})\leq 1\right\}$$

defines a norm in $L^{p(.)}(\Omega)$, called the Luxembourg norm. The space $(L^{p(.)}(\Omega), |.|_{p(.)})$ is a separable Banach space. Then, $L^{p(.)}(\Omega)$ is uniformly convex, hence reflexive and its dual space is isomorphic to $L^{p'(.)}(\Omega)$, where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$, for all $x \in \Omega$. We have the following properties (see [13]) on the modular $\rho_{p(.)}$.
If $u, u_n \in L^{p(.)}(\Omega)$ and $p_+ < \infty$, then

$$|u|_{p(.)} < 1 \Rightarrow |u|_{p(.)}^{p^+} \leq \rho_{p(.)}(u) \leq |u|_{p(.)}^{p^-}, \tag{2.1}$$

$$|u|_{p(.)} > 1 \Rightarrow |u|_{p(.)}^{p^-} \leq \rho_{p(.)}(u) \leq |u|_{p(.)}^{p^+}, \tag{2.2}$$

$$|u|_{p(.)} < 1 (= 1; > 1) \Rightarrow \rho_{p(.)}(u) < 1 (= 1; > 1), \tag{2.3}$$

and

$$|u_n|_{p(.)} \to 0 \ (|u_n|_{p(.)} \to \infty) \Leftrightarrow \rho_{p(.)}(u_n) \to 0 \ (\rho_{p(.)}(u_n) \to \infty). \tag{2.4}$$

If in addition, $(u_n)_{n\in\mathbb{N}} \subset L^{p(.)}(\Omega)$, then $\lim_{n\to\infty} |u_n - u|_{p(.)} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{p(.)}(u_n - u) = 0 \Leftrightarrow (u_n)_{n\in\mathbb{N}}$ converges to u in measure and $\lim_{n\to\infty} \rho_{p(.)}(u_n) = \rho_{p(.)}(u)$.
We introduce the definition of the isotropic Sobolev space with variable exponent,

$$W^{1,p(.)}(\Omega) := \left\{ u \in L^{p(.)}(\Omega) : |\nabla u| \in L^{p(.)}(\Omega) \right\},$$

which is a Banach space equipped with the norm

$$\|u\|_{1,p(.)} := |u|_{p(.)} + |\nabla u|_{p(.)}.$$

Now, we present the anisotropic Sobolev space with variable exponent which is used for the study of $P(\rho, \mu, d)$.
The anisotropic variable exponent Sobolev space $W^{1,\vec{p}(.)}(\Omega)$ is defined as follow.

$$W^{1,\vec{p}(.)}(\Omega) := \left\{ u \in L^{p_M(.)}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i(.)}(\Omega), \text{ for all } i \in \{1, ..., N\} \right\}.$$

Endowed with the norm

$$\|u\|_{\vec{p}(.)} := |u|_{p_M(.)} + \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|_{p_i(.)},$$

the space $\left(W^{1,\vec{p}(.)}(\Omega), \|.\|_{\vec{p}(.)}\right)$ is a reflexive Banach space (see [14], Theorem 2.1 and Theorem 2.2). As consequence, we have the following.

Theorem 2.1. *(see [14]) Let $\Omega \subset \mathbb{R}^{\mathbb{N}}$ $(N \geq 3)$ be a bounded open set and for all $i \in \{1, ..., N\}$, $p_i \in L^\infty(\Omega)$, $p_i(x) \geq 1$ a.e. in Ω. Then, for any $r \in L^\infty(\Omega)$ with $r(x) \geq 1$ a.e. in Ω such that*

$$ess \inf_{x\in\Omega} (p_M(x) - r(x)) > 0,$$

we have the compact embedding

$$W^{1,\vec{p}(.)}(\Omega) \hookrightarrow L^{r(.)}(\Omega).$$

We also need the following trace theorem due to [7].

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^{\mathbb{N}}$ $(N \geq 2)$ be a bounded open set with smooth boundary and let $\vec{p}(.) \in C(\bar{\Omega})$ satisfy the condition*

$$1 \leq r(x) < \min_{x \in \partial\Omega} \{p_1^{\partial}(x), ..., p_N^{\partial}(x)\}, \ \forall x \in \partial\Omega. \tag{2.5}$$

Then, there is a compact boundary trace embedding

$$W^{1,\vec{p}(.)}(\Omega) \hookrightarrow L^{r(.)}(\partial\Omega).$$

Let us introduce the following notation:

$$\vec{p}_- = (p_1^-, ..., p_N^-).$$

We will use in this paper, the Marcinkiewicz spaces $\mathcal{M}^q(\Omega)$ $(1 < q < \infty)$ with constant exponent. Note that the Marcinkiewicz spaces $\mathcal{M}^{q(.)}(\Omega)$ in the variable exponent setting was introduced for the first time by Sanchon and Urbano (see [37]).
Marcinkiewicz spaces $\mathcal{M}^q(\Omega)$ $(1 < q < \infty)$ contain all measurable function $h : \Omega \to \mathbb{R}$ for which the distribution function

$$\lambda_h(\gamma) := \text{meas}(\{x \in \Omega : |h(x)| > \gamma\}), \ \gamma \geq 0,$$

satisfies an estimate of the form $\lambda_h(\gamma) \leq C\gamma^{-q}$, for some finite constant $C > 0$.
The space $\mathcal{M}^q(\Omega)$ is a Banach space under the norm

$$\|h\|^*_{\mathcal{M}^q(\Omega)} = \sup_{t>0} t^{\frac{1}{q}} \left(\frac{1}{t} \int_0^t h^*(s)ds \right),$$

where h^* denotes the nonincreasing rearrangement of h.

$$h^*(t) := \inf \left\{ C : \lambda_h(\gamma) \leq C\gamma^{-q}, \ \forall \gamma > 0 \right\},$$

which is equivalent to the norm $\|h\|^*_{\mathcal{M}^q(\Omega)}$ (see [3]).
We need the following Lemma (see [4], Lemma A-2).

Lemma 2.3. *Let $1 \leq q < p < \infty$. Then, for every measurable function u on Ω,*

(i) $\dfrac{(p-1)^p}{p^{p+1}} \|u\|^p_{\mathcal{M}^p(\Omega)} \leq \sup\limits_{\lambda>0} \{\lambda^p meas[x \in \Omega : |u| > \lambda]\} \leq \|u\|^p_{\mathcal{M}^p(\Omega)}$.
Moreover,

(ii) $\displaystyle\int_K |u|^q dx \leq \frac{p}{p-q} (\frac{p}{q})^{\frac{q}{p}} \|u\|^q_{\mathcal{M}^p(\Omega)} (meas(K))^{\frac{p-q}{p}}$, *for every measurable subset $K \subset \Omega$.*

In particular, $\mathcal{M}^p(\Omega) \subset L^q_{loc}(\Omega)$, with continuous embedding and $u \in \mathcal{M}^p(\Omega)$ implies $|u|^q \in \mathcal{M}^{\frac{p}{q}}(\Omega)$.

The following result is due to Troisi (see [39]).

Theorem 2.4. *Let* $p_1, ..., p_N \in [1, \infty)$, $\vec{p} = (p_1, ..., p_N)$*;* $g \in W^{1,\vec{p}}(\Omega)$*, and let*

$$\begin{cases} q = \bar{p}^* & \text{if} \quad \bar{p}^* < N, \\ q \in [1, \infty) & \text{if} \quad \bar{p}^* \geq N; \end{cases} \tag{2.6}$$

where $p^* = \dfrac{N}{\sum_{i=1}^N \frac{1}{p_i} - 1}$, $\sum_{i=1}^N \frac{1}{p_i} > 1$ *and* $\bar{p}^* = \dfrac{N\bar{p}}{N - \bar{p}}$.
Then, there exists a constant $C > 0$ *depending on* N, $p_1, ..., p_N$ *if* $\bar{p} < N$ *and also on* q *and* $meas(\Omega)$ *if* $\bar{p} \geq N$ *such that*

$$\|g\|_{L^q(\Omega)} \leq c \prod_{i=1}^N \left[\|g\|_{L^{p_M}(\Omega)} + \|\frac{\partial g}{\partial x_i}\|_{L^{p_i}(\Omega)} \right]^{\frac{1}{N}}, \tag{2.7}$$

where $p_M = \max\{p_1, ..., p_N\}$ *and* $\frac{1}{\bar{p}} = \frac{1}{N}\sum_{i=1}^N \frac{1}{p_i}$*. In particular, if* $u \in W_0^{1,\vec{p}}(\Omega)$*, we have*

$$\|g\|_{L^q(\Omega)} \leq c \prod_{i=1}^N \left[\left\| \frac{\partial g}{\partial x_i} \right\|_{L^{p_i}(\Omega)} \right]^{\frac{1}{N}}. \tag{2.8}$$

In the sequel, we consider the following spaces.

$$W_D^{1,\vec{p}(.)}(\Omega) = \{\xi \in W^{1,\vec{p}(.)}(\Omega) \ : \ \xi = 0 \text{ on } \Gamma_D\}$$

and

$$W_{Ne}^{1,\vec{p}(.)}(\Omega) = \{\xi \in W_D^{1,\vec{p}(.)}(\Omega) \ : \ \xi \equiv \text{constant on } \Gamma_{Ne}\}.$$

$$\mathcal{T}_D^{1,\vec{p}(.)}(\Omega) = \{\xi \text{ measurable on } \Omega \text{ such that } \forall k > 0, \ T_k(\xi) \in \ W_D^{1,\vec{p}(.)}(\Omega)\}$$

and

$$\mathcal{T}_{Ne}^{1,\vec{p}(.)}(\Omega) = \{\xi \text{ measurable on } \Omega \text{ such that } \forall k > 0, \ T_k(\xi) \in \ W_{Ne}^{1,\vec{p}(.)}(\Omega)\},$$

where T_k is a truncation function defined by

$$T_k(s) = \begin{cases} k & \text{if } s > k, \\ s & \text{if } |s| \leq k, \\ -k & \text{if } s < -k. \end{cases}$$

For any $v \in W_{Ne}^{1,\vec{p}(.)}(\Omega)$, we set $v_N = v_{Ne} := v|_{\Gamma_{Ne}}$.

Definition 2.5. *A measurable function* $u : \Omega \to \mathbb{R}$ *is an entropy solution of* $P(\rho, \mu, d)$ *if* $u \in \mathcal{T}_{Ne}^{1,\vec{p}(.)}(\Omega)$ *and for every* $k > 0$,

$$\begin{cases} \displaystyle\int_\Omega \left(\sum_{i=1}^N a_i \left(x, \frac{\partial}{\partial x_i} u \right) \frac{\partial}{\partial x_i} T_k(u - \xi) \right) dx + \int_\Omega |u|^{p_M(x)-2} u T_k(u - \xi) dx \leq \\ \displaystyle\int_\Omega T_k(u - \xi) d\mu + (d - \rho(u_{Ne})) T_k(u_{Ne} - \xi), \end{cases} \tag{2.9}$$

for all $\xi \in W_{Ne}^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$.

Our main result in this paper is the following theorem.

Theorem 2.6. *Assume* (H_1)-(H_5)*. Then for any* $(\mu, d) \in \mathcal{M}_b^{p_m(\cdot)}(\Omega) \times \mathbb{R}$*, the problem* $P(\rho, \mu, d)$ *admits a unique entropy solution* u*.*

3 The approximated problem corresponding to $P(\rho, \mu, d)$

We define a new bounded domain $\tilde{\Omega}$ in $\mathbb{R}^N$ as follow.
We fix $\theta > 0$ and we set $\tilde{\Omega} = \Omega \cup \{x \in \mathbb{R}^N / dist(x, \Gamma_{Ne}) < \theta\}$. Then, $\partial\tilde{\Omega} = \Gamma_D \cup \tilde{\Gamma}_{Ne}$ is Lipschitz with $\Gamma_D \cap \tilde{\Gamma}_{Ne} = \emptyset$.

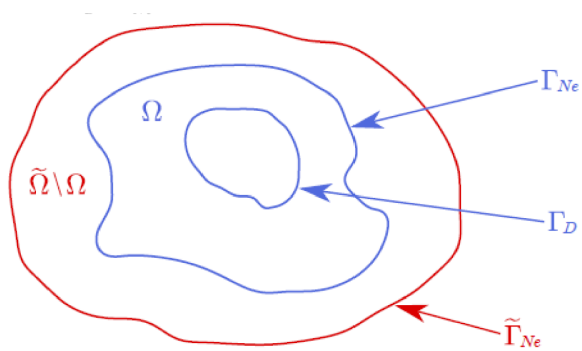

Figure 1: Domains representation

Let us consider $\tilde{a}_i(x, \xi)$ (to be defined later) Carathéodory and satisfying (1.4), (1.5), (1.6) and (1.7), for all $x \in \tilde{\Omega}$.
We also consider a function $\tilde{d}$ in $L^1(\tilde{\Gamma}_{Ne})$ such that

$$\int_{\tilde{\Gamma}_{Ne}} \tilde{d} d\sigma = d. \tag{3.1}$$

For any $\epsilon > 0$, we set $\mu_\epsilon = f_\epsilon - \text{div} F$, where $f_\epsilon = T_{\frac{1}{\epsilon}}(f) \in L^\infty(\Omega)$. Note that $f_\epsilon \to f$ as $\epsilon \to 0$ in $L^1(\Omega)$ and $\|f_\epsilon\|_1 \leq \|f\|_1$.

We set $\tilde{\mu}_\epsilon = f_\epsilon \chi_\Omega - \text{div} F \chi_\Omega$, $\tilde{d}_\epsilon = T_{\frac{1}{\epsilon}}(\tilde{d})$ and we consider the problem

$$P(\tilde{\rho}, \tilde{\mu}_\epsilon, \tilde{d}_\epsilon) \begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \tilde{a}_i(x, \frac{\partial}{\partial x_i} u_\epsilon) + |u_\epsilon|^{p_M(x)-2} u_\epsilon \chi_\Omega(x) = \tilde{\mu}_\epsilon & \text{in } \tilde{\Omega} \\ u_\epsilon = 0 & \text{on } \Gamma_D \\ \tilde{\rho}(u_\epsilon) + \displaystyle\sum_{i=1}^{N} \tilde{a}_i(x, \frac{\partial}{\partial x_i} u_\epsilon) \eta_i = \tilde{d}_\epsilon & \text{on } \tilde{\Gamma}_{Ne}, \end{cases} \tag{3.2}$$

where the function $\tilde{\rho}$ is defined as follow.

- $\tilde{\rho}(s) = \dfrac{1}{|\tilde{\Gamma}_{Ne}|}\rho(s)$, where $|\tilde{\Gamma}_{Ne}|$ denotes the Hausdorff measure of $\tilde{\Gamma}_{Ne}$.

We obviously have $\forall \epsilon > 0$, $\tilde{d}_\epsilon \in L^\infty(\tilde{\Gamma}_{Ne})$.
The following definition gives the notion of solution for the problem $P_\epsilon(\tilde{\rho}, \tilde{\mu}_\epsilon, \tilde{d}_\epsilon)$.

Definition 3.1. *A measurable function $u_\epsilon : \tilde{\Omega} \to \mathbb{R}$ is a solution to problem $P_\epsilon(\tilde{\rho}, \tilde{\mu}_\epsilon, \tilde{d}_\epsilon)$ if $u_\epsilon \in W_D^{1,\vec{p}(.)}(\tilde{\Omega})$ and*

$$\int_{\tilde{\Omega}} \sum_{i=1}^{N} \tilde{a}_i(x, \frac{\partial}{\partial x_i} u_\epsilon) \frac{\partial}{\partial x_i} \tilde{\xi} dx + \int_\Omega |u_\epsilon|^{p_M(x)-2} u_\epsilon \tilde{\xi} dx = \int_\Omega f_\epsilon \tilde{\xi} dx + \int_\Omega F.\nabla \tilde{\xi} + \int_{\tilde{\Gamma}_{Ne}} (\tilde{d}_\epsilon - \tilde{\rho}(u_\epsilon)) \tilde{\xi} d\sigma, \tag{3.3}$$

for any $\tilde{\xi} \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \cap L^\infty(\Omega)$.

Theorem 3.2. *The problem $P_\epsilon(\tilde{\rho}, \tilde{\mu}_\epsilon, \tilde{d}_\epsilon)$ admits at least one solution in the sense of Definition 3.1.*

Step 1: Approximated problem we study an existence result to the following problem. For any $k > 0$ we consider

$$P_{\epsilon,k}(\tilde{\rho}, \tilde{\mu}_\epsilon, \tilde{d}_\epsilon) \begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \tilde{a}_i(x, \frac{\partial}{\partial x_i} u_{\epsilon,k}) + T_k(b(u_{\epsilon,k}))\chi_\Omega(x) = \tilde{\mu}_\epsilon & \text{in } \tilde{\Omega} \\ u_{\epsilon,k} = 0 & \text{on } \Gamma_D \\ T_k(\tilde{\rho}(u_{\epsilon,k})) + \displaystyle\sum_{i=1}^{N} \tilde{a}_i(x, \frac{\partial}{\partial x_i} u_{\epsilon,k}) \eta_i = \tilde{d}_\epsilon & \text{on } \tilde{\Gamma}_{Ne}, \end{cases} \tag{3.4}$$

where $b(u) = |u|^{p_M(x)-2} u$.
We have to prove that $P_{\epsilon,k}(\tilde{\rho}, \tilde{\mu}_\epsilon, \tilde{d}_\epsilon)$ admits at least one solution in the following sense.

$$\begin{cases} u_{\epsilon,k} \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \text{ and for all } \tilde{\xi} \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}), \\ \displaystyle\int_{\tilde{\Omega}} \sum_{i=1}^{N} \tilde{a}_i(x, \frac{\partial}{\partial x_i} u_{\epsilon,k}) \frac{\partial}{\partial x_i} \tilde{\xi} dx + \int_\Omega T_k(b(u_{\epsilon,k})) \tilde{\xi} dx = \int_\Omega \tilde{\xi} d\mu_\epsilon + \int_{\tilde{\Gamma}_{Ne}} (\tilde{d}_\epsilon - T_k(\tilde{\rho}(u_{\epsilon,k}))) \tilde{\xi} d\sigma. \end{cases} \tag{3.5}$$

For any $k > 0$, let us introduce the operator $\Lambda_k : W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \to (W_D^{1,\vec{p}(.)}(\tilde{\Omega}))'$ such that for any $(u, v) \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \times W_D^{1,\vec{p}(.)}(\tilde{\Omega})$,

$$\langle \Lambda_k(u), v \rangle = \int_{\tilde{\Omega}} \left(\sum_{i=1}^{N} \tilde{a}_i(x, \frac{\partial}{\partial x_i} u) \frac{\partial}{\partial x_i} v \right) dx + \int_\Omega T_k(b(u)) v dx + \int_{\tilde{\Gamma}_{Ne}} T_k(\tilde{\rho}(u)) v d\sigma. \tag{3.6}$$

We need to prove that for any $k > 0$, the operator Λ_k is bounded, coercive, of type M and therefore, surjective.

(i) Boundedness of Λ_k. Let $(u, v) \in F \times W_D^{1,\vec{p}(.)}(\tilde{\Omega})$ with F a bounded subset of $W_D^{1,\vec{p}(.)}(\tilde{\Omega})$.

We have

$$\begin{cases} |\langle \Lambda_k(u), v\rangle| \leq \sum_{i=1}^{N} \left(\int_{\tilde{\Omega}} \left|\tilde{a}_i(x, \frac{\partial}{\partial x_i}u)\right| \left|\frac{\partial}{\partial x_i}v\right| dx \right) + \int_{\tilde{\Omega}} |T_k(b(u))||v|dx + \int_{\tilde{\Gamma}_{Ne}} |T_k(\tilde{\rho}(u))||v|d\sigma \\ = I_1 + I_2 + I_3, \end{cases}$$

where we denote by I_1, I_2 and I_3 the three terms on the right hand side of the first inequality. By (H_2) and the Hölder type inequality, we have

$$\begin{cases} I_1 \leq C_1 \sum_{i=1}^{N} \left(\int_{\tilde{\Omega}} |j_i(x)| \left|\frac{\partial}{\partial x_i}v\right| dx + \int_{\tilde{\Omega}} \left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)-1} \left|\frac{\partial}{\partial x_i}v\right| dx \right) \\ \leq C_1 \sum_{i=1}^{N} \left(\frac{1}{p_i'^-} + \frac{1}{p_i^-}\right) |j_i|_{p_i'(.)} \left|\frac{\partial}{\partial x_i}v\right|_{p_i(.)} + \sum_{i=1}^{N} \left(\frac{1}{p_i'^-} + \frac{1}{p_i^-}\right) \left|\left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)-1}\right|_{p_i'(.)} \left|\frac{\partial}{\partial x_i}v\right|_{p_i(.)}. \end{cases}$$

As $u \in F$, $\forall\, i \in \{1, ..., N\}$, there exists a constant $M > 0$ such that

$$\sum_{i=1}^{N} \left|\left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)-1}\right|_{p_i'(.)} < M;$$

so

$$\left|\left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)-1}\right|_{p_i'(.)} < M, \ \forall\, i \in \{1, ..., N\}.$$

Let $C_4 = \max_{i=1,\dots,N} \left\{ \left|\left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)-1}\right|_{p_i'(.)} \right\}$.
As $j_i \in L^{p_i'(.)}(\tilde{\Omega})$, we have

$$I_1 \leq C_5(C_1, p_i^-, (p_i')^-, C_3(j_i)) \sum_{i=1}^{N} \left|\frac{\partial}{\partial x_i}v\right|_{p_i(.)} + C_6(C_1, p_i^-, (p_i')^-, C_4) \sum_{i=1}^{N} \left|\frac{\partial}{\partial x_i}v\right|_{p_i(.)}.$$

It is easy to see that

$$I_2 \leq k \int_{\tilde{\Omega}} |v|dx.$$

Using Theorem 2.1, we have

$$\|v\|_{L^1(\tilde{\Omega})} \leq C_7 \|v\|_{W_D^{1,\vec{p}(.)}(\tilde{\Omega})}.$$

So,

$$I_2 \leq kC_7 \|v\|_{W_D^{1,\vec{p}(.)}(\tilde{\Omega})}.$$

Similarly, by using Theorem 2.2, we have

$$I_3 \leq kC_8 \|v\|_{W_D^{1,\vec{p}(.)}(\tilde{\Omega})} \square$$

Therefore, Λ_k maps bounded subsets of $W_D^{1,\vec{p}(.)}(\tilde{\Omega})$ into bounded subsets of $(W_D^{1,\vec{p}(.)}(\tilde{\Omega}))'$. Thus, Λ_k is bounded on $W_D^{1,\vec{p}(.)}(\tilde{\Omega})$.

(ii) Coerciveness of Λ_k. We have to show that for any $k > 0$, $\dfrac{\langle \Lambda_k(u), u\rangle}{\|u\|_{W_D^{1,\vec{p}(.)}(\tilde{\Omega})}} \to \infty$ as $\|u\|_{W_D^{1,\vec{p}(.)}(\tilde{\Omega})} \to \infty$.

For any $u \in W_D^{1,\vec{p}(.)}(\tilde{\Omega})$, we have

$$\langle \Lambda_k(u), u\rangle = \langle \Lambda(u), u\rangle + \int_\Omega T_k(b(u))u dx + \int_{\tilde{\Gamma}_{Ne}} T_k(\tilde{\rho}(u))u d\sigma, \tag{3.7}$$

where $\langle \Lambda(u), u\rangle = \displaystyle\sum_{i=1}^{N}\left(\int_{\tilde{\Omega}} \tilde{a}_i(x, \frac{\partial}{\partial x_i}u)\frac{\partial}{\partial x_i}u dx\right)$.

The last two terms on the right-hand side of (3.7) are non-negative by the monotonicity of T_k, b and $\tilde{\rho}$. We can assert that

$$\begin{cases} \langle \Lambda_k(u), u\rangle \geq \langle \Lambda(u), u\rangle \\ \geq \dfrac{1}{N^{p_m^- - 1}}\|u\|^{p_m^-}_{W_D^{1,\vec{p}(.)}(\tilde{\Omega})} - N. \end{cases}$$

Indeed, since $\displaystyle\int_{\tilde{\Omega}} |T_k(b(u))||u|dx + \int_{\tilde{\Gamma}_{Ne}} |T_k(\tilde{\rho}(u))||u|d\sigma \geq 0$, for all $u \in W_D^{1,\vec{p}(.)}(\tilde{\Omega})$, we have

$$\langle \Lambda_k(u), u\rangle \geq \langle \Lambda(u), u\rangle.$$

So,

$$\langle \Lambda_k(u), u\rangle \;\geq\; \sum_{i=1}^{N}\left(\int_{\tilde{\Omega}} \tilde{a}_i(x, \frac{\partial}{\partial x_i}u)\frac{\partial}{\partial x_i}u dx\right) \geq \sum_{i=1}^{N}\left(\int_{\tilde{\Omega}} \left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)} dx\right).$$

We make the following notations:

$$\mathcal{I} = \left\{ i \in \{1, ..., N\} : \left|\frac{\partial}{\partial x_i}u\right|_{p_i(.)} \leq 1 \right\} \text{ and } \mathcal{J} = \left\{ i \in \{1, ..., N\} : \left|\frac{\partial}{\partial x_i}u\right|_{p_i(.)} > 1 \right\}.$$

We have

$$\begin{aligned}
\langle \Lambda_k(u), u\rangle \;&\geq\; \sum_{i\in\mathcal{I}}\left(\int_{\tilde{\Omega}} \left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)} dx\right) + \sum_{i\in\mathcal{J}}\left(\int_{\tilde{\Omega}} \left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)} dx\right) \\
&\geq\; \sum_{i\in\mathcal{I}}\left(\left|\frac{\partial}{\partial x_i}u\right|^{p_i^+}_{p_i(.)}\right) + \sum_{i\in\mathcal{J}}\left(\left|\frac{\partial}{\partial x_i}u\right|^{p_i^-}_{p_i(.)}\right) \\
&\geq\; \sum_{i\in\mathcal{J}}\left(\left|\frac{\partial}{\partial x_i}u\right|^{p_i^-}_{p_i(.)}\right) \\
&\geq\; \sum_{i\in\mathcal{J}}\left(\left|\frac{\partial}{\partial x_i}u\right|^{p_m^-}_{p_i(.)}\right) \\
&\geq\; \sum_{i=1}^{N}\left(\left|\frac{\partial}{\partial x_i}u\right|^{p_m^-}_{p_i(.)}\right) - \sum_{i\in\mathcal{I}}\left(\left|\frac{\partial}{\partial x_i}u\right|^{p_m^-}_{p_i(.)}\right) \\
&\geq\; \sum_{i=1}^{N}\left(\left|\frac{\partial}{\partial x_i}u\right|^{p_m^-}_{p_i(.)}\right) - N.
\end{aligned}$$

We now use Jensen's inequality on the convex function $Z: \mathbb{R}^+ \to \mathbb{R}^+$, $Z(t) = t^{p_m^-}$, $p_m^- > 1$ to get

$$\begin{cases} \langle \Lambda_k(u), u\rangle \geq \langle \Lambda(u), u\rangle \\ \geq \dfrac{1}{N^{p_m^- -1}} \|u\|^{p_m^-}_{W_D^{1,\vec{p}(.)}(\tilde{\Omega})} - N. \end{cases}$$

Hence, Λ_k is coercive (as $p_m^- > 1$).

(iii) The operator Λ_k is of type M.

Lemma 3.3. *(cf [41]) Let $\mathcal{A}$ and $\mathcal{B}$ be two operators. If $\mathcal{A}$ is of type M and $\mathcal{B}$ is monotone and weakly continuous, then $\mathcal{A} + \mathcal{B}$ is of type M.*

Now , we set $\langle \mathcal{A}u, v\rangle := \langle \Lambda(u), v\rangle$ and $\langle \mathcal{B}_k u, v\rangle := \int_\Omega T_k(b(u))v dx + \int_{\tilde{\Gamma}_{Ne}} T_k(\tilde{\rho}(u))v d\sigma$.
Then, for every $k > 0$, we have $\Lambda_k = \mathcal{A} + \mathcal{B}_k$. We now have to show that for every $k > 0$, $\mathcal{B}_k$ is monotone and weakly continuous, because it is well-known that $\mathcal{A}$ is of type M. For the monotonicity of $\mathcal{B}_k$, we have to show that

$$\langle \mathcal{B}_k u - \mathcal{B}_k v, u - v\rangle \geq 0 \text{ for all } (u, v) \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \times W_D^{1,\vec{p}(.)}(\tilde{\Omega}).$$

We have

$$\begin{aligned} \langle \mathcal{B}_k u - \mathcal{B}_k v, u - v\rangle &= \int_\Omega (T_k(b(u)) - T_k(b(v)))(u - v) dx \\ &+ \int_{\tilde{\Gamma}_{Ne}} (T_k(\tilde{\rho}(u)) - T_k(\tilde{\rho}(v)))(u - v) d\sigma. \end{aligned}$$

From the monotonicity of b, $\tilde{\rho}$ and the map T_k, we conclude that

$$\langle \mathcal{B}_k u - \mathcal{B}_k v, u - v\rangle \geq 0. \tag{3.8}$$

We need now to prove that for each $k > 0$ the operator $\mathcal{B}_k$ is weakly continuous, that is, for all sequences $(u_n)_{n\in\mathbb{N}} \subset W_D^{1,\vec{p}(.)}(\tilde{\Omega})$ such that $u_n \rightharpoonup u$ in $W_D^{1,\vec{p}(.)}(\tilde{\Omega})$, we have $\mathcal{B}_k u_n \rightharpoonup \mathcal{B}_k u$ as $n \to \infty$. For all $\phi \in W_D^{1,\vec{p}(.)}(\tilde{\Omega})$, we have

$$\langle \mathcal{B}_k u_n, \phi\rangle := \int_\Omega T_k(b(u_n))\phi dx + \int_{\tilde{\Gamma}_{Ne}} T_k(\tilde{\rho}(u_n))\phi d\sigma. \tag{3.9}$$

Passing to the limit in (3.9) as n goes to ∞ and using the Lebesgue dominated convergence theorem, since $u_n \rightharpoonup u$ in $W_D^{1,\vec{p}(.)}(\tilde{\Omega})$; up to a subsequence, we have $u_n \to u$ in $L^1(\tilde{\Omega})$ and a.e. in $\tilde{\Omega}$. As $|T_k(b(u_n))\phi| \leq k|\phi|$ and $\phi \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \hookrightarrow L^1(\tilde{\Omega})$, for the first term on the right-hand side of (3.9), we obtain

$$\lim_{n\to\infty} \int_\Omega T_k(b(u_n))\phi dx = \int_\Omega T_k(b(u))\phi dx. \tag{3.10}$$

Furthermore, since $u_n \rightharpoonup u$ in $W_D^{1,\vec{p}(.)}(\tilde{\Omega})$; up to a subsequence, we have $u_n \to u$ *in* $L^1(\partial\tilde{\Omega})$ and a.e. on $\partial\tilde{\Omega}$. As $|T_k(\tilde{\rho}(u_n))\phi| \leq k|\phi|$ and $\phi \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \hookrightarrow L^1(\partial\tilde{\Omega})$, we deduce by the Lebesgue dominated convergence theorem that

$$\lim_{n\to\infty} \int_{\tilde{\Gamma}_{Ne}} T_k(\tilde{\rho}(u_n))\phi dx = \int_{\tilde{\Gamma}_{Ne}} T_k(\tilde{\rho}(u))\phi dx. \tag{3.11}$$

From (3.10) and (3.11) we conclude that for every $k > 0$, $\mathcal{B}_k(u_n) \to \mathcal{B}_k(u)$ as $n \to \infty$.

The operator $\mathcal{A}$ is type M and as $\mathcal{B}_k$ is monotone and weakly continuous, thanks to Lemma 3.3, we conclude that the operator Λ_k is of type M. Then for any $L \in (W_D^{1,\vec{p}(.)}(\tilde{\Omega}))'$, there exists $u_{\epsilon,k} \in W_D^{1,\vec{p}(.)}(\tilde{\Omega})$, such that $\Lambda_k(u_{\epsilon,k}) = L$.

We now consider $L \in (W_D^{1,\vec{p}(.)}(\tilde{\Omega}))'$ defined by $L(v) = \int_\Omega v d\mu_\epsilon + \int_{\tilde{\Gamma}_{Ne}} \tilde{d}_\epsilon v d\sigma$, for $v \in W_D^{1,\vec{p}(.)}(\tilde{\Omega})$ and we obtain (3.5)□

Step 2: A priori estimates

Lemma 3.4. *Let $u_{\epsilon,k}$ a solution of $P_{\epsilon,k}(\tilde{\rho}, \tilde{\mu}_\epsilon, \tilde{d}_\epsilon)$. Then*

$$\begin{cases} |\tilde{\rho}(u_{\epsilon,k})| \leq k_1 := \max\{\|\tilde{d}_\epsilon\|_\infty, (\tilde{\rho}_\epsilon \circ b^{-1})(\|\mu_\epsilon\|_\infty)\} \text{ a.e. on } \tilde{\Gamma}_{Ne}, \\ |b(u_{\epsilon,k})| \leq k_2 := \max\{|\mu_\epsilon\|_\infty; (b \circ \rho_0^{-1})(|\tilde{\Gamma}_{Ne}|\|\tilde{d}_\epsilon\|_\infty)\} \text{ a.e. in } \Omega. \end{cases} \tag{3.12}$$

Proof. For any $\tau > 0$, let us introduce the function $H_\tau : \mathbb{R} \to \mathbb{R}$ by

$$H_\tau(s) = \begin{cases} 0 & \text{if } s < 0, \\ \dfrac{s}{\tau} & \text{if } 0 \leq s \leq \tau, \\ 1 & \text{if } s > \tau. \end{cases}$$

In (3.5) we set $\tilde{\xi} = H_\tau(u_{\epsilon,k} - M)$, where $M > 0$ is to be fixed later. We get

$$\begin{cases} \displaystyle\int_{\tilde{\Omega}} \sum_{i=1}^N \tilde{a}_i(x, \frac{\partial}{\partial x_i} u_{\epsilon,k}) \frac{\partial}{\partial x_i} H_\tau(u_{\epsilon,k} - M)dx + \int_\Omega T_k(b(u_{\epsilon,k}))H_\tau(u_{\epsilon,k} - M)dx = \\ \displaystyle\int_\Omega H_\tau(u_{\epsilon,k} - M)d\mu_\epsilon + \int_{\tilde{\Gamma}_{Ne}} (\tilde{d}_\epsilon - T_k(\tilde{\rho}(u_\epsilon, k)))H_\tau(u_{\epsilon,k} - M)d\sigma. \end{cases} \tag{3.13}$$

The first term in (3.13) is non-negative. Indeed,

$$\int_{\tilde{\Omega}} \sum_{i=1}^N \tilde{a}_i(x, \frac{\partial}{\partial x_i} u_{\epsilon,k}) \frac{\partial}{\partial x_i} H_\tau(u_{\epsilon,k} - M)dx = \frac{1}{\tau} \int_{\{0 \leq u_{\epsilon,k} - M \leq \tau\}} \sum_{i=1}^N \tilde{a}_i(x, \frac{\partial}{\partial x_i} u_{\epsilon,k}) \frac{\partial}{\partial x_i} u_{\epsilon,k} dx \geq 0.$$

From (3.13) we obtain

$$\int_\Omega T_k(b(u_{\epsilon,k}))H_\tau(u_{\epsilon,k} - M)dx \leq \int_\Omega H_\tau(u_{\epsilon,k} - M)d\mu_\epsilon + \int_{\tilde{\Gamma}_{Ne}} (\tilde{d}_\epsilon - T_k(\tilde{\rho}(u_\epsilon, k)))H_\tau(u_{\epsilon,k} - M)d\sigma.$$

Then, one has

$$\begin{cases} \displaystyle\int_\Omega (T_k b(u_{\epsilon,k}) - T_k(b(M)))H_\tau(u_{\epsilon,k} - M)dx + \int_{\tilde{\Gamma}_{Ne}} (T_k(\tilde{\rho}(u_\epsilon, k)) - T_k(\tilde{\rho}(M)))H_\tau(u_{\epsilon,k} - M)dx \leq \\ \displaystyle\int_\Omega (\mu_\epsilon - T_k(b(M)))H_\tau(u_{\epsilon,k} - M)dx + \int_{\tilde{\Gamma}_{Ne}} (\tilde{d}_\epsilon - T_k(\tilde{\rho}(M)))H_\tau(u_{\epsilon,k} - M)d\sigma. \end{cases}$$

Letting τ go to 0 in the inequality above, we get

$$\begin{cases} \displaystyle\int_\Omega (T_k(b(u_{\epsilon,k})) - T_k(b(M)))^+ dx + \int_{\tilde{\Gamma}_{Ne}} (T_k(\tilde{\rho}(u_{\epsilon,k})) - T_k(\tilde{\rho}(M)))^+ d\sigma \leq \\ \displaystyle\int_\Omega (\mu_\epsilon - T_k(b(M)))sign_0^+(u_k - M)dx + \int_{\tilde{\Gamma}_{Ne}} (\tilde{d}_\epsilon - T_k(\tilde{\rho}(M)))sign_0^+(u_{\epsilon,k} - M)d\sigma. \end{cases}$$

As $Im(b) = Im(\rho) = \mathbb{R}$, we can fix $M = M_0 = \max\{b^{-1}(\|\mu_\epsilon\|_\infty), \rho_0^{-1}(|\tilde{\Gamma}_{Ne}|\|\tilde{d}_\epsilon\|_\infty)\}$. From the above inequality we obtain

$$\begin{cases} \displaystyle\int_\Omega (T_k(b(u_{\epsilon,k})) - T_k(b(M_0)))^+ dx + \int_{\tilde{\Gamma}_{Ne}} (T_k(\tilde{\rho}(u_{\epsilon,k}) - T_k(\tilde{\rho}(M_0)))^+ d\sigma \leq \\ \displaystyle\int_\Omega (\mu_\epsilon - T_k(\|\mu_\epsilon\|_\infty)) sign_0^+(u_{\epsilon,k} - M_0) dx + \int_{\tilde{\Gamma}_{Ne}} (\tilde{d} - T_k(\|\tilde{d}_\epsilon\|_\infty)) sign_0^+(u_{\epsilon,k} - M_0) d\sigma. \end{cases}$$

For $k > k_0 := \max\{\|\mu_\epsilon\|, \|\tilde{d}_\epsilon\|_\infty\}$, it follows that

$$\int_\Omega (T_k(b(u_{\epsilon,k})) - T_k(b(M_0)))^+ dx + \int_{\tilde{\Gamma}_{Ne}} (T_k(\tilde{\rho}(u_{\epsilon,k})) - T_k(\tilde{\rho}(M_0)))^+ d\sigma \leq 0. \tag{3.14}$$

From (3.14), we deduce that

$$\begin{cases} T_k(\tilde{\rho}(u_{\epsilon,k})) \leq T_k(\tilde{\rho}(M_0)) \text{ a.e. on } \tilde{\Gamma}_{Ne}, \\ T_k(b(u_{\epsilon,k})) \leq T_k(b(M_0)) \text{ a.e. in } \Omega. \end{cases} \tag{3.15}$$

From (3.15), we deduce that for every $k > k_1 := \max\{\|\tilde{d}_\epsilon\|_\infty, \|\mu_\epsilon\|_\infty,\ b(M_0),\ \tilde{\rho}(M_0)\}$,

$$\tilde{\rho}(u_{\epsilon,k}) \leq \tilde{\rho}(M_0) \text{ a.e. on } \tilde{\Gamma}_{Ne}$$

and

$$b(u_{\epsilon,k}) \leq b(M_0) \text{ a.e. in } \Omega.$$

Note that with the choice of M_0 and the fact that $D(\rho) = D(b) = \mathbb{R}$, for every $k > k_1 := \max\{\|\tilde{d}_\epsilon\|_\infty, \|\mu_\epsilon\|_\infty,\ b(M_0),\ \tilde{\rho}(M_0)\}$, we have

$$\begin{cases} b(u_{\epsilon,k}) \leq \max\{\|\mu_\epsilon\|_\infty, b \circ \rho_0^{-1}(|\tilde{\Gamma}_{Ne}|\|\tilde{d}_\epsilon\|_\infty)\ \} \text{ a.e. in } \Omega, \\ \tilde{\rho}(u_{\epsilon,k}) \leq \max\{\|\tilde{d}_\epsilon\|_\infty, (\tilde{\rho} \circ b^{-1})(\|\mu_\epsilon\|_\infty)\} \text{ a.e. on } \tilde{\Gamma}_{Ne}. \end{cases} \tag{3.16}$$

We need to show that for any k large enough,

$$\begin{cases} b(u_{\epsilon,k}) \geq -\max\{\|\mu_\epsilon\|_\infty, b \circ \rho_0^{-1}(|\tilde{\Gamma}_{Ne}|\|\tilde{d}_\epsilon\|_\infty)\} \text{ a.e. in } \Omega, \\ \tilde{\rho}(u_{\epsilon,k}) \geq -\max\{\|\tilde{d}_\epsilon\|_\infty, (\tilde{\rho} \circ b^{-1})(\|\mu_\epsilon\|_\infty)\} \text{ a.e. on } \tilde{\Gamma}_{Ne}. \end{cases} \tag{3.17}$$

It is easy to see that if $(u_{\epsilon,k})$ is a solution of $P_{\epsilon,k}(\tilde{\rho}, \tilde{\mu}_\epsilon, \tilde{d}_\epsilon)$, then $(-u_{\epsilon,k})$ is a solution of

$$P_{\epsilon,k}(\hat{\rho}, \hat{\mu}_\epsilon, \hat{d}_\epsilon) \begin{cases} -\displaystyle\sum_{i=1}^N \frac{\partial}{\partial x_i} \hat{a}_i(x, \frac{\partial}{\partial x_i} u_{\epsilon,k}) + T_k(\hat{b}(u_{\epsilon,k}))\chi_\Omega(x) = \hat{\mu}_\epsilon & \text{in } \tilde{\Omega} \\ u_{\epsilon,k} = 0 & \text{on } \Gamma_D \\ T_k(\hat{\rho}(u_{\epsilon,k})) + \displaystyle\sum_{i=1}^N \hat{a}_i(x, \frac{\partial}{\partial x_i} u_{\epsilon,k}) \eta_i = \hat{d}_\epsilon & \text{on } \tilde{\Gamma}_{Ne}, \end{cases}$$

where $\hat{a}_i(x,\xi) = -\tilde{a}_i(x, -\xi)$, $\hat{\rho}(s) = -\tilde{\rho}(-s)$, $\hat{b}(s) = -b(-s)$, $\hat{\mu}_\epsilon = -\tilde{\mu}_\epsilon$ and $\hat{d} = -\tilde{d}_\epsilon$. Then for every $k > k_2 := \max\{\|\tilde{d}_\epsilon\|_\infty, \|\mu_\epsilon\|_\infty,\ -b(-M_0),\ -\tilde{\rho}(-M_0)\}$, we have

$$\begin{cases} -b(u_{\epsilon,k}) \leq \max\{\|\mu_\epsilon\|_\infty, b \circ \rho_0^{-1}(|\tilde{\Gamma}_{Ne}|\|\tilde{d}_\epsilon\|_\infty)\} \text{ a.e. in } \Omega, \\ -\tilde{\rho}(u_{\epsilon,k}) \leq \max\{\|\tilde{d}_\epsilon\|_\infty, (\tilde{\rho} \circ b^{-1})(\|\mu_\epsilon\|_\infty)\} \text{ a.e. on } \tilde{\Gamma}_{Ne}, \end{cases}$$

which implies (3.17).

From (3.16) and (3.17), we deduce (3.12). □

Step 3. Convergence Since $u_{\epsilon,k}$ is a solution of $P_{\epsilon,k}(\tilde{\rho},\tilde{\mu}_\epsilon,\tilde{d}_\epsilon)$, thanks to Lemma 3.4 and the fact that Ω is bounded, we have $\tilde{\rho}(u_{\epsilon,k}) \in L^1(\tilde{\Gamma}_{Ne})$ and $b(u_{\epsilon,k}) \in L^1(\Omega)$. For $k = 1+\max(k_1,k_2)$ fixed, by Lemma 3.4, one sees that problem $P_\epsilon(\tilde{\rho},\tilde{\mu}_\epsilon,\tilde{d}_\epsilon)$ admits at least one solution u_ϵ □

Remark 3.5. *Using the relation* (3.12) *and the fact that the functions b and ρ are non-decreasing, it follows that for k large enough, the solution of the problem $P(\tilde{\rho},\tilde{\mu}_\epsilon,\tilde{d}_\epsilon)$ belongs to $L^\infty(\Omega)\cap L^\infty(\tilde{\Gamma}_{Ne})$ and $|u_\epsilon| \le c(b,k_1)$ a.e. in Ω and $|u_\epsilon| \le c(\rho,k_2)$ a.e. on $\tilde{\Gamma}_{Ne}$.*

Now, we set $\tilde{a}_i(x,\xi) = a_i(x,\xi)\chi_\Omega(x) + \dfrac{1}{\epsilon^{p_i(x)}}|\xi|^{p_i(x)-2}\xi\chi_{\tilde{\Omega}\setminus\Omega}(x)$ for all $(x,\xi)\in\tilde{\Omega}\times\mathbb{R}^N$ and we consider the following problem. $P_\epsilon(\tilde{\rho},\tilde{\mu}_\epsilon,\tilde{d}_\epsilon)$

$$\begin{cases} -\displaystyle\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(a_i\left(x,\frac{\partial}{\partial x_i}u_\epsilon\right)\chi_\Omega(x) + \frac{1}{\epsilon^{p_i(x)}}\left|\frac{\partial}{\partial x_i}u_\epsilon\right|^{p_i(x)-2}\frac{\partial}{\partial x_i}u_\epsilon\chi_{\tilde{\Omega}\setminus\Omega}(x)\right) + & \\ |u_\epsilon|^{p_M(x)-2}u_\epsilon\chi_\Omega = \tilde{\mu}_\epsilon & \text{in } \tilde{\Omega} \\ u_\epsilon = 0 & \text{on } \Gamma_D \\ \tilde{\rho}(u_\epsilon) + \displaystyle\sum_{i=1}^N \tilde{a}_i(x,\frac{\partial}{\partial x_i}u_\epsilon)\eta_i = \tilde{d}_\epsilon & \text{on } \tilde{\Gamma}_{Ne}. \end{cases} \tag{3.18}$$

Thanks to Theorem 3.2, $P_\epsilon(\tilde{\rho},\tilde{\mu}_\epsilon,\tilde{d}_\epsilon)$ has at least one solution. So, there exists at least one measurable function $u_\epsilon : \tilde{\Omega}\to\mathbb{R}$ such that

$$\begin{cases} \displaystyle\sum_{i=1}^N \int_\Omega a_i\left(x,\frac{\partial}{\partial x_i}u_\epsilon\right)\frac{\partial}{\partial x_i}\tilde{\xi}dx + \sum_{i=1}^N \int_{\tilde{\Omega}\setminus\Omega}\left(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial}{\partial x_i}u_\epsilon|^{p_i(x)-2}\frac{\partial}{\partial x_i}u_\epsilon.\frac{\partial}{\partial x_i}\tilde{\xi}\right)dx \\ + \displaystyle\int_\Omega |u_\epsilon|^{p_M(x)-2}u_\epsilon\tilde{\xi}dx = \int_\Omega \tilde{\xi}d\mu_\epsilon + \int_{\tilde{\Gamma}_{Ne}}(\tilde{d}_\epsilon - \tilde{\rho}(u_\epsilon)\tilde{\xi}d\sigma, \end{cases} \tag{3.19}$$

where $u_\epsilon \in W_D^{1,\vec{p}(.)}(\tilde{\Omega})$ and $\tilde{\xi}\in W_D^{1,\vec{p}(.)}(\tilde{\Omega})\cap L^\infty(\Omega)$.

Moreover $u_\epsilon \in L^\infty(\Omega)\cap L^\infty(\tilde{\Gamma}_{Ne})$.

Our aim is to prove that these approximated solutions u_ϵ tend, as ϵ goes to 0, to a measurable function u which is an entropy solution of the problem $P(\tilde{\rho},\tilde{\mu},\tilde{d})$. To start with, we establish some a priori estimates.

Proposition 3.6. *Let u_ϵ be a solution of the problem $P_\epsilon(\tilde{\rho},\tilde{\mu}_\epsilon,\tilde{d}_\epsilon)$. Then, the following statements hold.*

(i) $\forall k > 0$,

$$\sum_{i=1}^N \int_\Omega \left|\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right|^{p_i(x)}dx + \sum_{i=1}^N \int_{\tilde{\Omega}\setminus\Omega}\left(\frac{1}{\epsilon}\left|\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right|\right)^{p_i(x)}dx \le k(\|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})} + |\mu|(\Omega));$$

(ii)

$$\int_{\Omega} |u_\epsilon|^{p_M(x)-1} dx + \int_{\tilde{\Gamma}_{Ne}} |\tilde{\rho}(u_\epsilon)| dx \leq (\|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})} + |\mu|(\Omega));$$

(iii) $\forall k > 0$,

$$\sum_{i=1}^{N} \int_{\tilde{\Omega}} \left| \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right|^{p_i(x)} dx \leq k(\|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})} + |\mu|(\Omega)).$$

Proof. For any $k > 0$, we set $\tilde{\xi} = T_k(u_\epsilon)$ in (3.19), to get

$$\begin{cases} \sum_{i=1}^{N} \int_{\Omega} \left(a_i \left(x, \frac{\partial}{\partial x_i} u_\epsilon \right) \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right) dx + \sum_{i=1}^{N} \int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon^{p_i(x)}} \left| \frac{\partial}{\partial x_i} u_\epsilon \right|^{p_i(x)-2} \frac{\partial}{\partial x_i} u_\epsilon \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right) dx \\ \int_{\Omega} |u_\epsilon|^{p_M(x)-2} u_\epsilon T_k(u_\epsilon) dx = \int_{\Omega} T_k(u_\epsilon) d\mu_\epsilon + \int_{\tilde{\Gamma}_{Ne}} (\tilde{d}_\epsilon - \tilde{\rho}(u_\epsilon)) T_k(u_\epsilon) d\sigma. \end{cases} \tag{3.20}$$

(i) Obviously, we have

$$\sum_{i=1}^{N} \int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon^{p_i(x)}} \left| \frac{\partial}{\partial x_i} u_\epsilon \right|^{p_i(x)-2} \frac{\partial}{\partial x_i} u_\epsilon \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right) dx = \sum_{i=1}^{N} \int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon^{p_i(x)}} \left| \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right|^{p_i(x)} \right) dx \geq 0,$$

$\int_{\tilde{\Gamma}_{Ne}} \tilde{\rho}(u_\epsilon) T_k(u_\epsilon) d\sigma \geq 0$ and $\int_{\Omega} |u_\epsilon|^{p_M(x)-2} u_\epsilon T_k(u_\epsilon) dx \geq 0$.
Moreover,

$$\begin{cases} \int_{\Omega} T_k(u_\epsilon) d\mu_\epsilon + \int_{\tilde{\Gamma}_{Ne}} \tilde{d}_\epsilon T_k(u_\epsilon) d\sigma & \leq k \int_{\Omega} d\mu_\epsilon + k \int_{\tilde{\Gamma}_{Ne}} |\tilde{d}_\epsilon| d\sigma \\ & \leq k \left(|\mu|(\Omega) + \int_{\tilde{\Gamma}_{Ne}} |\tilde{d}| d\sigma \right). \end{cases} \tag{3.21}$$

Using the inequalities above and (1.7), it follows that

$$\sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial T_k(u_\epsilon)}{\partial x_i} \right|^{p_i(x)} dx \leq k \left(|\mu|(\Omega) + \int_{\tilde{\Gamma}_{Ne}} |\tilde{d}| d\sigma \right). \tag{3.22}$$

As $\sum_{i=1}^{N} \int_{\Omega} \left(a_i \left(x, \frac{\partial}{\partial x_i} u_\epsilon \right) \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right) dx \geq 0$, $\int_{\tilde{\Gamma}_{Ne}} \tilde{\rho}(u_\epsilon) T_k(u_\epsilon) d\sigma \geq 0$ and $\int_{\Omega} |u_\epsilon|^{p_M(x)-2} u_\epsilon T_k(u_\epsilon) dx \geq 0$, therefore, we get from (3.20),

$$\sum_{i=1}^{N} \int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon^{p_i(x)}} |\frac{\partial}{\partial x_i} T_k(u_\epsilon)|^{p_i(x)} \right) dx \leq k \left(|\mu|(\Omega) + \int_{\tilde{\Gamma}_{Ne}} |\tilde{d}| d\sigma \right) \tag{3.23}$$

Adding (3.22) and (3.23), we obtain *(i)*.

(ii) The first two terms in (3.20) are non-negative and using (3.21), we have from (3.20) the following

$$\int_{\tilde{\Gamma}_{Ne}} \tilde{\rho}(u_\epsilon) T_k(u_\epsilon) d\sigma + \int_{\Omega} |u_\epsilon|^{p_M(x)-2} u_\epsilon T_k(u_\epsilon) dx \leq k \left(|\mu|(\Omega) + \int_{\tilde{\Gamma}_{Ne}} |\tilde{d}| d\sigma \right).$$

We divide the above inequality by $k > 0$ and let k go to zero, to get

$$\begin{aligned} \int_{\tilde{\Gamma}_{Ne}} \tilde{\rho}(u_\epsilon) sign(u_\epsilon) d\sigma + \int_{\Omega} |u_\epsilon|^{p_M(x)-2} u_\epsilon sign(u_\epsilon) dx &= \int_{\tilde{\Gamma}_{Ne}} |\tilde{\rho}(u_\epsilon)| d\sigma + \int_{\Omega} |u_\epsilon|^{p_M(x)-1} dx \\ &\leq \left(|\mu|(\Omega) + \int_{\tilde{\Gamma}_{Ne}} |\tilde{d}| d\sigma \right). \end{aligned}$$

(iii) For all $k>0$, we have

$$\sum_{i=1}^{N}\int_{\tilde{\Omega}}\left|\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right|^{p_i(x)}dx\leq\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right|^{p_i(x)}dx+\sum_{i=1}^{N}\int_{\tilde{\Omega}\setminus\Omega}\left|\frac{1}{\epsilon}\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right|^{p_i(x)}dx,$$

for any $0<\epsilon<1$. According to (i), we deduce that

$$\sum_{i=1}^{N}\int_{\tilde{\Omega}}\left|\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right|^{p_i(x)}dx\leq k\left(|\mu|(\Omega)+\int_{\tilde{\Gamma}_{Ne}}|\tilde{d}|d\sigma\right).$$

□

Lemma 3.7. *There is a positive constant D such that*

$$meas\{|u_\epsilon|>k\}\leq D^{p_m^-}\frac{(1+k)}{k^{p_m^--1}},\ \forall k>0.$$

Proof. Let $k>0$; by using Proposition 3.6-*(iii)*, we have

$$\begin{aligned}\sum_{i=1}^{N}\int_{\tilde{\Omega}}\left|\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right|^{p_m^-(x)}dx &\leq \sum_{i=1}^{N}\int_{\left\{\left|\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right|>1\right\}}\left|\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right|^{p_m^-(x)}dx+Nmeas(\tilde{\Omega})\\ &\leq \sum_{i=1}^{N}\int_{\tilde{\Omega}}\left|\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right|^{p_i(x)}dx+Nmeas(\tilde{\Omega})\\ &\leq k\left(|\mu|(\Omega)+\int_{\tilde{\Gamma}_{Ne}}|\tilde{d}|d\sigma\right)+Nmeas(\tilde{\Omega})\\ &\leq C'(k+1),\end{aligned}$$

with $C'=max\left(\left(|\mu|(\Omega)+\int_{\tilde{\Gamma}_{Ne}}|\tilde{d}|d\sigma\right);Nmeas(\tilde{\Omega})\right)$.
We can write the above inequality as

$$\sum_{i=1}^{N}\left\|\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right\|_{p_m^-}^{p_m^-}\leq C'(1+k)\text{ or }\|T_k(u_\epsilon)\|_{W_D^{1,p_m^-}(\tilde{\Omega})}\leq[C'(1+k)]^{\frac{1}{p_m^-}}.$$

By the Poincaré inequality in constant exponent, we obtain

$$\|T_k(u_\epsilon)\|_{L^{p_m^-}(\tilde{\Omega})}\leq D(1+k)^{\frac{1}{p_m^-}}.$$

The above inequality implies that

$$\int_{\tilde{\Omega}}|T_k(u_\epsilon)|^{p_m^-}dx\leq D^{p_m^-}(1+k),$$

from which we obtain

$$meas\left\{|u_\epsilon|>k\right\}\leq D^{p_m^-}\frac{(1+k)}{k^{p_m^-}},$$

since

$$\int_{\tilde{\Omega}}|T_k(u_\epsilon)|^{p_m^-}dx=\int_{\{|u_\epsilon|>k\}}|T_k(u_\epsilon)|^{p_m^-}dx+\int_{\{|u_\epsilon|\leq k\}}|T_k(u_\epsilon)|^{p_m^-}dx,$$

we get

$$\int_{\{|u_\epsilon|>k\}} |T_k(u_\epsilon)|^{p_m^-} dx \le \int_{\tilde{\Omega}} |T_k(u_\epsilon)|^{p_m^-} dx$$

and

$$k^{p_m^-} meas\,\{|u_\epsilon| > k\} \le \int_{\tilde{\Omega}} |T_k(u_\epsilon)|^{p_m^-} dx \le D^{p_m^-}(1+k)$$

□

Lemma 3.8. *There is a positive constant C such that*

$$\sum_{i=1}^{N} \int_{\tilde{\Omega}} \left(\left| \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right|^{p_i^-} \right) dx \le C(k+1),\ \forall k > 0. \tag{3.24}$$

Proof. Let $k > 0$, we set $\Omega_1 = \left\{ |u| \le k; \left| \frac{\partial}{\partial x_i} u_\epsilon \right| \le 1 \right\}$ and $\Omega_2 = \left\{ |u| \le k; \left| \frac{\partial}{\partial x_i} u_\epsilon \right| > 1 \right\}$; using Proposition 3.6-*(iii)*, we have

$$\begin{aligned} \sum_{i=1}^{N} \int_{\tilde{\Omega}} \left(\left| \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right|^{p_i^-} \right) dx &= \sum_{i=1}^{N} \int_{\Omega_1} \left(\left| \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right|^{p_i^-} \right) dx + \sum_{i=1}^{N} \int_{\Omega_2} \left(\left| \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right|^{p_i^-} \right) dx \\ &\le N meas(\tilde{\Omega}) + \sum_{i=1}^{N} \int_{\tilde{\Omega}} \left(\left| \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right|^{p_i(x)} \right) dx \\ &\le N meas(\tilde{\Omega}) + k \left(|\mu|(\Omega) + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})} \right) \le C(k+1), \end{aligned}$$

with $C = \max \left\{ N meas(\tilde{\Omega}); \left(|\mu|(\Omega) + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})} \right) \right\}$. □

Lemma 3.9. *For all $k > 0$, there is two constants C_1 and C_2 such that*

(i) $\|u_\epsilon\|_{\mathcal{M}^{q^*}(\tilde{\Omega})} \le C_1$;

(ii) $\left\| \left| \frac{\partial}{\partial x_i} u_\epsilon \right| \right\|_{\mathcal{M}^{p_i^- q/p}(\tilde{\Omega})} \le C_2$.

Proof. *(i)* By Lemma 3.8, we have

$$\sum_{i=1}^{N} \int_{\tilde{\Omega}} \left| \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right|^{p_i^-} dx \le C(1+k),\ \forall k > 0 \text{ and } i = 1, ..., N.$$

• If $k > 1$, we have

$$\sum_{i=1}^{N} \int_{\tilde{\Omega}} \left| \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right|^{p_i^-} dx \le C' k,$$

which means $T_k(u_\epsilon) \in W^{1,(p_1^-,\dots,p_N^-)}(\tilde{\Omega})$. Using relation (2.8), we deduce that

$$\|T_k(u_\epsilon)\|_{L^{(\bar{p})^*}(\tilde{\Omega}} \le C_1 \prod_{i=1}^{N} \left\| \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right\|_{L^{p_i^-}(\tilde{\Omega})}^{\frac{1}{N}}.$$

So,

$$\begin{aligned}
\int_{\tilde{\Omega}} |T_k(u_\epsilon)|^{(\bar{p})^*}\,dx &\le C\left[\prod_{i=1}^{N}\left(\int_{\tilde{\Omega}}\left|\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right|^{p_i^-}dx\right)^{\frac{1}{Np_i^-}}\right]^{(\bar{p})^*}\\
&\le C''\left[\prod_{i=1}^{N}(k)^{\frac{1}{Np_i^-}}\right]^{(\bar{p})^*}\\
&\le C''\left[k^{\sum_{i=1}^{N}\frac{1}{Np_i^-}}\right]^{(\bar{p})^*}\\
&\le C''k^{\frac{(\bar{p})^*}{\bar{p}}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_{\{|u_\epsilon|>k\}} |T_k(u_\epsilon)|^{(\bar{p})^*}\,dx &\le \int_{\tilde{\Omega}} |T_k(u_\epsilon)|^{(\bar{p})^*}\,dx\\
&\le C'k^{\frac{(\bar{p})^*}{\bar{p}}}
\end{aligned}$$

and so,

$$(k)^{(\bar{p})^*} meas\{x\in\tilde{\Omega} : |u_\epsilon|>k\} \quad\le\quad C'k^{\frac{(\bar{p})^*}{\bar{p}}};$$

which means that

$$\lambda_{u_\epsilon}(k) \le C'k^{(\bar{p})^*(\frac{1}{\bar{p}}-1)} = C'k^{-q^*},\ \forall k\ge 1.$$

- If $0<k<1$, we have

$$\begin{aligned}
\lambda_{u_\epsilon}(k) &= meas\Big\{x\in\tilde{\Omega} : |u_\epsilon|>k\Big\}\\
&\le meas(\tilde{\Omega})\\
&\le meas(\tilde{\Omega})k^{-q^*}.
\end{aligned}$$

So,

$$\lambda_{u_\epsilon}(k) \le (C'+meas(\tilde{\Omega}))k^{-q^*} = C_1k^{-q^*}.$$

Therefore,

$$\|u_\epsilon\|_{\mathcal{M}^{q^*}(\tilde{\Omega})} \le C_1.$$

(ii) • Let $\alpha \geq 1$. For all $k \geq 1$, we have

$$\begin{aligned}
\lambda_{\frac{\partial u_\epsilon}{\partial x_i}}(\alpha) &= meas\left(\left\{\left|\frac{\partial u_\epsilon}{\partial x_i}\right| > \alpha\right\}\right) \\
&= meas\left(\left\{\left|\frac{\partial u_\epsilon}{\partial x_i}\right| > \alpha; |u_\epsilon| \leq k\right\}\right) + meas\left(\left\{\left|\frac{\partial u_\epsilon}{\partial x_i}\right| > \alpha;; |u_\epsilon| > k\right\}\right) \\
&\leq \int_{\left\{\left|\frac{\partial u_\epsilon}{\partial x_i}\right| > \alpha; |u_\epsilon| \leq k\right\}} dx + \lambda_{u_\epsilon}(k) \\
&\leq \int_{\{|u_\epsilon| \leq k\}} \left(\frac{1}{\alpha}\left|\frac{\partial u_\epsilon}{\partial x_i}\right|\right)^{p_i^-} dx + \lambda_{u_\epsilon}(k) \\
&\leq \alpha^{-p_i^-} C' k + C k^{-q^*} \\
&\leq B\left(\alpha^{-p_i^-} k + k^{-q^*}\right),
\end{aligned}$$

with $B = \max(C'; C)$.
Let $g : [1;\infty) \to \mathbb{R}$, $x \mapsto g(x) = \dfrac{x}{\alpha^{p_i^-}} + x^{-q^*}$.
We have $g'(x) = 0$ with $x = \left(q^* \alpha^{p^-}\right)^{\frac{1}{q^*+1}}$.
We set $k = \left(q^* \alpha^{p_i^-}\right)^{\frac{1}{q^*+1}} \geq 1$ in the above inequality to get,

$$\begin{aligned}
\lambda_{\frac{\partial u_\epsilon}{\partial x_i}}(\alpha) &\leq B\left[\alpha^{-p_i^-} \times \left(q^* \alpha^{p_i^-}\right)^{\frac{1}{q^*+1}} + \left(q^* \alpha^{p_i^-}\right)^{\frac{-q^*}{q^*+1}}\right] \\
&\leq B\left[(q^*)^{\frac{1}{q^*+1}} \times \alpha^{-p_i^-\left(1 - \frac{1}{q^*+1}\right)} + (q^*)^{\frac{-q^*}{q^*+1}} \times \alpha^{\frac{-p_i^- q^*}{q^*+1}}\right] \\
&\leq B\left[(q^*)^{\frac{1}{q^*+1}} \times \alpha^{-p_i^-\left(\frac{q^*}{q^*+1}\right)} + (q^*)^{\frac{-q^*}{q^*+1}} \times \alpha^{\frac{-p_i^- q^*}{q^*+1}}\right] \\
&\leq M \alpha^{-p_i^- \frac{q^*}{q^*+1}} \\
&\leq M \alpha^{-p_i^- \frac{q}{\bar{p}}},
\end{aligned}$$

where $M = B \times \max\left((q^*)^{\frac{1}{q^*+1}}; (q^*)^{\frac{-q^*}{q^*+1}}\right)$ and as $q^* = \dfrac{N(\bar{p}-1)}{N-\bar{p}}$, $q = \dfrac{N(\bar{p}-1)}{N-1}$.

So,

$$\begin{aligned}\frac{q^*}{q^*+1} &= \frac{q^*(N-\bar{p})}{N(\bar{p}-1)+N-\bar{p}}\\ &= \frac{q^*(N-\bar{p})}{N\bar{p}-\bar{p}}\\ &= \frac{N(\bar{p}-1)}{(N-1)\bar{p}}\\ &= \frac{q}{\bar{p}}.\end{aligned}$$

• If $0 \le \alpha < 1$, we have.

$$\begin{aligned}\lambda_{\frac{\partial u_\epsilon}{\partial x_i}}(\alpha) &= meas\left(\left\{x \in \tilde{\Omega} : \left|\frac{\partial u_\epsilon}{\partial x_i}\right| > \alpha\right\}\right)\\ &\le meas(\tilde{\Omega})\alpha^{-p_i^-\frac{q}{\bar{p}}}.\end{aligned}$$

Therefore,

$$\lambda_{\frac{\partial u_\epsilon}{\partial x_i}}(\alpha) \le \Big(M + meas(\tilde{\Omega})\Big)\,\alpha^{-p_i^-\frac{q}{\bar{p}}},\ \forall\, \alpha \ge 0.$$

So,

$$\left\|\frac{\partial u_\epsilon}{\partial x_i}\right\|_H \le C_2,$$

where $H = \mathcal{M}(\tilde{\Omega})^{\frac{p_i^- q}{\bar{p}}}$ □

Proposition 3.10. *Let u_ϵ be a solution of the problem $P(\tilde{\rho}, \tilde{\mu}_\epsilon, \tilde{d}_\epsilon)$. Then,*

(i) $u_\epsilon \to u$ in measure, a.e. in Ω and a.e. on $\tilde{\Gamma}_N$;

(ii) For all $i = 1, ...N$, $\frac{\partial T_k(u_\epsilon)}{\partial x_i} \rightharpoonup \frac{\partial T_k(u)}{\partial x_i} = 0$ in $L^{p_i^-}(\tilde{\Omega} \setminus \Omega)$.

Proof. (i) By Proposition 3.6 (i), we deduce that $(T_k(u_\epsilon))_{\epsilon>0}$ is bounded in $W_D^{1,\vec{p}(\cdot)}(\tilde{\Omega}) \hookrightarrow L^{p_m(\cdot)}(\tilde{\Omega}) \hookrightarrow L^{p_m^-}(\tilde{\Omega})$ (with compact embedding). Therefore, up to a subsequence, we can assume that as $\epsilon \to 0$, $(T_k(u_\epsilon))_{\epsilon>0}$ converges strongly to some function σ_k in $L^{p_m^-}(\tilde{\Omega})$, $a.e.$ in $\tilde{\Omega}$ and $a.e.$ on $\tilde{\Gamma}_{Ne}$.

Let us see that the sequence $(u_\epsilon)_{\epsilon>0}$ is Cauchy in measure.

Indeed, let $s > 0$ and define:

$E_1 = [|u_{\epsilon_1}| > k]$, $E_2 = [|u_{\epsilon_2}| > k]$ and $E_3 = [|T_k(u_{\epsilon_1}) - T_k(u_{\epsilon_2})| > s]$,

where $k > 0$ is fixed. We note that

$$[|u_{\epsilon_1} - u_{\epsilon_2}| > s] \subset E_1 \cup E_2 \cup E_3;$$

hence,

$$meas([|u_{\epsilon_1} - u_{\epsilon_2}| > s]) \leq \sum_{i=1}^{3} meas(E_i). \tag{3.25}$$

Let $\theta > 0$, using Lemma 3.7, we choose $k = k(\theta)$ such that

$$meas(E_1) \leq \frac{\theta}{3} \text{ and } meas(E_2) \leq \frac{\theta}{3}. \tag{3.26}$$

Since $(T_k(u_\epsilon))_{\epsilon>0}$ converges strongly in $L^{p_m^-}(\tilde{\Omega})$, then, it is a Cauchy sequence in $L^{p_m^-}(\tilde{\Omega})$. Thus,

$$meas(E_3) \leq \frac{1}{s^{p_m^-}} \int_{\Omega} |T_k(u_{\epsilon_1}) - T_k(u_{\epsilon_2})|^{p_m^-} dx \leq \frac{\theta}{3}, \tag{3.27}$$

for all $\epsilon_1, \epsilon_2 \geq n_0(s,\theta)$. Finally, from (3.25), (3.26) and (3.27), we obtain

$$meas([|u_{\epsilon_1} - u_{\epsilon_2}| > s]) \leq \theta \text{ for all } \epsilon_1, \epsilon_2 \geq n_0(s,\theta); \tag{3.28}$$

which means that the sequence $(u_\epsilon)_{\epsilon>0}$ is Cauchy in measure, so $u_\epsilon \to u$ in measure and up to a subsequence, we have $u_\epsilon \to u$ *a.e.* in $\tilde{\Omega}$. Hence, $\sigma_k = T_k(u)$ *a.e.* in $\tilde{\Omega}$ and so, $u \in \mathcal{T}_D^{1,\vec{p}(.)}(\Omega)$.

(ii) According to the proof of (i), we have $T_k(u_\epsilon) \rightharpoonup T_k(u)$ in $W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \hookrightarrow W_D^{1,\vec{p}_-}(\tilde{\Omega})$ which implies on one hand that for all $i = 1, ...N$, $\dfrac{\partial T_k(u_\epsilon)}{\partial x_i} \rightharpoonup \dfrac{\partial T_k(u)}{\partial x_i}$ in $L^{p_i(.)}(\tilde{\Omega})$ and on the other hand that for all $i = 1, ...N$, $\dfrac{\partial T_k(u_\epsilon)}{\partial x_i} \rightharpoonup \dfrac{\partial T_k(u)}{\partial x_i}$ in $L^{p_i(.)}(\tilde{\Omega})$ and then for all $i = 1, ...N$, $\dfrac{\partial T_k(u_\epsilon)}{\partial x_i} \rightharpoonup \dfrac{\partial T_k(u)}{\partial x_i}$ in $L^{p_i^-}(\tilde{\Omega} \setminus \Omega)$.

Let $i = 1, ..., N$, by Proposition 3.6-(i), we can assert that $\left(\dfrac{1}{\epsilon}\dfrac{\partial T_k(u_\epsilon)}{\partial x_i}\right)_{\epsilon>0}$ is bounded in $L^{p_i^-}(\tilde{\Omega} \setminus \Omega)$. Indeed, let $k > 0$, we set $\Omega^1 = \left\{x \in \tilde{\Omega} \setminus \Omega; |u(x)| \leq k; \left|\dfrac{\partial}{\partial x_i} u_\epsilon(x)\right| \leq \epsilon\right\}$ and $\Omega^2 = \left\{x \in \tilde{\Omega} \setminus \Omega; |u| \leq k; \left|\dfrac{\partial}{\partial x_i} u_\epsilon(x)\right| > \epsilon\right\}$; using Proposition 3.6-*(i)*, we have

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon}\left|\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right|^{p_i^-}\right) dx \\
= \;& \sum_{i=1}^{N} \int_{\Omega^1} \left(\frac{1}{\epsilon}\left|\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right|^{p_i^-}\right) dx + \sum_{i=1}^{N} \int_{\Omega^2} \left(\frac{1}{\epsilon}\left|\frac{\partial}{\partial x_i} T_k(u_\epsilon)\right|^{p_i^-}\right) dx \\
\leq \;& N meas(\tilde{\Omega} \setminus \Omega) + \sum_{i=1}^{N} \int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon}\left|\frac{\partial}{\partial x_i} T_k(u_\epsilon)\right|^{p_i(x)}\right) dx \\
\leq \;& N meas(\tilde{\Omega} \setminus \Omega) + k\left(|\mu|(\Omega) + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})}\right) \leq C'(k+1),
\end{aligned}$$

with $C' = \max\left\{N meas(\tilde{\Omega} \setminus \Omega); \left(|\mu|(\Omega) + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})}\right)\right\}$. To end, we have

$$\int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon}\left|\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right|^{p_i^-}\right) dx \leq \sum_{i=1}^{N} \int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon}\left|\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right|^{p_i^-}\right) dx, \text{ for any} i = 1, \ldots, N.$$

Therefore, there exists $\Theta_k \in L^{p_i^-}(\tilde{\Omega} \setminus \Omega)$ such that

$$\frac{1}{\epsilon}\frac{\partial T_k(u_\epsilon)}{\partial x_i} \rightharpoonup \Theta_k \text{ in } L^{p_i^-}(\tilde{\Omega} \setminus \Omega) \text{ as } \epsilon \to 0.$$

For any $\psi \in L^{(p_i')^-}(\tilde{\Omega} \setminus \Omega)$, we have

$$\int_{\tilde{\Omega}\setminus\Omega} \frac{\partial T_k(u_\epsilon)}{\partial x_i}\psi dx = \int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon}\frac{\partial T_k(u_\epsilon)}{\partial x_i} - \Theta_k\right)(\epsilon\psi)dx + \epsilon\int_{\tilde{\Omega}\setminus\Omega} \Theta_k \psi dx. \tag{3.29}$$

As $(\epsilon\psi)_{\epsilon>0}$ converges strongly to zero in $L^{(p_i')^-}(\tilde{\Omega}\setminus\Omega)$, we pass to the limit as $\epsilon \to 0$ in (3.29), to get

$$\frac{\partial T_k(u_\epsilon)}{\partial x_i} \rightharpoonup 0 \text{ in } L^{p_i^-}(\tilde{\Omega} \setminus \Omega).$$

Hence, one has

$$\frac{\partial T_k(u_\epsilon)}{\partial x_i} \rightharpoonup \frac{\partial T_k(u)}{\partial x_i} = 0 \text{ in } L^{p_i^-}(\tilde{\Omega} \setminus \Omega),$$

for any $i = 1, ..., N$.

□

Lemma 3.11. *$b(u) \in L^1(\Omega)$ and $\tilde{\rho}(u) \in L^1(\tilde{\Gamma}_{Ne})$.*

Proof. Having in mind that by Proposition 3.6-*(ii)*,

$$\int_\Omega |b(u_\epsilon)|dx + \int_{\tilde{\Gamma}_{Ne}} |\tilde{\rho}(u_\epsilon)|d\sigma \leq (|\mu|(\Omega) + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})}),$$

we deduce that

$$\int_\Omega |b(u_\epsilon)|dx \leq (|\mu|(\Omega) + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})}) \tag{3.30}$$

and

$$\int_{\tilde{\Gamma}_{Ne}} |\tilde{\rho}(u_\epsilon)|d\sigma \leq (|\mu|(\Omega) + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})}). \tag{3.31}$$

By Fatou's lemma, the continuity of b, $\tilde{\rho}$ and using Proposition 3.10, we have

$$\liminf_{\epsilon\to 0} \int_\Omega |b(u_\epsilon)|dx \geq \int_\Omega |b(u)|dx \tag{3.32}$$

and

$$\liminf_{\epsilon\to 0} \int_{\tilde{\Gamma}_{Ne}} |\tilde{\rho}(u_\epsilon)|d\sigma \geq \int_{\tilde{\Gamma}_{Ne}} |\tilde{\rho}(u)|d\sigma. \tag{3.33}$$

Using (3.30)-(3.33), we deduce that

$$\int_\Omega |b(u)|dx \leq (|\mu|(\Omega) + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})})$$

and

$$\int_{\tilde{\Gamma}_{Ne}} |\tilde{\rho}(u)|d\sigma \leq (|\mu|(\Omega) + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})}).$$

Therefore, $b(u) \in L^1(\Omega)$ and $\tilde{\rho}(u) \in L^1(\tilde{\Gamma}_{Ne})$. □

Lemma 3.12. *Assume* (1.4)-(1.8) *hold and* u_ϵ *be a weak solution of the problem* $P(\rho, \tilde{\mu}_\epsilon, \tilde{d}_\epsilon)$. *Then,*

(i) $\dfrac{\partial}{\partial x_i} u_\epsilon$ *converges in measure to* $\dfrac{\partial}{\partial x_i} u$.

(ii) $a_i\left(x, \dfrac{\partial T_k(u_\epsilon)}{\partial x_i}) \to a_i(x, \dfrac{\partial T_k(u)}{\partial x_i}\right)$ *strongly in* $L^1(\Omega)$ *and weakly in* $L^{p_i'(\cdot)}(\Omega)$, *for all i=1,...,N.*

In order to give the proof of Lemma 3.12, we need the following lemmas.

Lemma 3.13 (Cf [6])**.** *Let* $u \in \mathcal{T}^{1,\vec{p}(\cdot)}(\Omega)$. *Then, there exists a unique measurable function* $\nu_i : \Omega \to \mathbb{R}$ *such that*

$$\nu_i \chi_{\{|u|<k\}} = \frac{\partial}{\partial x_i} T_k(u) \text{ for a.e. } x \in \Omega,\ \forall k > 0 \text{ and } i = 1, ..., N;$$

where χ_A *denotes the characteristic function of a measurable set* A.
The functions ν_i *are denoted* $\dfrac{\partial}{\partial x_i} u$. *Moreover, if* u *belongs to* $W^{1,\vec{p}(\cdot)}(\Omega)$, *then* $\nu_i \in L^{p_i(\cdot)}(\Omega)$ *and coincides with the standard distributional gradient of* u *i.e.* $\nu_i = \dfrac{\partial}{\partial x_i} u$.

Lemma 3.14 (Cf [37], lemma 5.4)**.** *Let* $(v_n)_{n\in\mathbb{N}}$ *be a sequence of measurable functions. If* v_n *converges in measure to* v *and is uniformly bounded in* $L^{p(\cdot)}(\Omega)$ *for some* $1 << p(.) \in L^\infty(\Omega)$, *then* $v_n \to v$ *strongly in* $L^1(\Omega)$.

The third technical lemma is a standard fact in measure theory (Cf [16]).

Lemma 3.15. *Let* $(X, \mathcal{M}, \mu)$ *be a measurable space such that* $\mu(X) < \infty$.
Consider a measurable function $\gamma : X \to [0; \infty]$ *such that*

$$\mu(\{x \in X : \gamma(x) = 0\}) = 0.$$

Then, for every $\epsilon > 0$, *there exists* δ *such that*

$$\mu(A) < \epsilon, \text{ for all } A \in \mathcal{M} \text{ with } \int_A \gamma dx < \delta.$$

Proof of Lemma 3.12. (i) We claim that $\left(\dfrac{\partial}{\partial x_i} u_\epsilon\right)_{\epsilon\in\mathbb{N}}$ is Cauchy in measure. Indeed, let $s > 0$, consider
$A_{n,m} := \left\{\left|\dfrac{\partial}{\partial x_i} u_n\right| > h\right\} \cup \left\{\left|\dfrac{\partial}{\partial x_i} u_m\right| > h\right\}$, $B_{n,m} := \{|u_n - u_m| > k\}$ and
$C_{n,m} := \left\{\left|\dfrac{\partial}{\partial x_i} u_n\right| \leq h, \left|\dfrac{\partial}{\partial x_i} u_m\right| \leq h, |u_n - u_m| \leq k, \left|\dfrac{\partial}{\partial x_i} u_n - \dfrac{\partial}{\partial x_i} u_m > s\right|\right\}$, where h and k will be chosen later. One has

$$\left\{\left|\frac{\partial}{\partial x_i} u_n - \frac{\partial}{\partial x_i} u_m\right| > s\right\} \subset A_{n,m} \cup B_{n,m} \cup C_{n,m}. \tag{3.34}$$

Let $\vartheta > 0$. By Lemma 3.9, we can choose $h = h(\vartheta)$ large enough such that $meas(A_{n,m}) \leq \dfrac{\vartheta}{3}$ for all $n, m \geq 0$. On the other hand, by Proposition 3.10, we have that $meas(B_{n,m}) \leq \dfrac{\vartheta}{3}$

for all $n, m \geq n_0(k, \vartheta)$. Moreover, by assumption (H_3), there exists a real valued function $\gamma : \Omega \to [0, \infty]$ such that $meas\{x \in \Omega : \gamma(x) = 0\} = 0$ and

$$(a_i(x,\xi) - a_i(x,\xi')).(\xi - \xi') \geq \gamma(x), \tag{3.35}$$

for all $i = 1, ..., N$, $|\xi|, |\xi'| \leq h$, $|\xi - \xi'| \geq s$, for a.e. $x \in \Omega$. Indeed, let's set $K = \{(\xi, \eta) \in \mathbb{R} \times \mathbb{R} : |\xi| \leq h, |\eta| \leq h, |\xi - \eta| \geq s\}$. We have $K \subset B(0,h) \times B(0,h)$ and so K is a compact set because it is closed in a compact set.
For all $x \in \Omega$ and for all $i = 1, ..., N$, let us define $\psi : K \to [0; \infty]$ such that

$$\psi(\xi, \eta) = (a_i(x,\xi) - a_i(x,\eta)).(\xi - \eta).$$

As for *a.e.* $x \in \Omega$, $a_i(x, .)$ is continuous on $\mathbb{R}$, ψ is continuous on the compact K, by Weierstrass theorem, there exists $(\xi_0, \eta_0) \in K$ such that

$$\forall (\xi, \eta) \in K, \ \psi(\xi, \eta) \geq \psi(\xi_0, \eta_0).$$

Now let us define γ on Ω as follows.

$$\gamma(x) = \psi_i(\xi_0, \eta_0) = (a_i(x, \xi_0) - a_i(x, \eta)).(\xi - \eta_0).$$

Since $s > 0$, the function γ is such that $meas\left(\{x \in \Omega : \gamma(x) = 0\}\right) = 0$. Let $\delta = \delta(\epsilon)$ be given by Lemma 3.15, replacing ϵ and A by $\dfrac{\epsilon}{3}$ and $C_{n,m}$ respectively. Taking respectively $\tilde{\xi} = T_k(u_n - u_m)$ and $\tilde{\xi} = T_k(u_m - u_n)$ for the weak solutions u_n and u_m in (3.19) and after adding the two relations, we have

$$\begin{cases} \displaystyle\sum_{i=1}^{N} \int_{\{|u_n - u_m| < k\}} \left(a_i\left(x, \frac{\partial}{\partial x_i} u_n\right) - a_i\left(x, \frac{\partial}{\partial x_i} u_m\right)\right) \left(\frac{\partial}{\partial x_i}(u_n - u_m)\right) dx \\ \displaystyle + \int_Q \left(\left(\frac{1}{\epsilon^{p_i(x)}} \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial u_n}{\partial x_i}\right) - \left(\frac{1}{\epsilon^{p_i(x)}} \left|\frac{\partial u_m}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial u_m}{\partial x_i}\right)\right) \left(\frac{\partial (u_n - u_m)}{\partial x_i}\right) dx \\ \displaystyle + \int_\Omega (|u_n|^{p_M(x)-2} u_n - u_m|^{p_M(x)-2} u_m)(T_k(u_n - u_m) dx + \int_{\tilde{\Gamma}_{Ne}} (\tilde{\rho}(u_n) - \tilde{\rho}(u_m)) T_k(u_n - u_m) d\sigma \\ \displaystyle = 2\left(\int_\Omega T_k(u_n - u_m) d\mu_\epsilon + \int_{\tilde{\Gamma}_{Ne}} \tilde{d}_\epsilon T_k(u_n - u_m) d\sigma \right), \end{cases}$$

where $Q = \{\tilde{\Omega} \setminus \Omega \cap \{|u_n - u_m| < k\}\}$. As the three last terms on the left hand side are non-negative and

$$\int_\Omega T_k(u_n - u_m) d\mu_\epsilon + \int_{\tilde{\Gamma}_{Ne}} \tilde{d}_\epsilon T_k(u_n - u_m) d\sigma \leq k(|\mu|(\Omega) + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})}),$$

we deduce that

$$\sum_{i=1}^{N} \int_{\{|u_n - u_m| < k\}} \left(a_i\left(x, \frac{\partial u_n}{\partial x_i}\right) - a_i\left(x, \frac{\partial u_m}{\partial x_i}\right)\right) \left(\frac{\partial (u_n - u_m)}{\partial x_i}\right) dx \leq 2k(|\mu|(\Omega) + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})}).$$

Therefore, using (H_3) we have

$$\begin{aligned}\int_{C_{n,m}} \gamma dx &\leq \int_{C_{n,m}} \left(a_i\left(x, \frac{\partial}{\partial x_i}u_n\right) - a_i\left(x, \frac{\partial}{\partial x_i}u_m\right)\right)\frac{\partial}{\partial x_i}\left(u_n - u_m\right)dx \\ &\leq \sum_{i=1}^{N}\int_{C_{n,m}} \left(a_i\left(x, \frac{\partial}{\partial x_i}u_n\right) - a_i\left(x, \frac{\partial}{\partial x_i}u_m\right)\right)\frac{\partial}{\partial x_i}(u_n - u_m)dx \\ &\leq 2k(\|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})} + |\mu|(\Omega)) < \delta,\end{aligned}$$

by choosing $k = \delta/4\left(\|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})} + |\mu|(\Omega)\right)$. From Lemma 3.15 again, it follows that $meas(C_{n,m}) < \dfrac{\vartheta}{3}$. Thus, using (3.35) and the estimates obtained for $A_{n,m}$, $B_{n,m}$ and $C_{n,m}$, it follows that

$$meas\left(\left\{\left|\frac{\partial}{\partial x_i}u_n - \frac{\partial}{\partial x_i}u_m\right| > s\right\}\right) \leq \vartheta, \tag{3.36}$$

for all $n, m \geq n_0(s, \vartheta)$, and then the claim is proved.
As consequence, $\left(\dfrac{\partial}{\partial x_i}u_\epsilon\right)_{\epsilon\in\mathbb{N}}$ converges in measure to some measurable function ν_i.
In order to end the proof of Lemma 3.12, we need the following lemma.

Lemma 3.16. *(a) For a.e.* $k \in \mathbb{R}$, $\dfrac{\partial}{\partial x_i}T_k(u_\epsilon)$ *converges in measure to* $\nu_i\chi_{\{|u|<k\}}$.

(b) For a.e. $k \in \mathbb{R}$, $\dfrac{\partial}{\partial x_i}T_k(u) = \nu_i\chi_{\{|u|<k\}}$.

(c) $\dfrac{\partial}{\partial x_i}T_k(u) = \nu_i\chi_{\{|u|<k\}}$ *holds for all* $k \in \mathbb{R}$.

Proof. (a) We know that $\dfrac{\partial}{\partial x_i}u_\epsilon \to \nu_i$ in measure. Thus $\dfrac{\partial}{\partial x_i}u_\epsilon\chi_{\{|u|<k\}} \to \nu_i\chi_{\{|u|<k\}}$ in measure.

Now, let us show that $\left(\chi_{\{|u_\epsilon|<k\}} - \chi_{\{|u|<k\}}\right)\dfrac{\partial}{\partial x_i}u_\epsilon \to 0$ in measure.
For that, it is sufficient to show that $\left(\chi_{\{|u_\epsilon|<k\}} - \chi_{\{|u|<k\}}\right) \to 0$ in measure. Now, for all $\delta > 0$, $\left\{|\chi_{\{|u_\epsilon|<k\}} - \chi_{\{|u|<k\}}|\left|\dfrac{\partial}{\partial x_i}u_\epsilon\right| > \delta\right\} \subset \left\{|\chi_{\{|u_\epsilon|<k\}} - \chi_{\{|u|<k\}}| \neq 0\right\} \subset \{|u| = k\} \cup \{u_\epsilon < k < u\} \cup \{u < k < u_\epsilon\} \cup \{u_\epsilon < -k < u\} \cup \{u < -k < u_\epsilon\}$. Thus,

$$\left\{\begin{aligned} & meas\left(\left\{|\chi_{\{|u_\epsilon|<k\}} - \chi_{\{|u|<k\}}|\left|\frac{\partial}{\partial x_i}u_\epsilon\right| > \delta\right\}\right) \\ \leq\ & meas\left(\{|u| = k\}\right) + meas\left(\{u_\epsilon < k < u\}\right) \\ & + meas\left(\{u < k < u_\epsilon\}\right) \\ & + meas\left(\{u_\epsilon < -k < u\}\right) \\ & + meas\left(\{u < -k < u_\epsilon\}\right). \end{aligned}\right. \tag{3.37}$$

Note that

$meas\left(\{|u| = k\}\right) \leq meas\left(\{k - h < u < k + h\}\right) + meas\left(\{-k - h < u < -k + h\}\right) \to 0$ as $h \to 0$ for a.e. $k > 0$, since u is fixed function.

Next, $meas\left(\{u_\epsilon < k < u\}\right) \leq meas\left(\{k < u < k + h\}\right) + meas\left(\{|u_\epsilon - u| > h\}\right)$, for all

$h > 0$.

Due to Proposition 3.10, we have for all fixed $h > 0$, $meas\left(\{|u_\epsilon - u| > h\}\right) \to 0$ as $\epsilon \to 0$. Since $meas\left(\{k < u < k+h\}\right) \to 0$ as $h \to 0$, for all $\vartheta > 0$, one can find N such that for all $n > N$, $meas\left(\{|u| = k\}\right) < \frac{\vartheta}{2} + \frac{\vartheta}{2} = \vartheta$ by choosing h and then N. Each of the other terms on the right-hand side of (3.37) can be treated in the same way as for $meas\left(\{u_\epsilon < k < u\}\right)$. Thus, $meas\left(\left\{|\chi_{\{|u_\epsilon|<k\}} - \chi_{\{|u|<k\}}|\left|\frac{\partial}{\partial x_i}u_\epsilon\right| > \delta\}\right\}\right) \to 0$ as $\epsilon \to 0$. Finally, since $\frac{\partial}{\partial x_i}T_k(u_\epsilon) = \frac{\partial}{\partial x_i}u_\epsilon\chi_{\{|u_\epsilon|<k\}}$, the claim (a) follows.

(b) Using the Proof of Proposition 3.10-(ii) we have $\frac{\partial}{\partial x_i}T_k(u_\epsilon) \rightharpoonup \frac{\partial}{\partial x_i}T_k(u)$ weakly in $L^{p_i^-}(\tilde{\Omega})$. The previous convergence also ensures that $\frac{\partial}{\partial x_i}T_k(u_\epsilon)$ converges to $\frac{\partial}{\partial x_i}T_k(u)$ weakly in $L^1(\Omega)$. On the other hand, by (a), $\frac{\partial}{\partial x_i}T_k(u_\epsilon)$ converges to $\nu_i\chi_{\{|u|<k\}}$ in measure. By Lemma 3.14, since $\frac{\partial}{\partial x_i}T_k(u_\epsilon)$ is uniformly bounded in $L^{p_i^-}(\tilde{\Omega})$ (see Lemma 3.8) hence in $L^{p_i^-}(\Omega)$, the convergence is actually strong in $L^1(\Omega)$; thus it is also weak in $L^1(\Omega)$. By the uniqueness of the weak L^1-limit, $\nu_i\chi_{\{|u|<k\}}$ coincides with $\frac{\partial}{\partial x_i}T_k(u)$.

(c) Let $0 < k < s$, and s be such that $\nu_i\chi_{\{|u|<s\}}$ coincides with $\frac{\partial}{\partial x_i}T_s(u)$. Then,

$$
\begin{aligned}
\frac{\partial}{\partial x_i}T_k(u) &= \frac{\partial}{\partial x_i}T_k(T_s(u)) \\
&= \frac{\partial}{\partial x_i}T_s(u)\chi_{\{|T_s(u)|<k\}} \\
&= \nu_i\chi_{\{|u|<s\}}\chi_{\{|u|<k\}} \\
&= \nu_i\chi_{\{|u|<k\}}.
\end{aligned}
$$

□

□

Now, we can end the proof of Lemma 3.12. Indeed, combining lemmas 3.16 (c) and 3.13; *(i)* follows.

Next, by lemmas 3.14 and 3.16, we have for all $k > 0$, $i = 1, ..., N$, $a_i\left(x, \frac{\partial}{\partial x_i}T_k(u_\epsilon)\right)$ converges to $a_i\left(x, \frac{\partial}{\partial x_i}T_k(u)\right)$ in $L^1(\Omega)$ strongly. Indeed, let $s, k > 0$, consider $E_4 = \left\{\left|\frac{\partial u_n}{\partial x_i} - \frac{\partial u_m}{\partial x_i}\right| > s, |u_n| \le k, |u_m| \le k\right\}$, $E_5 = \left\{\left|\frac{\partial u_m}{\partial x_i}\right| > s, |u_n| > k, |u_m| \le k\right\}$, $E_6 = \left\{\left|\frac{\partial u_n}{\partial x_i}\right| > s, |u_n| \le k, |u_m| > k\right\}$.

We have

$$
\left\{\left|\frac{\partial T_k(u_n)}{\partial x_i} - \frac{\partial T_k(u_m)}{\partial x_i}\right| > s\right\} \subset E_4 \cup E_5 \cup E_6. \tag{3.38}
$$

$\forall\vartheta > 0$, by Lemma 3.7, there exists $k(\vartheta)$ such that

$$meas(E_5) \leq \frac{\vartheta}{3} \text{ and } meas(E_6) \leq \frac{\vartheta}{3}. \tag{3.39}$$

Using (3.36)-(3.39), we get

$$meas\left(\left\{\left|\frac{\partial}{\partial x_i}T_k(u_n) - \frac{\partial}{\partial x_i}T_k(u_m)\right| > s\right\}\right) \leq \vartheta, \tag{3.40}$$

for all $n, m \geq n_1(s,\vartheta)$. Therefore, $\frac{\partial T_k(u_\epsilon)}{\partial x_i}$ converges in measure to $\frac{\partial T_k(u)}{\partial x_i}$. Using lemmas 3.8 and 3.14, we deduce that $\frac{\partial T_k(u_\epsilon)}{\partial x_i}$ converges to $\frac{\partial T_k(u)}{\partial x_i}$ in $L^1(\Omega)$. So, after passing to a suitable subsequence of $\left(\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right)_{\epsilon>0}$, we can assume that $\frac{\partial T_k(u_\epsilon)}{\partial x_i}$ converges to $\frac{\partial T_k(u)}{\partial x_i}$ *a.e.* in Ω. By the continuity of $a_i(x,.)$, we deduce that $a_i\left(x, \frac{\partial T_k(u_\epsilon)}{\partial x_i}\right)$ converges to $a_i\left(x, \frac{\partial T_k(u)}{\partial x_i}\right)$ *a.e.* in Ω. As Ω is bounded, this convergence is in measure. Using lemmas 3.14 and 3.16, we deduce that for all $k > 0$, $i = 1, ..., N$, $a_i\left(x, \frac{\partial}{\partial x_i}T_k(u_\epsilon)\right)$ converges to $a_i\left(x, \frac{\partial}{\partial x_i}T_k(u)\right)$ in $L^1(\Omega)$ strongly and $a_i\left(x, \frac{\partial}{\partial x_i}T_k(u_\epsilon)\right)$ converges to $\chi_k \in L^{p'_i(\cdot)}(\Omega)$ weakly in $L^{p'_i(\cdot)}(\Omega)$. Since each of the convergences implies the weak L^1-convergence, χ_k can be identified with $a_i\left(x, \frac{\partial}{\partial x_i}T_k(u)\right)$; thus, $a_i\left(x, \frac{\partial}{\partial x_i}T_k(u)\right) \in L^{p'_i(\cdot)}(\Omega)$

By using Lebesgue generalized convergence theorem and above results, we deduce the following result.

Proposition 3.17. *For any $k > 0$ and any $i = 1, ..., N$, as ϵ tends to 0, we have*

(i) $\frac{\partial T_k(u_\epsilon)}{\partial x_i} \to \frac{\partial T_k(u)}{\partial x_i}$ *a.e. in* Ω,

(ii) $a_i\left(x, \frac{\partial T_k(u_\epsilon)}{\partial x_i}\right)\frac{\partial T_k(u_\epsilon)}{\partial x_i} \to a_i\left(x, \frac{\partial T_k(u)}{\partial x_i}\right)\frac{\partial T_k(u)}{\partial x_i}$ *a.e. in* Ω *and strongly in* $L^1(\Omega)$,

(iii) $\frac{\partial T_k(u_\epsilon)}{\partial x_i} \to \frac{\partial T_k(u)}{\partial x_i}$ *strongly in* $L^{p_i(x)}(\Omega)$.

4 Existence and uniqueness of solution to $P(\rho, \mu, d)$

We are now able to prove Theorem 2.6.

Proof of Theorem 2.6

Thanks to the Proposition 3.10 and as $\forall k > 0$, $\forall i = 1, ..., N$, $\frac{\partial T_k(u)}{\partial x_i} = 0$ in $L^{p_i^-}(\tilde{\Omega} \setminus \Omega)$, then, $\forall k > 0$, $T_k(u) = constant$ *a.e.* on $\tilde{\Omega} \setminus \Omega$. Hence, we conclude that $u \in \mathcal{T}^{1,\vec{p}(\cdot)}_{Ne}(\Omega)$.

We already state that $b(u) \in L^1(\Omega)$.

To show that u is an entropy solution of $P(\rho, \mu, d)$, we only have to prove the inequality in (2.9). Let $\varphi \in W_D^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$. We consider the function $\varphi_1 \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \cap L^\infty(\Omega)$ such that

$$\varphi_1 = \varphi \chi_\Omega + \varphi_N \chi_{\tilde{\Omega}\setminus\Omega}.$$

We set $\tilde{\xi} = T_k(u_\epsilon - \varphi_1)$ in (3.19) to get

$$\begin{cases} \displaystyle\sum_{i=1}^N \int_\Omega \left(a_i\left(x, \frac{\partial}{\partial x_i} u_\epsilon\right) . \frac{\partial}{\partial x_i} T_k(u_\epsilon - \varphi) \right) dx \\ \displaystyle + \sum_{i=1}^N \int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon^{p_i(x)}} \left| \frac{\partial}{\partial x_i} u_\epsilon \right|^{p_i(x)-2} \frac{\partial}{\partial x_i} u_\epsilon . \frac{\partial}{\partial x_i} T_k(u_\epsilon - \varphi_N) \right) dx \\ \displaystyle \int_\Omega b(u_\epsilon) T_k(u_\epsilon - \varphi) dx = \int_\Omega T_k(u_\epsilon - \varphi) d\mu_\epsilon + \int_{\tilde{\Gamma}_{Ne}} (\tilde{d}_\epsilon - \tilde{\rho}(u_\epsilon)) T_k(u_\epsilon - \varphi_N) d\sigma. \end{cases} \tag{4.1}$$

The following convergence result hold true.

Lemma 4.1. *For any $k > 0$, for all $i = 1, ..., N$, as $\epsilon \to 0$,*

$$\frac{\partial}{\partial x_i} T_k(u_\epsilon - \varphi) \to \frac{\partial}{\partial x_i} T_k(u - \varphi) \text{ strongly in } L^{p_i(.)}(\Omega).$$

Proof. Let $k > 0$, $i = 1, ..., N$. We have

$$\begin{aligned} & \int_\Omega \left| \frac{\partial}{\partial x_i} T_k(u_\epsilon - \varphi) - \frac{\partial}{\partial x_i} T_k(u - \varphi) \right|^{p_i(x)} dx \\ = & \int_{\Omega \cap [|u_\epsilon - \varphi| \le k, |u - \varphi| \le k]} \left| \frac{\partial}{\partial x_i} u_\epsilon - \frac{\partial}{\partial x_i} u \right|^{p_i(x)} dx \\ \le & \int_{\Omega \cap [|u_\epsilon| \le l, |u| \le l]} \left| \frac{\partial u_\epsilon}{\partial x_i} - \frac{\partial u}{\partial x_i} \right|^{p_i(x)} dx, \text{ with } l = k + \|\varphi\|_\infty \\ = & \int_\Omega \left| \frac{\partial}{\partial x_i} T_l(u_\epsilon) - \frac{\partial}{\partial x_i} T_l(u) \right|^{p_i(x)} dx \\ \to & \ 0 \text{ as } \epsilon \to 0 \text{ by Proposition 3.17} - (iii). \end{aligned}$$

□

We need to pass to the limit in (4.1) as $\epsilon \to 0$. We have

$$\sum_{i=1}^N \int_\Omega \left(a_i\left(x, \frac{\partial}{\partial x_i} u_\epsilon\right) \frac{\partial}{\partial x_i} T_k(u_\epsilon - \varphi) \right) dx = \sum_{i=1}^N \int_\Omega \left(a_i\left(x, \frac{\partial T_l(u_\epsilon)}{\partial x_i}\right) \frac{\partial}{\partial x_i} T_k(u_\epsilon - \varphi) \right) dx,$$

with $l = k + \|\varphi\|_\infty$, then, by Lemma 3.12- (ii) and Lemma 4.1, we have

$$\lim_{\epsilon \to 0} \sum_{i=1}^N \int_\Omega \left(a_i\left(x, \frac{\partial T_l(u_\epsilon)}{\partial x_i}\right) \frac{\partial}{\partial x_i} T_k(u_\epsilon - \varphi) \right) dx = \sum_{i=1}^N \int_\Omega \left(a_i\left(x, \frac{\partial T_l(u)}{\partial x_i}\right) \frac{\partial}{\partial x_i} T_k(u - \varphi) \right) dx;$$

that is

$$\lim_{\epsilon \to 0} \sum_{i=1}^N \int_\Omega \left(a_i\left(x, \frac{\partial}{\partial x_i} u_\epsilon\right) \frac{\partial}{\partial x_i} T_k(u_\epsilon - \varphi) \right) dx = \sum_{i=1}^N \int_\Omega \left(a_i\left(x, \frac{\partial T_l(u)}{\partial x_i}\right) \frac{\partial}{\partial x_i} T_k(u - \varphi) \right) dx. \tag{4.2}$$

For the second term in the left hand side of (4.1), we have

$$\limsup_{\epsilon\to 0}\sum_{i=1}^{N}\int_{\tilde{\Omega}\setminus\Omega}\left(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial}{\partial x_i}u_\epsilon|^{p_i(x)-2}\frac{\partial}{\partial x_i}u_\epsilon\frac{\partial}{\partial x_i}T_k(u_\epsilon-\varphi_N)\right)dx\geq 0. \tag{4.3}$$

Indeed

$$\begin{cases} & \sum_{i=1}^{N}\int_{\tilde{\Omega}\setminus\Omega}\left(\frac{1}{\epsilon^{p_i(x)}}\left|\frac{\partial}{\partial x_i}u_\epsilon\right|^{p_i(x)-2}\frac{\partial}{\partial x_i}u_\epsilon\frac{\partial}{\partial x_i}T_k(u_\epsilon-\varphi_N)\right)dx \\ = & \sum_{i=1}^{N}\int_{\tilde{\Omega}\setminus\Omega\cap[|u_\epsilon-\varphi|\leq k]}\left(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial}{\partial x_i}u_\epsilon|^{p_i(x)-2}\frac{\partial}{\partial x_i}u_\epsilon\frac{\partial}{\partial x_i}(u_\epsilon-\varphi_N)\right)dx \\ = & \sum_{i=1}^{N}\int_{\tilde{\Omega}\setminus\Omega\cap[|u_\epsilon-\varphi|\leq k]}\left(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial}{\partial x_i}u_\epsilon|^{p_i(x)}\right)dx\geq 0. \end{cases}$$

Hence, we get (4.3).

Let us examine the last term in the left hand side of (4.1).

we have

$$\int_\Omega b(u_\epsilon)T_k(u_\epsilon-\varphi)dx=\int_\Omega(b(u_\epsilon)-b(\varphi))T_k(u_\epsilon-\varphi)dx+\int_\Omega b(\varphi)T_k(u_\epsilon-\varphi)dx.$$

As b is non-decreasing,

$$(b(u_\epsilon)-b(\varphi))T_k(u_\epsilon-\varphi)\geq 0 \text{ a.e. in } \Omega$$

and we get by Fatou's lemma that

$$\liminf_{\epsilon\to 0}\int_\Omega(b(u_\epsilon)-b(\varphi))T_k(u_\epsilon-\varphi)dx\geq\int_\Omega(b(u)-b(\varphi))T_k(u-\varphi)dx.$$

As $\varphi\in L^\infty(\Omega)$, we obtain $b(\varphi)\in L^\infty(\Omega)$ and so $b(\varphi)\in L^1(\Omega)$ (as Ω is bounded) and by Lebesgue dominated convergence theorem, we deduce that

$$\lim_{\epsilon\to 0}\int_\Omega b(\varphi)T_k(u_\epsilon-\varphi)dx=\int_\Omega b(\varphi)T_k(u-\varphi)dx.$$

Consequently,

$$\limsup_{\epsilon\to 0}\int_\Omega b(u_\epsilon)T_k(u_\epsilon-\varphi)dx\geq\int_\Omega b(u)T_k(u-\varphi)dx. \tag{4.4}$$

As $f_\epsilon\to f$ strongly in $L^1(\Omega)$ and $T_k(u_\epsilon-v)\rightharpoonup^* T_k(u-v)$ in $L^\infty(\Omega)$, using the Lebesgue generalized convergence theorem we have

$$\begin{cases}\lim_{\epsilon\to 0}\int_\Omega f_\epsilon T_k(u_\epsilon-\varphi)dx=\int_\Omega T_k(u-\varphi)dx,\\ \lim_{\epsilon\to 0}\int_{\tilde{\Gamma}_{Ne}}\tilde{d}_\epsilon T_k(u_\epsilon-\varphi_N)d\sigma=\int_\Omega\tilde{d}T_k(u-\varphi_N)d\sigma.\end{cases} \tag{4.5}$$

Since $\nabla T_k(u_\epsilon-\varphi)\rightharpoonup\nabla T_k(u-\varphi)$ in $(L^{p_m(\cdot)}(\Omega))^N$ and $F\in(L^{p'_m(\cdot)}(\Omega))^N$,

$$\lim_{\epsilon\to 0}\int_\Omega F.\nabla T_k(u_\epsilon-\varphi)dx=\int_\Omega F.\nabla T_k(u-\varphi)dx. \tag{4.6}$$

We know that $\forall k>0$, $T_k(u)=constant$ on $\tilde{\Omega}\setminus\Omega$, then, it yields that $u=constant$ on $\tilde{\Omega}\setminus\Omega$. So, one has

$$\lim_{\epsilon\to 0}\int_{\tilde{\Gamma}_{Ne}}\tilde{d}_\epsilon T_k(u_\epsilon-\varphi)dx=dT_k(u_N-\varphi_N). \tag{4.7}$$

At last, we have

$$\int_{\tilde{\Gamma}_{Ne}} \tilde{\rho}(u_\epsilon)T_k(u_\epsilon - \varphi_N)d\sigma = \int_{\tilde{\Gamma}_{Ne}} (\tilde{\rho}(u_\epsilon) - \tilde{\rho}(\varphi_N))T_k(u_\epsilon - \varphi_N)d\sigma + \int_{\tilde{\Gamma}_{Ne}} \tilde{\rho}(\varphi_N)T_k(u_\epsilon - \varphi_N)d\sigma.$$

As $\tilde{\rho}$ is non-decreasing,

$$(\tilde{\rho}(u_\epsilon) - \tilde{\rho}(\varphi_N))T_k(u_\epsilon - \varphi_N) \geq 0 \text{ a.e. in } \tilde{\Gamma}_{Ne}$$

and we get by Fatou's lemma that

$$\begin{aligned}\liminf_{\epsilon\to 0} \int_{\tilde{\Gamma}_{Ne}} (\tilde{\rho}(u_\epsilon) - \tilde{\rho}(\varphi_N))T_k(u_\epsilon - \varphi_N)d\sigma \quad &\geq \quad \int_{\tilde{\Gamma}_{Ne}} (\tilde{\rho}(u_N) - \tilde{\rho}(\varphi_N))T_k(u_N - \varphi_N)d\sigma \\ &= \quad (\rho(u_N) - \rho(\varphi_N))T_k(u_N - \varphi_N).\end{aligned}$$

As $\varphi_N \in L^\infty(\tilde{\Gamma}_{Ne})$, we obtain $\tilde{\rho}(\varphi_N) \in L^\infty(\tilde{\Gamma}_{Ne})$ and so $\tilde{\rho}(\varphi_N) \in L^1(\tilde{\Gamma}_{Ne})$ (as $\tilde{\Gamma}_{Ne}$ is bounded) and by the Lebesgue dominated convergence theorem, we deduce that

$$\lim_{\epsilon\to 0} \int_{\tilde{\Gamma}_{Ne}} \tilde{\rho}(\varphi_N)T_k(u_\epsilon - \varphi_N)d\sigma = \int_{\tilde{\Gamma}_{Ne}} \tilde{\rho}(\varphi_N)T_k(u_N - \varphi_N)d\sigma = \rho(\varphi_N)T_k(u_N - \varphi_N).$$

Hence,

$$\limsup_{\epsilon\to 0} \int_{\tilde{\Gamma}_{Ne}} \tilde{\rho}(u_\epsilon)T_k(u_\epsilon - \varphi_N)d\sigma \geq \rho(\varphi_N)T_k(u_N - \varphi_N). \tag{4.8}$$

Passing to the limit as $\epsilon \to 0$ in (4.1) and using (4.2)-(4.8), we see that u is an entropy solution of $P(\rho, \mu, d)$.

We now prove the uniqueness part of Theorem 2.6.
Let u and v be two entropy solutions of $P(\rho, \mu, d)$.
Let $h > 0$. For u, we take $\xi = T_h(v)$ as test function and for v, we take $\xi = T_h(u)$ as test function in (2.9), to get for any $k > 0$ with $k < h$,

$$\begin{cases} \displaystyle\int_\Omega \left(\sum_{i=1}^N a_i\left(x, \frac{\partial}{\partial x_i}u\right)\frac{\partial}{\partial x_i}T_k(u - T_h(v))\right)dx + \int_\Omega b(u)T_k(u - T_h(v))dx \leq \\ \displaystyle\int_\Omega fT_k(u - T_h(v))dx + \int_\Omega F.\nabla T_k(u - T_h(v))dx + (d - \rho(u_{Ne}))T_k(u_{Ne} - T_h(v)) \end{cases} \tag{4.9}$$

and

$$\begin{cases} \displaystyle\int_\Omega \left(\sum_{i=1}^N a_i\left(x, \frac{\partial}{\partial x_i}v\right)\frac{\partial}{\partial x_i}T_k(v - T_h(u))\right)dx + \int_\Omega b(v)T_k(v - T_h(u))dx \leq \\ \displaystyle\int_\Omega fT_k(v - T_h(u))dx + \int_\Omega F.\nabla T_k(v - T_h(u))dx + (d - \rho(v_{Ne}))T_k(v_{Ne} - T_h(u)). \end{cases} \tag{4.10}$$

By adding (4.9) and (4.10), we obtain

$$\begin{cases} \int_\Omega \left(\sum_{i=1}^N a_i \left(x, \frac{\partial}{\partial x_i} u \right) \frac{\partial}{\partial x_i} T_k(u - T_h(v)) \right) dx \\ + \int_\Omega \left(\sum_{i=1}^N a_i \left(x, \frac{\partial}{\partial x_i} v \right) \frac{\partial}{\partial x_i} T_k(v - T_h(u)) \right) dx & := A(h,k) \\ + \int_\Omega b(u) T_k(u - T_h(v)) dx + \int_\Omega b(v) T_k(v - T_h(u)) dx & := B(h,k) \\ + \rho(u_{Ne}) T_k(u_{Ne} - T_h(v)) + \rho(v_{Ne}) T_k(v_{Ne} - T_h(u)) & := C(h,k) \\ \leq \int_\Omega f T_k(u - T_h(v)) dx + \int_\Omega f T_k(v - T_h(u)) dx & := D(h,k) \\ + \int_\Omega F.\nabla T_k(u - T_h(v)) dx + \int_\Omega F.\nabla T_k(v - T_h(u)) dx & := T(h,k) \\ + d T_k(u_{Ne} - T_h(v)) + d T_k(v_{Ne} - T_h(u)) & := E(h,k). \end{cases} \tag{4.11}$$

Let us introduce the following subsets of Ω.

$$\begin{aligned} A_0 &:= [|u - v| < k, |u| < h, |v| < h] \\ A_1 &:= [|u - T_h(v)| < k, |v| \geq h] \\ A_1' &:= [|v - T_h(u)| < k, |u| \geq h] \\ A_2 &:= [|u - T_h(v)| < k, |u| \geq h, |v| < h] \\ A_2' &:= [|v - T_h(u)| < k, |v| \geq h, |u| < h]. \end{aligned}$$

We have the following assertion (see [22] for the proof).

Assertion 4.2. *If u is an entropy solution of $P(\rho, \mu, d)$, then $A_2 \subset F_{h,k}$ and $A_1 \subset F_{h-k,2k}$, where*

$$F_{h,k} = \{h \leq |u| < h + k, h > 0, k > 0\}.$$

Assertion 4.3. *Let u be an entropy solution of $P(\rho, \mu, d)$. On A_2 (and on A_1) we have according to Hölder inequality.*

(1)
$$\int_{A_2} F.\nabla u dx \leq \left(\int_{A_2} |F|^{(p_m')^-} dx \right)^{\frac{1}{(p_m')^-}} \left(\int_{A_2} |\nabla u|^{p_m^-} \right)^{\frac{1}{p_m^-}} dx, \tag{4.12}$$

with $\lim_{h \to \infty} \left(\int_{A_2} |F|^{(p_m')^-} dx \right)^{\frac{1}{(p_m')^-}} \left(\int_{A_2} |\nabla u|^{p_m^-} dx \right)^{\frac{1}{p_m^-}} = 0.$

(2)
$$\int_{A_1} F.\nabla u dx \leq \left(\int_{A_1} |F|^{(p_m')^-} dx \right)^{\frac{1}{(p_m')^-}} \left(\int_{A_1} |\nabla u|^{p_m^-} dx \right)^{\frac{1}{p_m^-}}, \tag{4.13}$$

with $\lim_{h \to \infty} \left(\int_{A_1} |F|^{(p_m')^-} dx \right)^{\frac{1}{(p_m')^-}} \left(\int_{A_1} |\nabla u|^{p_m^-} dx \right)^{\frac{1}{p_m^-}} = 0.$

Proof. (1) $\lim_{h\to\infty}\left(\int_{A_2}|F|^{(p'_m)^-}dx\right)^{\frac{1}{(p'_m)^-}}=0$ (see [22]).

Now, it remains to prove that $\left(\int_{A_2}|\nabla u|^{p_m^-}dx\right)^{\frac{1}{p_m^-}}$ is bounded with respect to h.

We make the following notations:

$\mathcal{I}=\left\{i\in\{1,...,N\}:\left\{\left|\frac{\partial}{\partial x_i}u\right|\right\}\leq 1\right\}$ and $\mathcal{J}=\left\{i\in\{1,...,N\}:\left\{\left|\frac{\partial}{\partial x_i}u\right|\right\}>1\right\}$.

We have

$$\begin{aligned}
\sum_{i=1}^{N}\int_{F_{h,k}}\left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)}dx &= \sum_{i\in\mathcal{I}}\left(\int_{F_{h,k}}\left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)}dx\right)+\sum_{i\in\mathcal{J}}\left(\int_{F_{h,k}}\left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)}dx\right)\\
&\geq \sum_{i\in\mathcal{J}}\left(\int_{F_{h,k}}\left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)}dx\right)\\
&\geq \sum_{i\in\mathcal{J}}\left(\int_{F_{h,k}}\left|\frac{\partial}{\partial x_i}u\right|^{p_m^-}dx\right)\\
&\geq \sum_{i=1}^{N}\left(\int_{F_{h,k}}\left|\frac{\partial}{\partial x_i}u\right|^{p_m^-}dx\right)-\sum_{i\in\mathcal{I}}\left(\int_{F_{h,k}}\left|\frac{\partial}{\partial x_i}u\right|^{p_m^-}dx\right)\\
&\geq \sum_{i=1}^{N}\left(\int_{F_{h,k}}\left|\frac{\partial}{\partial x_i}u\right|^{p_m^-}\right)-Nmeas(\Omega)\\
&\geq \sum_{i=1}^{N}\left\|\frac{\partial}{\partial x_i}u\right\|^{p_m^-}_{(L^{p_m^-}(F_{h,k}))^N}-Nmeas(\Omega)\\
&\geq C\|\nabla u\|^{p_m^-}_{(L^{p_m^-}(F_{h,k}))^N}-Nmeas(\Omega).
\end{aligned}$$

We deduce that

$$\sum_{i=1}^{N}\int_{F_{h,k}}\left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)}dx\geq C\int_{F_{h,k}}|\nabla u|^{p_m^-}dx-Nmeas(\Omega). \tag{4.14}$$

Choosing $T_h(u)$ as test function in (2.9), we get

$$\begin{cases}\int_\Omega\left(\sum_{i=1}^{N}a_i\left(x,\frac{\partial}{\partial x_i}u\right))\frac{\partial}{\partial x_i}T_k(u-T_h(u))\right)dx+\int_\Omega|u|^{p_M(x)-2}uT_k(u-T_h(u))dx\leq\\ \int_\Omega fT_k(u-T_h(u))dx+\int_\Omega F.\nabla T_k(u-T_h(u))dx+(d-\rho(u_{Ne}))T_k(u_{Ne}-T_h(u_{Ne})).\end{cases} \tag{4.15}$$

According to the fact that $\nabla T_k(u-T_h(u))=\nabla u$ on $\{h\leq|u|<h+k\}$ and zero elsewhere, $\int_\Omega|u|^{p_M(x)-2}uT_k(u-T_h(u))dx\geq 0$ and $\rho(u_{Ne})T_k(u_{Ne}-T_h(u_{Ne}))\geq 0$, we deduce from (4.15) that

$$\begin{cases}\int_{F_{h,k}}\left(\sum_{i=1}^{N}a_i\left(x,\frac{\partial}{\partial x_i}u\right)\frac{\partial}{\partial x_i}T_k(u-T_h(u))\right)dx\leq\\ k\int_{|u|\geq h}|f|dx+\int_{F_{h,k}}\left|\left(\frac{2}{Cp_m^-}\right)^{\frac{1}{p_m^-}}F\right|\left|\left(\frac{Cp_m^-}{2}\right)^{\frac{1}{p_m^-}}\nabla u\right|dx+k|d|.\end{cases} \tag{4.16}$$

Using (1.7) (in the left hand side of (4.16)), Young inequality (in the right hand side of(4.16)) and setting

$$c = \left(\frac{2}{Cp_m^-}\right)^{\dfrac{(p'_m)^-}{p_m^-}} \frac{p_m^- - 1}{p_m^-},$$

we obtain

$$\begin{cases} \displaystyle\sum_{i=1}^{N} \int_{F_{h,k}} \left|\frac{\partial}{\partial x_i} u\right|^{p_i(x)} dx \leq \\ \displaystyle k \int_{|u| \geq h} |f| dx + c \int_{F_{h,k}} |F|^{(p')_m^-} dx + \frac{C}{2} \int_{F_{h,k}} |\nabla u|^{p_m^-} dx + k|d|. \end{cases} \tag{4.17}$$

From (4.14) and (4.17), we deduce

$$\begin{cases} \displaystyle C \int_{F_{h,k}} |\nabla u|^{p_m^-} dx \leq \\ \displaystyle k \int_{|u| \geq h} |f| dx + c \int_{F_{h,k}} |F|^{(p')_m^-} dx + \frac{C}{2} \int_{F_{h,k}} |\nabla u|^{p_m^-} dx + k|d| + N meas(\Omega). \end{cases}$$

Therefore,

$$\begin{cases} \displaystyle \frac{C}{2} \int_{F_{h,k}} |\nabla u|^{p_m^-} dx \leq \\ \displaystyle k \int_{\{|u| \geq h\}} |f| dx + c \int_{F_{h,k}} |F|^{(p')_m^-} dx + k|d| + N meas(\Omega). \end{cases} \tag{4.18}$$

Since $A_2 \subset F_{h,k}$, we deduce from (4.18) that $\displaystyle\int_{A_2} |\nabla u|^{p_m^-} dx$ is bounded.

(2) $\displaystyle\lim_{h\to\infty} \left(\int_{A_1} |F|^{(p'_m)^-} dx\right)^{\frac{1}{(p'_m)^-}} = 0$ (see [22]).

Now, it remains to prove that $\displaystyle\left(\int_{A_1} |\nabla u|^{p_m^-} dx\right)^{\frac{1}{p_m^-}}$ is bounded with respect to h.

Since $A_1 \subset F_{h-k,2k}$, we deduce from (4.18) that $\displaystyle\int_{A_2} |\nabla u|^{p_m^-} dx$ is bounded.

□

Remark 4.4. *Similarly, we prove that if v is an entropy solution of $P(\rho, f, d)$, then*

$$\lim_{h\to\infty} \int_{A'_2} F.\nabla v dx \leq 0$$

and

$$\lim_{h\to\infty} \int_{A'_1} F.\nabla v dx \leq 0.$$

Now, we have

$$\begin{cases} \displaystyle A(h,k) = \int_{A_0} \left(\sum_{i=1}^{N} \left(a_i\left(x, \frac{\partial}{\partial x_i} u\right) - a_i\left(x, \frac{\partial}{\partial x_i} v\right)\right) \frac{\partial}{\partial x_i}(u-v)\right) dx & := I_0(h,k) \\ \displaystyle + \int_{A_1} \left(\sum_{i=1}^{N} a_i\left(x, \frac{\partial}{\partial x_i} u\right) \frac{\partial}{\partial x_i} u\right) dx + \int_{A'_1} \left(\sum_{i=1}^{N} a_i\left(x, \frac{\partial}{\partial x_i} v\right) \frac{\partial}{\partial x_i} v\right) dx & := I_1(h,k) \\ \displaystyle + \int_{A_2} \left(\sum_{i=1}^{N} a_i\left(x, \frac{\partial}{\partial x_i} u\right) \frac{\partial}{\partial x_i}(u-v)\right) dx + \int_{A'_2} \left(\sum_{i=1}^{N} a_i\left(x, \frac{\partial}{\partial x_i} v\right) \frac{\partial}{\partial x_i}(v-u)\right) dx & := I_2(h,k). \end{cases}$$

The term $I_1(h,k)$ is non-negative since each term in $I_1(h,k)$ is non-negative.
For the term $I_2(h,k)$, as

$$I_2(h,k) + \int_{A_2} \left(\sum_{i=1}^N a_i \left(x, \frac{\partial}{\partial x_i} u \right) \frac{\partial}{\partial x_i} v \right) dx + \int_{A_2'} \left(\sum_{i=1}^N a_i \left(x, \frac{\partial}{\partial x_i} v \right) \frac{\partial}{\partial x_i} u \right) dx = I_1(h,k),$$

so,

$$I_2(h,k) \geq - \left(\int_{A_2} \left(\sum_{i=1}^N a_i \left(x, \frac{\partial}{\partial x_i} u \right) \frac{\partial}{\partial x_i} v \right) dx + \int_{A_2'} \left(\sum_{i=1}^N a_i \left(x, \frac{\partial}{\partial x_i} v \right) \frac{\partial}{\partial x_i} u \right) dx \right).$$

Let us show that $- \left(\int_{A_2} \left(\sum_{i=1}^N a_i \left(x, \frac{\partial}{\partial x_i} u \right) \frac{\partial}{\partial x_i} v \right) dx \right)$ goes to 0 as $h \to \infty$.
We have

$$\begin{cases} \left| \int_{A_2} \left(\sum_{i=1}^N a_i \left(x, \frac{\partial}{\partial x_i} u \right) \frac{\partial}{\partial x_i} (v) \right) dx \right| \leq \\ C \sum_{i=1}^N \left(|j_i|_{p_i'(.)} + \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)-1}_{L^{p_i(.)}(\{h<|u|\leq h+k\})} \right) \left| \frac{\partial v}{\partial x_i} \right|_{L^{p_i(.)}(\{h-k<|v|\leq h\})} \end{cases}.$$

For all $i = 1, ...N$, the quantity $\left(|j_i|_{p_i'(.)} + \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)-1}_{L^{p_i(.)}(\{h<|u|\leq h+k\})} \right)$ is finite since $u = T_{h+k}(u) \in \mathcal{T}^{1,\vec{p}(.)}_{Ne}(\Omega)$ and $j_i \in L^{p_i'(.)}(\Omega)$; then by Lemma 3.8, the last expression converges to zero as h tends to infinity.
Similarly we can show that $- \left(\int_{A_2} \left(\sum_{i=1}^N a_i \left(x, \frac{\partial}{\partial x_i} v \right) \frac{\partial}{\partial x_i} (u) \right) dx \right)$ goes to 0 as $h \to \infty$, hence, we obtain

$$\limsup_{h\to\infty} A(h,k) \geq \int_{[|u-v|<k]} \left[\sum_{i=1}^N \left(a_i \left(x, \frac{\partial}{\partial x_i} u \right) - a_i \left(x, \frac{\partial}{\partial x_i} v \right) \right) \frac{\partial}{\partial x_i} (u-v) \right] dx. \tag{4.19}$$

By using the Lebesgue dominated convergence theorem, it yields that

$$\lim_{h\to\infty} B(h,k) = \int_\Omega (b(u) - b(v)) T_k(u-v) dx \text{ and } \lim_{h\to\infty} D(h,k) = 0. \tag{4.20}$$

For h large enough, we get

$$\lim_{h\to\infty} C(h,k) = (\rho(u_N) - \rho(v_N)) T_k(u_N - v_N) \text{ and } \lim_{h\to\infty} E(h,k) = 0. \tag{4.21}$$

$$\begin{cases} T(h,k) = \int_{A_1} F.\nabla u dx + \int_{A_1'} F.\nabla v dx \\ + \int_{A_2} F.\nabla (u-v) dx + \int_{A_2'} F.\nabla (v-u) dx. \end{cases}$$

$$\begin{cases} T(h,k) = \int_{A_1} F.\nabla u dx + \int_{A_1'} F.\nabla v dx \\ + \int_{A_2} F.\nabla u dx - \int_{A_2} F.\nabla v dx + \int_{A_2'} F.\nabla v dx - \int_{A_2'} F.\nabla u dx. \end{cases}$$

Using Assertion 4.3 and Remark 4.4, it is easy to see that $\lim_{h\to\infty} |T(h,k)| = 0$.
Letting h go to ∞ in (4.11) and combining (4.20)-(4.21), we obtain

$$\begin{cases} \int_{[|u-v|<k]} \left[\sum_{i=1}^{N} \left(a_i \left(x, \frac{\partial}{\partial x_i} u \right) - a_i \left(x, \frac{\partial}{\partial x_i} v \right) \right) \frac{\partial}{\partial x_i}(u-v) \right] dx \\ + \int_{\Omega} (b(u) - b(v))\, T_k(u-v) dx + (\rho(u_N) - \rho(v_N))\, T_k(u_N - v_N) \leq 0. \end{cases} \tag{4.22}$$

All the terms in the left hand side of (4.22) are non-negative so that we get $\forall k > 0$,

$$\int_{[|u-v|<k]} \left[\sum_{i=1}^{N} \left(a_i \left(x, \frac{\partial}{\partial x_i} u \right) - a_i \left(x, \frac{\partial}{\partial x_i} v \right) \right) \frac{\partial}{\partial x_i}(u-v) \right] dx = 0 \tag{4.23}$$

and

$$\begin{cases} \int_{\Omega} (b(u) - b(v))\, T_k(u-v) dx = 0 \\ (\rho(u_N) - \rho(v_N))\, T_k(u_N - v_N) = 0. \end{cases} \tag{4.24}$$

Relation (4.23) gives $\frac{\partial}{\partial x_i}(u-v) = 0$ a.e. in Ω; we deduce that there exists a constant c such that $u - v = c$ a.e. in Ω. According to (4.24), $b(u) = b(v)$. Since b is invertible, we deduce that $u = v$ in Ω and so

$$\begin{cases} u = v \text{ a.e. in } \Omega \\ \rho(u_N) = \rho(v_N); \end{cases}$$

which prove the uniqueness part.

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